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Research Article

Nonoscillatory Solutions for Higher-Order Neutral Dynamic Equations on Time Scales

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We study the higher-order neutral dynamic equation $\{a(t)[(x(t)-p(t)x(\tau(t)))^{\Delta^m}]^{\alpha}\}^{\Delta}+f(t, x(\delta(t)))=0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ and obtain some necessary and sufficient conditions for the existence of nonoscillatory bounded solutions for this equation.

1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus, $\mathbb{R},\mathbb{Z}$, and $\mathbb{N}$, that is, the real numbers, the integers, and the natural numbers, are examples of time scales. We assume throughout that the time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The theory of time scale, which has recently received a lot of attention, was introduced by Hilger's landmark paper [1], a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is a nonempty closed subset of the real numbers, and the cases when this time scale is equal to the real numbers or to the integers represent the classical theories of differential and difference equations. Many other interesting time scales exist, and they give rise to many applications (see [2]). Not only the new theory of the so-called "dynamic equations" unifies the theories of differential equations and difference equations but also extends these classical cases to cases "in between", for example, to the so-called q-difference equations when $\mathbb{T}=\{1,q,q^2,\dots\}$, which has important applications in quantum theory (see [3]).

On a time scale $\mathbb{T}$, the forward jump operator, the backward jump operator and the graininess function are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t, \tag{1.1}$$

respectively. We refer the reader to [2, 4] for further results on time scale calculus.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the papers of [5–20].

In [21] Zhu and Wang studied the existence of nonoscillatory solutions to neutral dynamic equation

$$\left[x(t) + p(t)x(g(t))\right]^{\Delta} + f(t, x(h(t))) = 0. \tag{1.2}$$

Karpuz [22] studied the asymptotic behavior of delay dynamic equations having the following form:

$$[x(t) + A(t)x(\alpha(t))]^{\Delta} + B(t)F\big(x\big(\beta(t)\big)\big) - C(t)G\big(x\big(\gamma(t)\big)\big) = \varphi(t). \tag{1.3}$$

Furthermore, Karpuz in [23] obtained necessary and sufficient conditions for the asymptotic behaviour of all bounded solutions of the following higher-order nonlinear forced neutral dynamic equation

$$[x(t) + A(t)x(\alpha(t))]^{\Delta^n} + f\big(t, x\big(\beta(t)\big), x\big(\gamma(t)\big)\big) = \varphi(t) \tag{1.4}$$

and also studied in [24] oscillation of unbounded solutions of a similar type of equations.

Li et al. [25] considered the existence of nonoscillatory solutions to the second-order neutral delay dynamic equation of the form

$$\left[x(t) + p(t)x(\tau_0(t))\right]^{\Delta\Delta} + q_1(t)x(\tau_1(t)) - q_2(t)x(\tau_2(t)) = e(t). \tag{1.5}$$

In [26, 27], Zhang et al. obtained some sufficient conditions for the existence of nonoscillatory solutions for the following higher-order equations:

$$\begin{gathered} \left[x(t) + p(t)x(\tau(t))\right]^{\Delta^n} + f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_k(t))) = 0\;, \\ \left[x(t) + p(t)x(\tau(t))\right]^{\Delta^n} + f(t, x(\tau_1(t)), \ldots, x(\tau_k(t))) = 0, \end{gathered} \tag{1.6}$$

respectively.

Motivated by the above studies, in this paper, we investigate the existence of nonoscillatory solutions of the following higher order neutral dynamic equation:

$$\left\{a(t)\left[(x(t) - p(t)x(\tau(t)))^{\Delta^m}\right]^{\alpha}\right\}^{\Delta} + f(t, x(\delta(t))) = 0 \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.7}$$

where $m \in \mathbb{N}$, α is the quotient of odd positive integers, $t_0 \in \mathbb{T}$, the time scale interval $[t_0,\infty)_{\mathbb{T}} = \{t \in \mathbb{T} : t \geq t_0\}$, $a \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $p \in C([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $\tau,\delta \in C(\mathbb{T},\mathbb{T})$ with $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$ and $f \in C([t_0,\infty)_{\mathbb{T}} \times \mathbb{R},\mathbb{R})$ satisfying the following conditions:

(i) $uf(t,u) > 0$ for any $t \in [t_0,\infty)_{\mathbb{T}}$ and $u \neq 0$.

(ii) $f(t,u)$ is nondecreasing in u for any $t \in [t_0,\infty)_{\mathbb{T}}$.

Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $\sup\mathbb{T} = \infty$. By a solution of (1.7) we mean a nontrivial real-valued function $x \in C_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R}), T_x \geq t_0$, such that $a(t)[(x(t)-p(t)x(\tau(t)))^{\Delta^m}]^{\alpha} \in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies (1.7) on $[T_x,\infty)$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.7) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

2. Auxiliary Results

We state the following conditions, which are needed in the sequel:

(H_1) $\int_{t_0}^{\infty}(a(t))^{-1/\alpha}\Delta t = \infty$;

(H_2) there exist constants $a_1, b_1 \in [0,1)$ with $a_1 + b_1 < 1$ such that $-a_1 \leq p(t) \leq b_1$ for all $t \in [t_0,\infty)_{\mathbb{T}}$;

(H_3) there exist constants $a_2, b_2 \in (1,\infty)$ such that $-a_2 \leq p(t) \leq -b_2$ for all $t \in [t_0,\infty)_{\mathbb{T}}$;

