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Research Article

General Linear Boundary Value Problem for the Second-Order Integro-Differential Loaded Equation with Boundary Conditions Containing Both Nonlocal and Global Terms

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The paper is devoted to obtaining the sufficient conditions for Fredholm property for the general boundary value problem of the second-order linear integro-differential equation. Here, the boundary conditions corresponding with the boundary value problem contain both nonlocal and global terms.

1. Introduction

As is known, the boundary value problems are studied in differential equations theory and related areas in mathematical physics with local boundary conditions for linear elliptic partial differential equations [1–4]. Also the number of boundary conditions for linear ordinary differential equation coincide with the order of the [5], and for a partial differential equation (in a bounded domain with smooth boundaries) this number coincides with half of the order of the considered [6]. Then, boundary value problems for linear ordinary differential equations sharply are different from the same problems for linear partial differential equations. Using nonlocal boundary conditions, we remove the misunderstandings given above when passing from boundary value problems for an ordinary differential equation to the problems for partial [7–9].

The investigation method is as follows. Employing fundamental solution of two-dimensional Laplace equation, Green's second formula [10] and analogy of this formula [7–9] are constructed for the current problem. Further, necessary conditions are chosen from these formulas. Singular terms contained in necessary conditions are separated. Taking into account the fact that for the obtained singular integral equations we are on the spectrum, these singularities cannot be regularized by standard methods [11, 12]. These singularities are regularized by peculiar method proceeding from the given boundary conditions [7, 8]. Finally, joining the obtained regular relations with the given boundary conditions, we get sufficient condition on Fredholm property for the stated problems.

2. Problem Statement

Let $D \subset R^2$ be a bounded convex domain with respect to x_2 with boundary Γ supposed to be a Lyapinov line [10]. We assume that the boundary of D is divided into two parts Γ_k, $k = 1, 2$ with the equations $x_2 = \gamma_k(x_1), k = 1, 2,$ and $x_1 \in [a_1, b_1]$. So, consider the following problem:

$$\begin{aligned}
lu \equiv & \Delta u(x) + \sum_{j=1}^{2} a_j(x) \frac{\partial u(x)}{\partial x_j} + a_0(x) u(x) \\
& + \int_{a_1}^{b_1} \sum_{n=1}^{2} \left[\sum_{j=1}^{2} K_{jn}(x, \eta_1) \frac{\partial u(\eta)}{\partial \eta_j} + K_{0n}(x, \eta_1) u(\eta) \right] \Bigg|_{\eta_2 = \gamma_n(\eta_1)} d\eta_1 \\
& + \int_D \left[\sum_{j=1}^{2} K_j(x, \eta) \frac{\partial u(\eta)}{\partial \eta_j} + K_0(x, \eta) u(\eta) \right] d\eta = f(x), \\
x = & (x_1, x_2) \in D \subset R^2,
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
l_m u \equiv & \sum_{n=1}^{2} \left[\sum_{j=1}^{2} \alpha_{mjn}(x_1) \frac{\partial u(x)}{\partial x_j} + \alpha_{m0n}(x_1) u(x) \right] \Bigg|_{x_2 = \gamma_n(x_1)} \\
& + \int_{a_1}^{b_1} \sum_{n=1}^{2} \left[\sum_{j=1}^{2} A_{mjn}(x_1, \eta_1) \frac{\partial u(\eta)}{\partial \eta_j} + A_{m0n}(x_1, \eta_1) u(\eta) \right] \Bigg|_{\eta_2 = \gamma_n(\eta_1)} d\eta_1 \\
& + \int_D \left[\sum_{j=1}^{2} A_{mj}(x_1, \eta) \frac{\partial u(\eta)}{\partial \eta_j} + A_{m0}(x_1, \eta) u(\eta) \right] d\eta = \alpha_m(x_1), \\
& \qquad m = 1, 2, \; x_1 \in [a_1, b_1],
\end{aligned} \tag{2.2}$$

where Δ is two-dimensional Laplace operator and all the data (the coefficients, kernel of integrals, and the right-hand sides) are continuous functions. Boundary conditions (2.2) are assumed to be linearly independent.

3. Fundamental Solution

As is known, while investigating boundary value problems by potential theory, a fundamental solution of adjoint equation is used. If the considered equation is self-adjoint, then fundamental solution of the same equation is chosen. Taking into account that (2.1) is sufficiently general, it is very difficult to construct fundamental solution of the adjoint equation. Therefore, we will choose fundamental solution of the principal part of equation (2.1), that is, for two-dimensional Laplace equation as follows:

$$U(x-\xi) = \frac{1}{2\pi}\ln|x-\xi|. \tag{3.1}$$

4. Basic Relations

At first, we construct Green's second formula connected with (2.1) and fundamental solution (3.1), that is, we multiply (2.1) by fundamental solution (3.1) and integrate it with respect to domain D. Further, applying Ostrogradskii-Gauss formula, we get the following:

$$\begin{aligned}
&\int_{\Gamma}\left[u(x)\frac{dU(x-\xi)}{d\nu_x} - \frac{du(x)}{d\nu}U(x-\xi)\right]dx - \sum_{j=1}^{2}\int_{D} a_j(x)\frac{\partial u(x)}{\partial x_j}U(x-\xi)dx \\
&\quad - \int_{D} a_0(x)u(x)U(x-\xi)dx \\
&\quad - \int_{D}\left\{\int_{a_1}^{b_1}\sum_{n=1}^{2}\left[\sum_{j=1}^{2}K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j} + K_{0n}(x,\eta_1)u(\eta)\right]\Bigg|_{\eta_2=\gamma_n(\eta_1)} d\eta_1\right\} \\
&\quad \times U(x-\xi)dx - \int_{D}\left\{\int_{D}\left[\sum_{j=1}^{2}K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j} + K_0(x,\eta)u(\eta)\right]d\eta\right\} \\
&\quad \times U(x-\xi)dx + \int_{D} f(x)U(x-\xi)dx = \begin{cases} u(\xi), & \xi\in D, \\ \dfrac{1}{2}u(\xi), & \xi\in\Gamma, \end{cases}
\end{aligned} \tag{4.1}$$

where $\nu = \nu_x$ is an external normal to the boundary Γ of domain D at the point $x \in \Gamma$.

Now, proceeding from the relations obtained in [7–9], we derive Green's second formula for (2.1) as follows:

$$\begin{aligned}
&\int_{\Gamma}\frac{\partial u(x)}{\partial x_1}\frac{dU(x-\xi)}{d\nu_x}dx - \int_{\Gamma}\frac{\partial u(x)}{\partial x_2}\frac{dU(x-\xi)}{d\tau_x}dx \\
&\quad + \int_{D}\sum_{j=1}^{2}a_j(x)\,\frac{\partial u(x)}{\partial x_j}\frac{\partial U(x-\xi)}{\partial x_1}dx + \int_{D} a_0(x)u(x)\frac{\partial U(x-\xi)}{\partial x_1}dx \\
&\quad + \int_{D}\left\{\int_{a_1}^{b_1}\sum_{n=1}^{2}\left[\sum_{j=1}^{2}K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j} + K_{0n}(x,\eta_1)u(\eta)\right]\Bigg|_{\eta_2=\gamma_n(\eta_1)} d\eta_1\right. \\
&\quad \left. + \int_{D}\left[\sum_{j=1}^{2}K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j} + K_0(x,\eta)u(\eta)\right]d\eta\right\}\frac{\partial U(x-\xi)}{\partial x_1}dx \\
&\quad - \int_{D} f(x)\frac{\partial U(x-\xi)}{\partial x_1}dx = \begin{cases} \dfrac{\partial u(\xi)}{\partial \xi_1} & \xi\in D, \\ \dfrac{1}{2}\dfrac{\partial u(\xi)}{\partial \xi_1}, & \xi\in\Gamma, \end{cases}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\int_{\Gamma}\frac{\partial u(x)}{\partial x_1}\frac{dU(x-\xi)}{d\tau_x}dx+\int_{\Gamma}\frac{\partial u(x)}{\partial x_2}\frac{dU(x-\xi)}{d\nu_x}dx\\
&\quad+\int_D\sum_{j=1}^{2}a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial U(x-\xi)}{\partial x_2}dx+\int_D a_0(x)u(x)\frac{\partial U(x-\xi)}{\partial x_2}dx\\
&\quad+\int_D\left\{\int_{a_1}^{b_1}\sum_{n=1}^{2}\left[\sum_{j=1}^{2}K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial\eta_j}+K_{0n}(x,\eta_1)u(\eta)\right]\Bigg|_{\eta_2=\gamma_n(\eta_1)}d\eta_1\right.\\
&\quad\left.+\int_D\left[\sum_{j=1}^{2}K_j(x,\eta)\frac{\partial u(\eta)}{\partial\eta_j}+K_0(x,\eta)u(\eta)\right]d\eta\right\}\frac{\partial U(x-\xi)}{\partial x_2}dx\\
&\quad-\int_D f(x)\frac{\partial U(x-\xi)}{\partial x_2}dx=\begin{cases}\dfrac{\partial u(\xi)}{\partial\xi_2}, & \xi\in D,\\[2mm] \dfrac{1}{2}\dfrac{\partial u(\xi)}{\partial\xi_2}, & \xi\in\Gamma,\end{cases}
\end{aligned}\tag{4.3}$$