(H_4) there exist constants $a_3, b_3 \in (1,\infty)$ such that $a_3 \leq p(t) \leq b_3$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Let k be a nonnegative integer and $s,t \in \mathbb{T}$; we define two sequences of functions $h_k(t,s)$ and $g_k(t,s)$ as follows:

$$h_k(t,s) = \begin{cases} 1 & \text{if } k = 0, \\ \displaystyle\int_s^t h_{k-1}(\tau,s)\Delta\tau & \text{if } k \geq 1, \end{cases}$$
$$g_k(t,s) = \begin{cases} 1 & \text{if } k = 0, \\ \displaystyle\int_s^t g_{k-1}(\sigma(\tau),s)\Delta\tau & \text{if } k \geq 1. \end{cases} \tag{2.1}$$

By Theorems 1.112 and 1.60 of [2], we have

$$h_k(t,s) = (-1)^k g_k(s,t) \quad \text{for all } t,s \in \mathbb{T},$$
$$h_k^{\Delta_t}(t,s) = \begin{cases} 0 & \text{if } k = 0, \\ h_{k-1}(t,s) & \text{if } k \geq 1, \end{cases} \tag{2.2}$$
$$g_k^{\Delta_t}(t,s) = \begin{cases} 0 & \text{if } k = 0, \\ g_{k-1}(\sigma(t),s) & \text{if } k \geq 1, \end{cases}$$

where $g_k^{\Delta_t}(t,s)$ and $h_k^{\Delta_t}(t,s)$ denote for each fixed s the derivatisve of $g_k(t,s)$ and $h_k(t,s)$ with respect to t, respectively.

Lemma 2.1 (see [23, 24]). *Assume that $s, t \in \mathbb{T}$ and $g \in C_{rd}(\mathbb{T}\times\mathbb{T},\mathbb{R})$, then*

$$\int_s^t \left[\int_\eta^t g(\eta,\zeta)\Delta\zeta\right]\Delta\eta = \int_s^t \left[\int_s^{\sigma(\zeta)} g(\eta,\zeta)\Delta\eta\right]\Delta\zeta. \tag{2.3}$$

Lemma 2.2 (see [23, 24]). *Let n be a nonnegative integer, $h \in C_{rd}(\mathbb{T},[0,\infty))$, and $s \in \mathbb{T}$. Then each of the following is true:*

(i) $\int_s^\infty g_n(\sigma(\theta),s)h(\theta)\Delta\theta < \infty$ *implies that* $\int_t^\infty g_n(\sigma(\theta),t)h(\theta)\Delta\theta < \infty$ *for all* $t \in \mathbb{T}$;

(ii) $\int_s^\infty g_n(\sigma(\theta),s)h(\theta)\Delta\theta = \infty$ *implies that* $\int_t^\infty g_n(\sigma(\theta),t)h(\theta)\Delta\theta = \infty$ *for all* $t \in \mathbb{T}$.

Lemma 2.3 (see [23]). *Let n be a nonnegative integer, $h \in C_{rd}(\mathbb{T},[0,\infty))$ and $s \in \mathbb{T}$. Then*

$$\int_s^\infty g_n(\sigma(\theta),s)h(\theta)\Delta\theta < \infty \tag{2.4}$$

implies that each of the following is true:

(i) $\int_t^\infty g_j(\sigma(\theta),t)h(\theta)\Delta\theta$ *is decreasing for all* $t \in \mathbb{T}$ *and all* $0 \le j \le n$.

(ii) $\lim_{t\to\infty}\int_t^\infty g_j(\sigma(\theta),t)h(\theta)\Delta\theta = 0$ *for all* $0 \le j \le n$.

(iii) $\int_t^\infty g_j(\sigma(\theta),t)h(\theta)\Delta\theta < \infty$ *for all* $t \in \mathbb{T}$ *and all* $0 \le j \le n-1$.

Lemma 2.4 (see [28]). *Let $m \in \mathbf{N}$ and $f^{\Delta^m} \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. Then*

(1) $\liminf_{t\to\infty} f^{\Delta^m}(t) > 0$ *implies* $\lim_{t\to\infty} f^{\Delta^i}(t) = \infty$ *for all* $0 \le i \le m-1$.

(2) $\limsup_{t\to\infty} f^{\Delta^m}(t) < 0$ *implies* $\lim_{t\to\infty} f^{\Delta^i}(t) = -\infty$ *for all* $0 \le i \le m-1$.

Lemma 2.5 (see [29]). *Let $z(t)$ be bounded for $t \in [t_0,\infty)_{\mathbb{T}}$ with $z^{\Delta^n}(t) > 0$, where $n \in \mathbf{N}$. Then $(-1)^{n-i}z^{\Delta^i}(t) > 0$ for $1 \le i \le n$ and*

$$\lim_{t\to\infty} z^{\Delta^i}(t) = 0 \quad \text{for } 1 \le i \le n-1. \tag{2.5}$$

In the sequel, write

$$y(t) = x(t) - p(t)x(\tau(t)). \tag{2.6}$$

Lemma 2.6. *Assume that $p(t)$ is bounded and (H_1) holds. If $x(t)$ is a bounded nonoscillatory solution of (1.7), then $x(t)y^{\Delta^m}(t) > 0$ eventually.*

Proof. Without loss of generality, assume that there is some $t_1 \ge t_0$ such that $x(t) > 0$ and $x(\delta(t)) > 0$ for $t \ge t_1$. From (1.7) we have

$$\left\{a(t)\left[y^{\Delta^m}(t)\right]^\alpha\right\}^\Delta = -f(t,x(\delta(t))) < 0 \quad \text{for } t \ge t_1. \tag{2.7}$$

Thus, $R(t) = a(t)[y^{\Delta^m}(t)]^\alpha$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. If there exists $t_2 \geq t_1$ such that $R(t_2) < 0$, then

$$y^{\Delta^m}(t) \leq \left[\frac{R(t_2)}{a(t)}\right]^{1/\alpha} \quad \text{for } t \geq t_2. \tag{2.8}$$

Therefore, we have

$$y^{\Delta^{m-1}}(t) - y^{\Delta^{m-1}}(t_2) \leq \int_{t_2}^{t} \left[\frac{R(t_2)}{a(s)}\right]^{1/\alpha} \Delta s. \tag{2.9}$$

By condition (H_1), we obtain $\lim_{t\to\infty} y^{\Delta^{m-1}}(t) = -\infty$. Then it follows from Lemma 2.4 that $\lim_{t\to\infty} y(t) = -\infty$, which is a contradiction since $x(t)$ and $p(t)$ are bounded. Hence, $y^{\Delta^m}(t) = [R(t)/a(t)]^{1/\alpha} > 0$ for all $t \geq t_1$. The proof is completed.