where τ_x are tangential directions to the boundary Γ of domain D at the point $x\in\Gamma$.

Theorem 4.1. *If the data of* (2.1) *are continuous functions (coefficients, kernels of integrals, and the right-hand side), the domain D is convex in the direction of x_2, and the boundary is Lyapinov's Γ-line, then each solution of* (2.1) *satisfies the basic relations* (4.1)–(4.3).

Remark 4.2. The second relations obtained from (4.1)–(4.3)*, that is, the expressions corresponding for $\xi\in\Gamma$, are necessary conditions. Expressions* (4.1)–(4.3) *themselves are basic relations.*

5. Necessary Conditions

As it was noted above, necessary conditions are obtained from relations (4.1)–(4.3). For reducing these conditions, at first we notice that

$$\frac{dU(x-\xi)}{d\nu_x}dx=\frac{1}{2\pi\left[1+\gamma_1'^2(\sigma_1)\right]}\cdot\frac{\gamma_1'(\sigma_1)-\gamma_1'(x_1)}{x_1-\xi_1}dx_1,\quad x\in\Gamma_1,\ \xi\in\Gamma_1,\tag{5.1}$$

$$\frac{dU(x-\xi)}{d\nu_x}dx=\frac{1}{2\pi\left[1+\gamma_2'^2(\sigma_2)\right]}\cdot\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{x_1-\xi_1}dx_1,\quad x\in\Gamma_2,\ \xi\in\Gamma_2,\tag{5.2}$$

$$\frac{dU(x-\xi)}{d\tau_x}dx=\frac{1}{2\pi(x_1-\xi_1)}\cdot\frac{1+\gamma_1'(\sigma_1)\gamma_1'(x_1)}{1+\gamma_1'^2(\sigma_1)}dx_1,\quad x\in\Gamma_1,\ \xi\in\Gamma_1,\tag{5.3}$$

$$\frac{dU(x-\xi)}{d\tau_x}dx=\frac{1}{2\pi(x_1-\xi_1)}\cdot\frac{1+\gamma_2'(\sigma_2)\gamma_2'(x_1)}{1+\gamma_2'^2(\sigma_2)}dx_1,\quad x\in\Gamma_2,\ \xi\in\Gamma_2.\tag{5.4}$$

For Lyapinov boundary, boundary values of normal derivative of fundamental solution (5.1) and (5.2) may contain only weak singularity (5.3), (5.4) and the same values of tangential derivative contain singularity (5.5), (5.6). If the points x and ξ belong to different parts of the boundary Γ, then there are no singularities in boundary values from derivative of fundamental solution (both in norms and tangential directions).

Thus, we obtain the following necessary conditions:

$$
\begin{aligned}
\frac{1}{2}u(\xi_1,\gamma_1(\xi_1)) = &-\frac{1}{2\pi}\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))\frac{\gamma_1'(\sigma_1)-\gamma_1'(x_1)}{\left[1+\gamma_1'^2(\sigma_1)\right](x_1-\xi_1)}dx_1 \\
&+\frac{1}{2\pi}\int_{a_1}^{b_1} u(x_1,\gamma_2(x_1))\frac{\gamma_2(x_1)-\gamma_1(\xi_1)-(x_1-\xi_1)\gamma_2'(x_1)}{(x_1-\xi_1)^2+\left[\gamma_2(x_1)-\gamma_1(\xi_1)\right]^2}dx_1 \\
&-\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_1(x_1)} U(x_1-\xi_1,\gamma_1(x_1)-\gamma_1(\xi_1))\gamma_1'(x_1)dx_1 \\
&+\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} U(x_1-\xi_1,\gamma_1(x_1)-\gamma_1(\xi_1))dx_1 \\
&+\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_2(x_1)} U(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1))\gamma_2'(x_1)dx_1 \\
&-\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} U(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1))dx_1 \\
&-\sum_{j=1}^{2}\int_D a_j(x)\frac{\partial u(x)}{\partial x_j}U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dx \\
&-\int_D a_0(x)u(x)U(x_1-\xi_1,x_1-\gamma_1(\xi_1))dx \\
&-\int_D\left\{\int_{a_1}^{b_1}\sum_{n=1}^{2}\left.\left[\sum_{j=1}^{2}K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j}+K_{0n}(x,\eta_1)u(\eta)\right]\right|_{\eta_2=\gamma_n(\eta_1)}d\eta_1\right\} \\
&\times U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dx \\
&-\int_D\left\{\int_D\left[\sum_{j=1}^{2}K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j}+K_0(x,\eta)u(\eta)\right]d\eta\right\} \\
&\times U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dx+\int_D f(x)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dx,
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
\frac{1}{2}u(\xi_1,\gamma_2(\xi_1)) = &-\frac{1}{2\pi}\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))\frac{\gamma_1'(x_1)-\gamma_2(\xi_1)-(x_1-\xi_1)\gamma_1'(x_1)}{(x_1-\xi_1)^2+\left[\gamma_1(x_1)-\gamma_2(\xi_1)\right]^2}dx_1 \\
&+\frac{1}{2\pi}\int_{a_1}^{b_1} u(x_1,\gamma_2(x_1))\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{\left[1+\gamma'^2_2(\sigma_1)\right](x_1-\xi_1)}dx_1 \\
&-\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_1(x_1)} U(x_1-\xi_1,\gamma_1(x_1)-\gamma_2(\xi_1))\gamma_1'(x_1)dx_1 \\
&+\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} U(x_1-\xi_1,\gamma_1(x_1)-\gamma_2(\xi_1))dx_1 \\
&+\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} U(x_1-\xi_1,\gamma_2(x_1)-\gamma_2(\xi_1))\gamma_2'(x_1)dx_1 \\
&-\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} U(x_1-\xi_1,\gamma_2(x_1)-\gamma_2(\xi_1))dx_1 \\
&-\sum_{j=1}^{2}\int_D a_j(x)\frac{\partial u(x)}{\partial x_j}U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dx \\
&-\int_D a_0(x)u(x)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dx \\
&-\int_D\left\{\int_{a_1}^{b_1}\sum_{n=1}^{2}\left[\sum_{j=1}^{2}K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j}+K_{0n}(x,\eta_1)u(\eta)\right]\Bigg|_{\eta_2=\gamma_n(\eta_1)}d\eta_1\right\} \\
&\times U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dx \\
&-\int_D\left\{\int_D\left[\sum_{j=1}^{2}K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j}+K_0(x,\eta)u(\eta)\right]d\eta\right\} \\
&\times U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dx+\int_D f(x)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dx.
\end{aligned}
\tag{5.6}
$$