Let $BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_0,\infty)_{\mathbb{T}}$ with sup norm $\|x\| = \sup_{t\geq t_0}|x(t)|$. Let $X \subset BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, we say that X is uniformly Cauchy if for any given $\varepsilon > 0$, there exists $t_1 > t_0$ such that for any $x \in X$, $|x(u) - x(v)| < \varepsilon$ for all $u,v \in [t_1,\infty)_{\mathbb{T}}$. X is said to be equicontinuous on $[a,b]_{\mathbb{T}}$ if, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x \in X$ and $u,v \in [a,b]_{\mathbb{T}}$ with $|u - v| < \delta$, $|x(u)-x(v)| < \varepsilon$. $S : X \to BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is called completely continuous if it is continuous and maps bounded sets into relatively compact sets. □

Lemma 2.7 (see [21]). *Suppose that $X \subset BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[t_0,t_1]_{\mathbb{T}}$ for any $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then X is relatively compact.*

Lemma 2.8 (see [21]). *Suppose that X is a Banach space and Ω is a bounded, convex, and closed subset of X. Suppose further that there exist two operators U and $S : \Omega \to X$ such that*

(i) *$Ux + Sy \in \Omega$ for all $x,y \in \Omega$,*

(ii) *U is a contraction mapping,*

(iii) *S is completely continuous.*

Then $U + S$ has a fixed point in Ω.

3. Main Results and Examples

Now, we state and prove our main results.

Theorem 3.1. *Assume that (H_1) and (H_2) hold. Then (1.7) has a nonoscillatory bounded solution $x(t)$ with $\liminf_{t\to\infty}|x(t)| > 0$ if and only if there exists some constant $K \neq 0$ such that*

$$\int_{t_0}^{\infty} g_{m-1}(\sigma(s),t_0) \left[\frac{1}{a(s)} \int_{s}^{\infty} f(\theta,|K|)\Delta\theta\right]^{1/\alpha} \Delta s < \infty. \tag{3.1}$$

Proof. Sufficiency. Assume that (1.7) has a nonoscillatory bounded solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$ with $\liminf_{t\to\infty}|x(t)| > 0$. Without loss of generality, we assume that there is a constant $K > 0$

and some $t_1 \geq t_0$ such that $x(t) > K$ and $x(\delta(t)) > K$ for $t \geq t_1$. It follows from Lemma 2.6 that $y^{\Delta^m}(t) > 0$ for $t \geq t_1$. By assumption that $x(t)$ is bounded and condition (H_2), we see that $y(t)$ is bounded. Thus, by Lemma 2.5 we get that there exists $t_2 \geq t_1$ such that

$$(-1)^{m-k} y^{\Delta^k}(t) > 0 \quad \text{for } t \geq t \geq t_2\,,\ 1 \leq k \leq m. \tag{3.2}$$

Integrating (1.7) from $t(\geq t_2)$ to ∞, we have

$$a(t)\left(y^{\Delta^m}(t)\right)^{\alpha} \geq \int_t^{\infty} f(s, x(\delta(s)))\Delta s \geq \int_t^{\infty} f(s, K)\Delta s \quad \text{for } t \geq t_2. \tag{3.3}$$

Therefore, it follows from (3.2) and (3.3) that, for $t \geq t_2$,

$$\begin{aligned}
&\int_t^{\infty} g_{m-1}(\sigma(\theta), t)\left[\frac{1}{a(\theta)}\int_{\theta}^{\infty} f(s, K)\Delta s\right]^{1/\alpha}\Delta\theta \\
&\quad \leq \int_t^{\infty} y^{\Delta^m}(\theta) g_{m-1}(\sigma(\theta), t)\Delta\theta \\
&\quad = y^{\Delta^{m-1}}(\theta) g_{m-1}(\theta, t)\Big|_t^{\infty} - \int_t^{\infty} y^{\Delta^{m-1}}(\theta) g_{m-2}(\sigma(\theta), t)\Delta\theta \\
&\quad = -\lim_{\theta\to\infty} (-1)^{m-(m-1)} y^{\Delta^{m-1}}(\theta) g_{m-1}(\theta, t) - \int_t^{\infty} y^{\Delta^{m-1}}(\theta) g_{m-2}(\sigma(\theta), t)\Delta\theta \\
&\quad \leq -\int_t^{\infty} y^{\Delta^{m-1}}(\theta) g_{m-2}(\sigma(\theta), t)\Delta\theta \\
&\quad = -y^{\Delta^{m-2}}(\theta) g_{m-2}(\theta, t)\Big|_t^{\infty} + (-1)^2\int_t^{\infty} y^{\Delta^{m-2}}(\theta) g_{m-3}(\sigma(\theta), t)\Delta\theta \\
&\quad \leq (-1)^2\int_t^{\infty} y^{\Delta^{m-2}}(\theta) g_{m-3}(\sigma(\theta), t)\Delta\theta \\
&\quad \cdots\cdots\cdots\cdots \\
&\quad \leq (-1)^{m-1}\int_t^{\infty} y^{\Delta}(\theta)\Delta\theta \\
&\quad = (-1)^{m-1} y(\theta)\Big|_t^{\infty} < \infty.
\end{aligned} \tag{3.4}$$

By Lemma 2.2, we see that (3.1) holds.