Theorem 5.1. *Under the conditions of theorem 1, the necessary conditions* (5.7) *and* (5.8) *are regular, that is, they do not contain singular terms.*

Now, we give necessary conditions obtained from relations (4.2) and (4.3) as follows

$$\begin{aligned}
&\frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_1}\bigg|_{\xi_2=\gamma_1(\xi_1)} \\
&\quad= -\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_1}\bigg|_{x_2=\gamma_1(x_1)}\frac{1}{1+\gamma_1'^2(\sigma_1)}\frac{\gamma_1'(\sigma_1)-\gamma_1'(x_1)}{x_1-\xi_1}dx_1 \\
&\qquad+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_1}\bigg|_{x_2=\gamma_2(x_1)}\frac{\gamma_2(x_1)-\gamma_1(\xi_1)+(x_1-\xi_1)\gamma_2'(x_1)}{(x_1-\xi_1)^2+[\gamma_2(x_1)-\gamma_1(\xi_1)]^2}dx_1 \\
&\qquad-\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_2}\bigg|_{x_2=\gamma_1(x_1)}\frac{1+\gamma_1'(\sigma_1)\gamma_1'(x_1)}{1+\gamma_1'^2(\sigma_1)}\frac{dx_1}{x_1-\xi_1} \\
&\qquad-\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_2}\bigg|_{x_2=\gamma_2(x_1)}\frac{x_1-\xi_1+[\gamma_2(x_1)-\gamma_1(\xi_1)]\gamma_2'(x)}{(x_1-\xi_1)^2+[\gamma_2(x_1)-\gamma_1(\xi_1)]^2}dx_1 \\
&\qquad+\int_D\sum_{j=1}^{2}a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial U(x-\xi)}{\partial x_1}\bigg|_{\xi_2=\gamma_1(\xi_1)}dx \\
&\qquad+\int_D a_0(x)u(x)\frac{\partial U(x-\xi)}{\partial x_1}\bigg|_{\xi_2=\gamma_1(\xi_1)}dx \\
&\qquad+\int_D\left\{\int_{a_1}^{b_1}\sum_{n=1}^{2}\left[\sum_{j=1}^{2}K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j}+K_{0n}(x,\eta_1)u(\eta)\right]\bigg|_{\eta_2=\gamma_n(\eta_1)}d\eta_1\right. \\
&\qquad+\left.\int_D\left[\sum_{j=1}^{2}K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j}+K_0(x,\eta)u(\eta)\right]d\eta\right\}\frac{\partial U(x-\xi)}{\partial x_1}\bigg|_{\xi_2=\gamma_1(\xi_1)}dx \\
&\qquad-\int_D f(x)\frac{\partial U(x-\xi)}{\partial x_1}\bigg|_{\xi_2=\gamma_1(\xi_1)}dx,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_1}\bigg|_{\xi_2=\gamma_2(\xi_1)} \\
&\quad= -\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_1}\bigg|_{x_2=\gamma_1(x_1)}\frac{\gamma_1(x_1)-\gamma_2(\xi_1)+(x_1-\xi_1)\gamma_1'(x_1)}{(x_1-\xi_1)^2+[\gamma_1(x_1)-\gamma_2(\xi_1)]^2}dx_1 \\
&\qquad+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_1}\bigg|_{x_2=\gamma_2(x_1)}\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{1+\gamma_2'^2(\sigma_2)}\frac{dx_1}{x_1-\xi_1} \\
&\qquad-\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_2}\bigg|_{x_2=\gamma_1(x_1)}\frac{x_1-\xi_1+[\gamma_1(x_1)-\gamma_2(\xi_1)]\gamma_1'(x)}{(x_1-\xi_1)^2+[\gamma_1(x_1)-\gamma_2(\xi_1)]^2}dx_1
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2\pi}\int_{a_1}^{b_1} \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} \frac{1+\gamma_2'(\sigma_2)\ \gamma_2'(x_1)}{1+\gamma_2'^2(\sigma_2)} \frac{dx_1}{x_1-\xi_1} \\
&+ \int_D \sum_{j=1}^{2} a_j(x)\frac{\partial u(x)}{\partial x_j} \left.\frac{\partial U(x-\xi)}{\partial x_1}\right|_{\xi_2=\gamma_2(\xi_1)} dx \\
&+ \int_D a_0(x)u(x) \left.\frac{\partial U(x-\xi)}{\partial x_1}\right|_{\xi_2=\gamma_2(\xi_1)} dx \\
&+ \int_D \left\{ \int_{a_1}^{b_1} \sum_{n=1}^{2} \left.\left[\sum_{j=1}^{2} K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j} + K_{0n}(x,\eta_1)u(\eta)\right]\right|_{\eta_2=\gamma_n(\eta_1)} d\eta_1 \right. \\
&\left. + \int_D \left[\sum_{j=1}^{2} K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j} + K_0(x,\eta)u(\eta)\right] d\eta \right\} \left.\frac{\partial U(x-\xi)}{\partial x_1}\right|_{\xi_2=\gamma_2(\xi_1)} dx \\
&- \int_D f(x) \left.\frac{\partial U(x-\xi)}{\partial x_1}\right|_{\xi_2=\gamma_2(\xi_1)} dx,
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
&\frac{1}{2}\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_1(\xi_1)} \\
&\quad = -\frac{1}{2\pi}\int_{a_1}^{b_1} \left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_1(x_1)} \frac{1+\gamma_1'(\sigma_1)\gamma_1'(x_1)}{1+\gamma_1'^2(\sigma_1)} \frac{dx_1}{x_1-\xi_1} \\
&\quad + \frac{1}{2\pi}\int_{a_1}^{b_1} \left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_2(x_1)} \frac{x_1-\xi_1+\left[\gamma_2(x_1)-\gamma_1(\xi_1)\right]\gamma_2'(x_1)}{(x_1-\xi_1)^2+\left[\gamma_2(x_1)-\gamma_1(\xi_1)\right]^2} dx_1 \\
&\quad - \frac{1}{2\pi}\int_{a_1}^{b_1} \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} \frac{\gamma_1'(\sigma_1)-\gamma_1'(x_1)}{x_1-\xi_1} \frac{dx_1}{1+\gamma_1'^2(\sigma_1)} \\
&\quad + \frac{1}{2\pi}\int_{a_1}^{b_1} \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} \frac{\gamma_2(x_1)-\gamma_1(\xi_1)+(x_1-\xi_1)\gamma_2'(x_1)}{(x_1-\xi_1)^2+\left[\gamma_2(x_1)-\gamma_1(\xi_1)\right]^2} dx_1 \\
&\quad + \int_D \sum_{j=1}^{2} a_j(x)\frac{\partial u(x)}{\partial x_j} \left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\xi_2=\gamma_1(\xi_1)} dx \\
&\quad + \int_D a_0(x)u(x) \left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\xi_2=\gamma_1(\xi_1)} dx \\
&\quad + \int_D \left\{ \int_{a_1}^{b_1} \sum_{n=1}^{2} \left.\left[\sum_{j=1}^{2} K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j} + K_{0n}(x,\eta_1)u(\eta)\right]\right|_{\eta_2=\gamma_n(\eta_1)} d\eta_1 \right.
\end{aligned}
$$