Necessity. Suppose that there exists some constant $K > 0$ such that

$$\int_{t_0}^{\infty} g_{m-1}(\sigma(s), t_0) A(s)\Delta s < \infty, \tag{3.5}$$

where $A(s) = [\int_s^\infty f(\theta,K)\Delta\theta]^{1/\alpha}/a(s)$. Then by Lemma 2.3 there exists $t_1 \ge t_0$ such that

$$\int_t^\infty g_{m-1}(\sigma(s),t)A(s)\Delta s < \frac{1-b_1}{n}K \tag{3.6}$$

and $\min\{\delta(t),\tau(t)\} \ge t_0$ for $t \ge t_1$, where $n > 2(1-b_1)/(1-b_1-a_1)$ is a constant. Let

$$\Omega = \left\{ x \in BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}) : \frac{n(1-b_1-a_1)-2(1-b_1)}{n}K \le x(t) \le K \quad \text{for } t \ge t_0 \right\}. \tag{3.7}$$

It is easy to verify that Ω is a bounded, convex, and closed subset of $BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$.

Now we define two operators S and $T : \Omega \to BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ as follows:

$$\begin{gathered} (Sx)(t) = p(t^*)x(\tau(t^*)), \\ (Tx)(t) = \frac{(n-1)(1-b_1)}{n}K + (-1)^m \int_{t^*}^\infty g_{m-1}(\sigma(s),t^*)A(s,x)\Delta s, \end{gathered} \tag{3.8}$$

where $u^* = \max\{u,t_1\}$ for any $u \in [t_0,\infty)_{\mathbb{T}}$ and $A(s,z) = [\int_s^\infty f(\theta,z(\delta(\theta)))\Delta\theta]^{1/\alpha}/a(s)$ for any $z \in \Omega$. Now we show that S and T satisfy the conditions in Lemma 2.8.

(1) We will show that $Sx + Ty \in \Omega$ for any $x,y \in \Omega$. In fact, for any $x,y \in \Omega$ and $t \ge t_0$, $x(t),y(t) \in [[n(1-b_1-a_1)-2(1-b_1)]K/n, K]$,

$$\begin{aligned} (Sx)(t) + (Ty)(t) &= \frac{(n-1)(1-b_1)}{n}K + p(t^*)x(\tau(t^*)) \\ &\quad + (-1)^m \int_{t^*}^\infty g_{m-1}(\sigma(s),t^*)A(s,x)\Delta s \\ &\le \frac{(n-1)(1-b_1)}{n}K + b_1K + \frac{1-b_1}{n}K \\ &= K, \\ (Sx)(t) + (Ty)(t) &= \frac{(n-1)(1-b_1)}{n}K + p(t^*)x(\tau(t^*)) \\ &\quad + (-1)^m \int_{t^*}^\infty g_{m-1}(\sigma(s),t^*)A(s,x)\Delta s \\ &\ge \frac{(n-1)(1-b_1)}{n}K - a_1K - \frac{1-b_1}{n}K \\ &= \frac{n(1-b_1-a_1)-2(1-b_1)}{n}K, \end{aligned} \tag{3.9}$$

which implies that $Sx + Ty \in \Omega$ for any $x,y \in \Omega$.

(2) We will show that S is a contraction mapping. Indeed, for any $x, y \in \Omega$ and $t \geq t_0$, we have

$$\begin{aligned} \left|(Sx)(t) - (Sy)(t)\right| &= \left|p(t^*)x(\tau(t^*)) - p(t^*)y(\tau(t^*))\right| \\ &\leq \max\{a_1, b_1\}\left\|x - y\right\|. \end{aligned} \tag{3.10}$$

Therefore, we have

$$\left\|Sx - Sy\right\| \leq \max\{a_1, b_1\}\left\|x - y\right\|, \tag{3.11}$$

which implies that S is a contraction mapping.

(3) We will show that T is a completely continuous mapping.

(i) By the proof of (1), we see that $[n(1 - b_1 - a_1) - 2(1 - b_1)]K/n \leq (Tx)(t) \leq K$ for $t \in [t_0, \infty)_{\mathbb{T}}$. That is, $T\Omega \subset \Omega$.

(ii) We consider the continuity of T. Let $x_n \in \Omega$ and $\|x_n - x\| \to 0$ as $n \to \infty$, then $x \in \Omega$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [t_0, \infty)_{\mathbb{T}}$. Consequently, for any $s \in [t_1, \infty)_{\mathbb{T}}$, we have

$$\lim_{n \to \infty} \left|g_{m-1}(\sigma(s), t_1)[A(s, x_n) - A(s, x)]\right| = 0. \tag{3.12}$$

Since

$$\left|g_{m-1}(\sigma(s), t_1)[A(s, x_n) - A(s, x)]\right| \leq 2g_{m-1}(\sigma(s), t_1)A(s) \tag{3.13}$$

and, for any $t \in [t_0, \infty)_{\mathbb{T}}$,

$$|(Tx_n)(t) - (Tx)(t)| \leq \int_{t_1}^{\infty} g_{m-1}(\sigma(s), t_1)|A(s, x_n) - A(s, x)|\Delta s, \tag{3.14}$$

we have

$$\|Tx_n - Tx\| \leq \int_{t_1}^{\infty} g_{m-1}(\sigma(s), t_1)|A(s, x_n) - A(s, x)|\Delta s. \tag{3.15}$$

By Chapter 5 in [4], we see that the Lebesgue dominated convergence theorem holds for the integral on time scales. Then

$$\lim_{n \to \infty} \|Tx_n - Tx\| = 0, \tag{3.16}$$

which implies that T is continuous on Ω.