$$+\int_D \left[\sum_{j=1}^{2} K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j} + K_0(x,\eta)u(\eta)\right]d\eta\Bigg\}\frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\xi_2=\gamma_1(\xi_1)} dx$$

$$-\int_D f(x)\frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\xi_2=\gamma_1(\xi_1)} dx. \tag{5.9}$$

$$\frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_2}\Bigg|_{\xi_2=\gamma_2(\xi_1)}$$

$$= \frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_1}\Bigg|_{x_2=\gamma_1(x_1)}\frac{x_1-\xi_1+[\gamma_1(x_1)-\gamma_2(\xi_1)]\gamma_1'(x_1)}{(x_1-\xi_1)^2+[\gamma_1(x_1)-\gamma_2(\xi_1)]^2}dx_1$$

$$+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_1}\Bigg|_{x_2=\gamma_2(x_1)}\frac{1+\gamma_2'(\sigma_2)\gamma_2'(x_1)}{1+\gamma_2'^2(\sigma_2)}\frac{dx_1}{x_1-\xi_1}$$

$$-\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_1(x_1)}\frac{\gamma_1(x_1)-\gamma_2(\xi_1)-(x_1-\xi_1)\gamma_1'(x_1)}{(x_1-\xi_1)^2+[\gamma_1(x_1)-\gamma_2(\xi_1)]^2}dx_1$$

$$+\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_2(x_1)}\frac{\gamma_2'(\sigma_2)-\gamma_2'(x_1)}{1+\gamma_2'^2(\sigma_2)}\frac{dx_1}{x_1-\xi_1}$$

$$+\int_D \sum_{j=1}^{2} a_j(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\xi_2=\gamma_2(\xi_1)} dx \tag{5.10}$$

$$+\int_D a_0(x)u(x)\frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\xi_2=\gamma_2(\xi_1)} dx$$

$$+\int_D \Bigg\{\int_{a_1}^{b_1}\sum_{n=1}^{2}\left[\sum_{j=1}^{2}K_{jn}(x,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j}+K_{0n}(x,\eta_1)u(\eta)\right]\Bigg|_{\eta_2=\gamma_n(\eta_1)} d\eta_1$$

$$+\int_D \left[\sum_{j=1}^{2} K_j(x,\eta)\frac{\partial u(\eta)}{\partial \eta_j} + K_0(x,\eta)u(\eta)\right]d\eta\Bigg\}\frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\xi_2=\gamma_2(\xi_1)} dx$$

$$-\int_D f(x)\frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\xi_2=\gamma_2(\xi_1)} dx.$$

Thus, we prove the following statement.

Theorem 5.2. *Under the conditions of Theorem 4.1, each of the necessary conditions* (5.7), (5.8), (5.9), *and* (5.10) *contains one singular term.*

6. Separation of Singularities

As it was noted in Theorem 5.2, the last four of six necessary conditions have singularities. Let us separate these singularities, that is, determine their coefficients. It is easy to check that

$$\frac{1+\gamma_k'(\sigma_k)\gamma_k'(x_1)}{1+\gamma_k'^2(\sigma_k)} = 1+\left[\frac{1+\gamma_k'(\sigma_k)\gamma_k'(x_1)}{1+\gamma_k'^2(\sigma_k)}-1\right] = 1+\frac{\gamma_k'(\sigma_k)\left[\gamma_k'(x_1)-\gamma_k'(\sigma_k)\right]}{1+\gamma_k'^2(\sigma_k)}, \quad k=1,2. \tag{6.1}$$

Therefore, we write the necessary conditions in theorem 3 in the following form, where pure singular terms are noted

$$\begin{aligned}
\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_k(\xi_1)} &= -\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_k(x_1)}\frac{dx_1}{x_1-\xi_1}+\dots, \quad k=1,2,\\
\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_k(\xi_1)} &= \frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_k(x_1)}\frac{dx_1}{x_1-\xi_1}+\dots, \quad k=1,2.
\end{aligned} \tag{6.2}$$

The dots denote the sum of nonsingular terms.

7. Fredholm Property

Considering necessary conditions (6.2), we construct the following linear combination:

$$\begin{aligned}
&\sum_{n=1}^{2}\left[-\alpha_{m2n}(\xi_1)\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_n(\xi_1)}+\alpha_{m1n}(\xi_1)\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_n(\xi_1)}\right]\\
&\quad=\frac{1}{\pi}\int_{a_1}^{b_1}\sum_{n=1}^{2}\sum_{j=1}^{2}\alpha_{mjn}(\xi_1)\left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_n(x_1)}\frac{dx_1}{x_1-\xi_1}+\dots, \quad m=1,2.
\end{aligned} \tag{7.1}$$