(iii) We show that $T\Omega$ is uniformly Cauchy. In fact, for any $\varepsilon > 0$, take $t_2 > t_1$ so that

$$\int_{t_2}^{\infty} g_{m-1}(\sigma(s), t_2) A(s) \Delta s < \varepsilon. \tag{3.17}$$

Then for any $x \in \Omega$ and $u, v \in [t_2, \infty)_{\mathbb{T}}$, we have

$$|(Tx)(u) - (Tx)(v)| < 2\varepsilon, \tag{3.18}$$

which implies that $T\Omega$ is uniformly Cauchy.

(iv) We show that $T\Omega$ is equicontinuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume $t_2 \geq t_1$. For any $\varepsilon > 0$, choose $\delta = \varepsilon/(1 + \int_{t_0}^{\infty} g_{m-2}(\sigma(s), t_0) A(s) \Delta s)$, then when $u, v \in [t_0, t_2]$ with $|u - v| < \delta$, we have by Lemma 2.1 that for any $x \in \Omega$,

$$\begin{aligned}
|(Tx)(u) - (Tx)(v)| &= \left| \int_{u^*}^{\infty} g_{m-1}(\sigma(s), u^*) A(s,x) \Delta s - \int_{v^*}^{\infty} g_{m-1}(\sigma(s), v^*) A(s,x) \Delta s \right| \\
&= \left| \int_{u^*}^{\infty} h_{m-1}(u^*, \sigma(s)) A(s,x) \Delta s - \int_{v^*}^{\infty} h_{m-1}(v^*, \sigma(s)) A(s,x) \Delta s \right| \\
&= \left| \int_{u^*}^{\infty} \left[\int_{\sigma(s)}^{u^*} h_{m-2}(\theta, \sigma(s)) A(s,x) \Delta\theta \right] \Delta s \right. \\
&\quad \left. - \int_{v^*}^{\infty} \left[\int_{\sigma(s)}^{v^*} h_{m-2}(\theta, \sigma(s)) A(s,x) \Delta\theta \right] \Delta s \right| \\
&= \left| \int_{u^*}^{\infty} \left[\int_{\theta}^{\infty} h_{m-2}(\theta, \sigma(s)) A(s,x) \Delta s \right] \Delta\theta \right. \\
&\quad \left. - \int_{v^*}^{\infty} \left[\int_{\theta}^{\infty} h_{m-2}(\theta, \sigma(s)) A(s,x) \Delta s \right] \Delta\theta \right| \\
&= \left| \int_{u^*}^{\infty} \left[\int_{\theta}^{\infty} g_{m-2}(\sigma(s), \theta) A(s,x) \Delta s \right] \Delta\theta \right. \\
&\quad \left. - \int_{v^*}^{\infty} \left[\int_{\theta}^{\infty} g_{m-2}(\sigma(s), \theta) A(s,x) \Delta s \right] \Delta\theta \right| \\
&= \left| \int_{u^*}^{v^*} \left[\int_{\theta}^{\infty} g_{m-2}(\sigma(s), \theta) A(s,x) \Delta s \right] \Delta\theta \right| \\
&\leq \left| \int_{u^*}^{v^*} \left[\int_{t_0}^{\infty} g_{m-2}(\sigma(s), t_0) A(s,x) \Delta s \right] \Delta\theta \right| \\
&= |u^* - v^*| \int_{t_0}^{\infty} g_{m-2}(\sigma(s), t_0) A(s,x) \Delta s \\
&\leq \delta \int_{t_0}^{\infty} g_{m-2}(\sigma(s), t_0) A(s) \Delta s < \varepsilon,
\end{aligned} \tag{3.19}$$

which implies that $T\Omega$ is equi-continuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$.

By Lemma 2.7, we see that T is a completely continuous mapping. It follows from Lemma 2.8 that there exists $x \in \Omega$ such that $(U+S)x = x$, which is the desired bounded solution of (1.7) with $\liminf_{t\to\infty}|x(t)| > 0$. The proof is completed. □

Theorem 3.2. *Assume that* (H_1) *and* (H_3) *hold, and that* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T},\mathbb{T})$*. Then* (1.7) *has a nonoscillatory bounded solution* $x(t)$ *with* $\liminf_{t\to\infty}|x(t)| > 0$ *if and only if there exists some constant* $K \neq 0$ *such that* (3.1) *holds.*

Proof. The proof of sufficiency is similar to that of Theorem 3.1.

Necessity. Suppose that there exists some constant $K > 0$ such that

$$\int_{t_0}^{\infty} g_{m-1}(\sigma(s),t_0)A(s)\Delta s < \infty, \tag{3.20}$$

where $A(s) = [\int_s^\infty f(\theta,K)\Delta\theta]^{1/\alpha}/a(s)$. Then by Lemma 2.3 there exists $t_1 \geq t_0$ such that

$$\int_{\tau^{-1}(t)}^{\infty} g_{m-1}\left(\sigma(s),\tau^{-1}(t)\right)A(s)\Delta s < \frac{b_2}{n}K, \tag{3.21}$$

and $\min\{\delta(\tau^{-1}(t)),\tau^{-1}(t)\} \geq t_0$ for $t \geq t_1$, where $n > 2b_2/(b_2-1)$ is a constant. Let

$$\Omega = \left\{x \in BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}) : \frac{(n-2)b_2-n}{a_2 n}K \leq x(t) \leq K \quad \text{for } t \geq t_0\right\}. \tag{3.22}$$

It is easy to verify that Ω is a bounded, convex and closed subset of $BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$.