To the right-hand side under the sign of integral we substitute their expression from boundary conditions (2.2). Considering the schemes of [8, 9], we obtain the following:

$$\begin{aligned}
&\sum_{n=1}^{2}\left[-\alpha_{m2n}(\xi_1)\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_n(\xi_1)}+\alpha_{m1n}(\xi_1)\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_n(\xi_1)}\right]\\
&\quad=\frac{1}{\pi}\int_{a_1}^{b_1}\Bigg\{-\sum_{n=1}^{2}\alpha_{m0n}(x_1)u(x)\Big|_{x_2=\gamma_n(x_1)}\\
&\qquad-\int_{a_1}^{b_1}\sum_{n=1}^{2}\left[\left[\sum_{j=1}^{2}A_{mjk}(x_1,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j}+A_{m0k}(x_1,\eta_1)u(\eta)\right]\right]\Bigg|_{\eta_2=\gamma_n(\eta_1)}d\eta_1\\
&\qquad-\int_D\left[\sum_{j=1}^{2}A_{mj}(x_1,\eta)\frac{\partial u(\eta)}{\partial \eta_j}+A_{m0}(x_1,\eta)u(\eta)\right]d\eta+\alpha_m(x_1)\Bigg\}\\
&\qquad\times\frac{dx_1}{x_1-\xi_1}+\cdots, \quad m=1,2.
\end{aligned} \tag{7.2}$$

Thus, we proved the following.

Theorem 7.1. *Under the conditions of theorem 1, if boundary conditions* (2.2) *are linear independent, the coefficients* $\alpha_{mjn}(x_1)$, m, j, *and* $n = 1, 2$ *belong to some Holder class, all the kernels of the integrals in boundary conditions* (2.2) *are continuous, and the right-hand side* $\alpha_m(x_1)$ *for* $m = 1, 2$ *are continuously differentiable functions vanishing at the end of the interval* (a_1, b_1)*, then relations* (7.2) *are regular.*

Really, for regularization of the first term in the right-hand side of (7.2), it suffices to consider regular relations (5.5) and (5.6) for $u(x_1, \gamma_1(x_1))$, $u(x_1, \gamma_2(x_1))$, and interchange integrals after substitution.

The last term does not contain unknown functions and is understood in the sense of the principal value and is regularized in (7.2). All intermediate terms are regularized by permutation of integrals contained in these terms.

Finally, combining (2.2) and (7.2), from the obtained four relations we arrive at the following restriction:

$$\begin{vmatrix} \alpha_{111}(\xi_1) & \alpha_{112}(\xi_1) & \alpha_{121}(\xi_1) & \alpha_{122}(\xi_1) \\ \alpha_{211}(\xi_1) & \alpha_{212}(\xi_1) & \alpha_{221}(\xi_1) & \alpha_{222}(\xi_1) \\ -\alpha_{121}(\xi_1) & -\alpha_{122}(\xi_1) & \alpha_{111}(\xi_1) & \alpha_{112}(\xi_1) \\ -\alpha_{221}(\xi_1) & -\alpha_{222}(\xi_1) & \alpha_{211}(\xi_1) & \alpha_{212}(\xi_1) \end{vmatrix} \neq 0, \tag{7.3}$$

that is, a sufficient condition for reduction of (2.2), (7.2) with respect to boundary conditions to $\partial u(x)/\partial x'_1$ and $\partial u(x)/\partial x_2$ the normal form.

Thus, combining (5.5), (5.6), (2.2), and (7.2), we get (for boundary values) for $u(x)$, $\partial u(x)/\partial x'_1$, and $\partial u(x)/\partial x_2$ six second type Fredholm regular integral equations. Finally, considering three integral equations for $u(x)$, $\partial u(x)/\partial x'_1$, and $\partial u(x)/\partial x_2$ for $x \in D$, obtained from the first expressions of principal relations (4.1)–(4.3), we get the following result.

Theorem 7.2. *Under the conditions of Theorem 7.1 and condition* (7.3)*, boundary value problem* (2.1)–(2.2) *is of Fredholm type.*

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