Now we define two operators S and $T : \Omega \to BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ as follows:

$$\begin{aligned}(Sx)(t) &= \frac{x(\tau^{-1}(t^*))}{p(\tau^{-1}(t^*))} + \frac{(n-1)b_2K}{-np(\tau^{-1}(t^*))}\\ (Tx)(t) &= \frac{1}{p(\tau^{-1}(t^*))}(-1)^{m-1}\int_{\tau^{-1}(t^*)}^{\infty} g_{m-1}\left(\sigma(s),\tau^{-1}(t^*)\right)A(s,x)\Delta s,\end{aligned} \tag{3.23}$$

where $u^* = \max\{u,t_1\}$ for any $u \in [t_0,\infty)_{\mathbb{T}}$ and $A(s,z) = [\int_s^\infty f(\theta,z(\delta(\theta)))\Delta\theta]^{1/\alpha}/a(s)$ for any $z \in \Omega$. Now we show that S and T satisfy the conditions in Lemma 2.8.

We will show that $Sx + Ty \in \Omega$ for any $x, y \in \Omega$. In fact, for any $x, y \in \Omega$ and $t \geq t_0$, $x(t), y(t) \in [[(n-2)b_2 - n]K/a_2 n, K]$,

$$\begin{aligned}
(Sx)(t) + (Ty)(t) &= \frac{x\left(\tau^{-1}(t^*)\right)}{p\left(\tau^{-1}(t^*)\right)} + \frac{1}{-p\left(\tau^{-1}(t^*)\right)}\left[\frac{(n-1)b_2}{n}K \right. \\
&\quad \left. +(-1)^m \int_{\tau^{-1}(t^*)}^{\infty} g_{m-1}\left(\sigma(s), \tau^{-1}(t^*)\right) A(s,x)\Delta s\right] \\
&\leq \frac{1}{b_2}\left[\frac{(n-1)b_2}{n}K + \frac{b_2}{n}K\right] \\
&= K, \\
(Sx)(t) + (Ty)(t) &= \frac{x\left(\tau^{-1}(t^*)\right)}{p\left(\tau^{-1}(t^*)\right)} + \frac{1}{-p\left(\tau^{-1}(t^*)\right)}\left[\frac{(n-1)b_2}{n}K \right. \\
&\quad \left. +(-1)^m \int_{\tau^{-1}(t^*)}^{\infty} g_{m-1}\left(\sigma(s), \tau^{-1}(t^*)\right) A(s,x)\Delta s\right] \\
&\geq \frac{1}{a_2}\left[\frac{(n-1)b_2}{n}K - K - \frac{b_2}{n}K\right] \\
&= \frac{(n-2)b_2 - n}{a_2 n}K,
\end{aligned} \tag{3.24}$$

and $|(Tx)(t)| \leq b_2 K/n$, which implies that $Sx + Ty \in \Omega$ for any $x, y \in \Omega$ and $T\Omega$ is uniformly bounded.

Now we show that $T\Omega$ is equicontinuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $t_2 \geq t_1$. Since $1/p(\tau^{-1}(t)), \tau^{-1}(t)$ are continuous on $[t_0, t_2]_{\mathbb{T}}$, so they are uniformly continuous on $[t_0, t_2]_{\mathbb{T}}$. For any $\varepsilon > 0$, choose $\delta > 0$ such that when $u, v \in [t_0, t_2]_{\mathbb{T}}$ with $|u - v| < \delta$, we have

$$\begin{aligned}
\left|\frac{1}{p\left(\tau^{-1}(u)\right)} - \frac{1}{p\left(\tau^{-1}(v)\right)}\right| &< \frac{\varepsilon}{1 + \int_{t_0}^{\infty} g_{m-1}(\sigma(s), t_0)A(s)\Delta s}, \\
\left|\tau^{-1}(u) - \tau^{-1}(v)\right| &< \frac{\varepsilon}{1 + \int_{t_0}^{\infty} g_{m-2}(\sigma(s), t_0)A(s)\Delta s}.
\end{aligned} \tag{3.25}$$

Then, it follows from Lemma 2.1 that, for any $x \in \Omega$,

$$\begin{aligned}
|(Tx)(u) - (Tx)(v)| = \Bigg|&\frac{1}{p\left(\tau^{-1}(u^*)\right)} \int_{\tau^{-1}(u^*)}^{\infty} g_{m-1}\left(\sigma(s), \tau^{-1}(u^*)\right) A(s,x)\Delta s \\
&- \frac{1}{p\left(\tau^{-1}(v^*)\right)} \int_{\tau^{-1}(v^*)}^{\infty} g_{m-1}\left(\sigma(s), \tau^{-1}(v^*)\right) A(s,x)\Delta s\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\left[\frac{1}{p(\tau^{-1}(u^*))}-\frac{1}{p(\tau^{-1}(v^*))}\right]\int_{\tau^{-1}(u^*)}^{\infty} g_{m-1}\Big(\sigma(s),\tau^{-1}(u^*)\Big)A(s,x)\Delta s\right| \\
&\quad+\left|\frac{1}{p(\tau^{-1}(v^*))}\left[\int_{\tau^{-1}(u^*)}^{\infty} g_{m-1}\Big(\sigma(s),\tau^{-1}(u^*)\Big)A(s,x)\Delta s\right.\right. \\
&\quad\left.\left.-\int_{\tau^{-1}(v^*)}^{\infty} g_{m-1}\Big(\sigma(s),\tau^{-1}(v^*)\Big)A(s,x)\Delta s\right]\right| \\
&= \left|\left[\frac{1}{p(\tau^{-1}(u^*))}-\frac{1}{p(\tau^{-1}(v^*))}\right]\int_{\tau^{-1}(u^*)}^{\infty} g_{m-1}\Big(\sigma(s),\tau^{-1}(u^*)\Big)A(s,x)\Delta s\right| \\
&\quad+\left|\frac{1}{p(\tau^{-1}(v^*))}\int_{\tau^{-1}(u^*)}^{\tau^{-1}(v^*)}\left[\int_{\theta}^{\infty} g_{m-2}(\sigma(s),\theta)A(s,x)\Delta s\right]\Delta\theta\right| \\
&\leq \left|\left[\frac{1}{p(\tau^{-1}(u^*))}-\frac{1}{p(\tau^{-1}(v^*))}\right]\int_{t_0}^{\infty} g_{m-1}(\sigma(s),t_0)A(s)\Delta s\right| \\
&\quad+\left|\frac{1}{p(\tau^{-1}(v^*))}\int_{\tau^{-1}(u^*)}^{\tau^{-1}(v^*))}\left[\int_{t_0}^{\infty} g_{m-2}(\sigma(s),t_0)A(s)\Delta s\right]\Delta\theta\right| \\
&\leq \varepsilon+\left|\tau^{-1}(u^*)-\tau^{-1}(v^*)\right|\int_{t_0}^{\infty} g_{m-2}(\sigma(s),t_0)A(s)\Delta s \\
&< 2\varepsilon,
\end{aligned}
\tag{3.26}
$$

which implies that $T\Omega$ is equi-continuous on $[t_0,t_2]_{\mathbb{T}}$ for any $t_2\in[t_0,\infty)_{\mathbb{T}}$.

The rest of the proof is similar to that of Theorem 3.1. The proof is completed. □

Theorem 3.3. *Assume that* (H_1) *and* (H_4) *hold and that* τ *has the inverse* $\tau^{-1}\in C(\mathbb{T},\mathbb{T})$. *Then* (1.7) *has a nonoscillatory bounded solution* $x(t)$ *with* $\liminf_{t\to\infty}|x(t)|>0$ *if and only if there exists some constant* $K\neq 0$ *such that* (3.1) *holds.*

Proof. The proof of sufficiency is similar to that of Theorem 3.1.

Necessity. Suppose that there exists some constant $K>0$ such that

$$\int_{t_0}^{\infty} g_{m-1}(\sigma(s),t_0)A(s)\Delta s<\infty, \tag{3.27}$$

where $A(s)=[\int_s^{\infty} f(\theta,K)\Delta\theta]^{1/\alpha}/a(s)$. Then by Lemma 2.3 there exists $t_1\geq t_0$ such that

$$\int_{\tau^{-1}(t)}^{\infty} g_{m-1}\Big(\sigma(s),\tau^{-1}(t)\Big)A(s)\Delta s<\frac{a_3-1}{n}K \tag{3.28}$$

and $\min\{\delta(\tau^{-1}(t)),\tau^{-1}(t)\}\geq t_0$ for $t\geq t_1$, where $n>2$ is a constant. Let

$$\Omega=\left\{x\in BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}):\frac{(n-2)(a_3-1)}{n(b_3-1)}K\leq x(t)\leq K\quad \text{for } t\geq t_0\right\}. \tag{3.29}$$

It is easy to verify that Ω is a bounded, convex, and closed subset of $BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$.

Now we define two operators S and $T:\Omega\to BC_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ as follows:

$$\begin{aligned}(Sx)(t)&=\frac{x(\tau^{-1}(t^*))}{p(\tau^{-1}(t^*))}+\frac{(n-1)(a_3-1)K}{np(\tau^{-1}(t^*))},\\(Tx)(t)&=\frac{1}{p(\tau^{-1}(t^*))}(-1)^{m-1}\int_{\tau^{-1}(t^*)}^{\infty}g_{m-1}\Big(\sigma(s),\tau^{-1}(t^*)\Big)A(s,x)\Delta s,\end{aligned} \tag{3.30}$$

where $u^*=\max\{u,t_1\}$ for any $u\in[t_0,\infty)_{\mathbb{T}}$ and $A(s,z)=[\int_s^\infty f(\theta,z(\delta(\theta)))\Delta\theta]^{1/\alpha}/a(s)$ for any $z\in\Omega$. Now we show that S and T satisfy the conditions in Lemma 2.8.

We will show that $Sx+Ty\in\Omega$ for any $x,y\in\Omega$. In fact, for any $x,y\in\Omega$ and $t\geq t_0$, $x(t),y(t)\in[(n-2)(a_3-1)K/n(b_3-1),K]$,

$$\begin{aligned}(Sx)(t)+(Ty)(t)&=\frac{x(\tau^{-1}(t^*))}{p(\tau^{-1}(t^*))}+\frac{1}{p(\tau^{-1}(t^*))}\\&\quad\times\left[\frac{(n-1)(a_3-1)}{n}K+(-1)^{m-1}\int_{\tau^{-1}(t^*)}^{\infty}g_{m-1}\Big(\sigma(s),\tau^{-1}(t^*)\Big)A(s,x)\Delta s\right]\\&\leq\frac{1}{a_3}\left[\frac{(n-1)(a_3-1)}{n}K+K+\frac{a_3-1}{n}K\right]\\&=K,\\(Sx)(t)+(Ty)(t)&=\frac{x(\tau^{-1}(t^*))}{p(\tau^{-1}(t^*))}+\frac{1}{p(\tau^{-1}(t^*))}\\&\quad\times\left[\frac{(n-1)(a_3-1)}{n}K+(-1)^{m-1}\int_{\tau^{-1}(t^*)}^{\infty}g_{m-1}\Big(\sigma(s),\tau^{-1}(t^*)\Big)A(s,x)\Delta s\right]\\&\geq\frac{1}{b_3}\left[\frac{(n-1)(a_3-1)}{n}K+\frac{(n-2)(a_3-1)K}{n(b_3-1)}-\frac{a_3-1}{n}K\right]\\&=\frac{(n-2)(a_3-1)K}{n(b_3-1)}\end{aligned} \tag{3.31}$$

and $|(Tx)(t)|\leq(a_3-1)K/n$, which implies that $Sx+Ty\in\Omega$ for any $x,y\in\Omega$ and $T\Omega$ is uniformly bounded. The rest of the proof is similar to that of Theorem 3.2. The proof is completed.

According to the proofs of Theorem 3.1, Theorem 3.2, and Theorem 3.3 in [23], we have □

Lemma 3.4. *Assume that* (H_1) *holds. Suppose that one of the following holds:*

(i) $p(t)$ *satisfies condition* (H_2),

(ii) $p(t)$ *satisfies condition* (H_3) *and* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$,

(iii) $p(t)$ *satisfies condition* (H_4) *and* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$.

If $x(t)$ *is a bounded nonoscillatory solution of* (1.7), *then* $\liminf_{t\to\infty}|x(t)| = 0$ *implies that* $\lim_{t\to\infty} x(t) = 0$.

By Lemma 3.4, Theorem 3.1,Theorem 3.2 and Theorem 3.3, we have.

Corollary 3.5. *Assume that* (H_1) *holds. Suppose that one of the following holds:*

(i) $p(t)$ *satisfies condition* (H_2),

(ii) $p(t)$ *satisfies condition* (H_3) *and* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$,

(iii) $p(t)$ *satisfies condition* (H_4) *and* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$.

Then every bounded solution of (1.7) *is oscillation or converges to zero at infinity if and only if there exists some constant* $K \neq 0$ *such that*

$$\int_{t_0}^{\infty} g_{m-1}(\sigma(s), t_0) \left[\frac{1}{a(s)} \int_{s}^{\infty} f(\theta, |K|)\Delta\theta\right]^{1/\alpha} \Delta s = \infty. \tag{3.32}$$

Example 3.6. Let $\mathbb{T} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ with $q > 1$. Consider the following higher order dynamic equation:

$$\begin{aligned} &\left\{t^{\alpha}\left[(x(t)-p_k(t)x(qt))^{\Delta^m}\right]^{\alpha}\right\}^{\Delta} \\ &\quad+\frac{qg_{m-1}^{\alpha}(q^2t,0)-g_{m-1}^{\alpha}(qt,0)}{(q-1)qt^2g_{m-1}^{\alpha}(q^2t,0)g_{m-1}^{\alpha}(qt,0)}x^{2r+1}\left(q^{r+3}t\right)=0 \quad \text{for } t\in[1,\infty)_{\mathbb{T}}, \end{aligned} \tag{3.33}$$

where $m, r \in \mathbf{N}$, α is the quotient of odd positive integers, $p_k(t) = -2[(-1)^k k^2 + (-1)^{\log_q t}]/5$,, where $k = 1, 2, 3$, $a(t) = t^{\alpha}$, $\tau(t) = qt$, $\delta(t) = q^{3+r}t$ and $f(t, u) = (qg_{m-1}^{\alpha}(q^2t, 0) - g_{m-1}^{\alpha}(qt, 0)/(q-1)qt^2 g_{m-1}^{\alpha}(q^2t, 0)g_{m-1}^{\alpha}(qt, 0))u^{2r+1}$.

It is easy to verify that $p_k(t)$ satisfies condition (H_{k+1}) and $\tau^{-1} \in C(\mathbb{T}, \mathbb{T})$. On the other hand, we have

$$\int_{1}^{\infty} \left[\frac{1}{a(t)}\right]^{1/\alpha} \Delta t = \int_{1}^{\infty} \left[\frac{1}{t^{\alpha}}\right]^{1/\alpha} \Delta t = \infty \tag{3.34}$$

andfor any constant $K > 0$,

$$\begin{aligned}
&\int_1^{\infty} g_{m-1}(\sigma(s),1)\left[\frac{1}{a(s)}\int_s^{\infty} f(\theta,K)\Delta\theta\right]^{1/\alpha}\Delta s \\
&= K^{(2r+1)/\alpha}\int_1^{\infty} g_{m-1}(\sigma(s),1)\left\{\frac{1}{s^{\alpha}}\int_s^{\infty}\left[\frac{-1}{\theta g_{m-1}^{\alpha}(\sigma(\theta),0)}\right]^{\Delta}\right\}^{1/\alpha}\Delta s \\
&= K^{(2r+1)/\alpha}\int_1^{\infty} g_{m-1}(\sigma(s),1)\left[\frac{1}{s^{\alpha}}\frac{1}{s g_{m-1}^{\alpha}(\sigma(s),0)}\right]^{1/\alpha}\Delta s \\
&= K^{(2r+1)/\alpha}\int_1^{\infty}\frac{1}{s^{1+1/\alpha}}\Delta s \\
&= K^{\frac{2r+1}{\alpha}}\frac{q^{1/\alpha}(q-1)}{q^{1/\alpha}-1} < \infty.
\end{aligned} \tag{3.35}$$

That is, conditions (H_1) and (3.1) hold. By Theorem 3.1, Theorem 3.2, and Theorem 3.3, we see that (3.33) has a nonoscillatory bounded solution $x(t)$ with $\liminf_{t\to\infty}|x(t)| > 0$.

Remark 3.7. Results of this paper can be extended to the case with several delays easily.

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