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Research Article

Convergence of a Sequence of Sets in a Hadamard Space and the Shrinking Projection Method for a Real Hilbert Ball

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We propose a new concept of set convergence in a Hadamard space and obtain its equivalent condition by using the notion of metric projections. Applying this result, we also prove a convergence theorem for an iterative scheme by the shrinking projection method in a real Hilbert ball.

1. Introduction

A Hadamard space is defined as a complete geodesic metric space satisfying the CAT(0) inequality for each pair of points in every triangle. Since this concept includes various important spaces, it has been widely studied by a large number of researchers. In 2004, Kirk [1] proved a fixed point theorem for a nonexpansive mapping defined on a subset of a Hadamard space, and, since then, the study of approximation theory for fixed points of nonlinear mappings has been rapidly developed. See [2–4] and references therein. Kirk and Panyanak [5] proposed a concept of convergence called Δ-convergence, which was originally introduced by Lim [6]. This notion corresponds to usual weak convergence in Banach spaces, and they share many useful properties.

On the other hand, the notion of set convergence for a reflexive Banach space has also been investigated by many researchers. In this paper, we will focus on the Mosco convergence. The relationship between convergence of a sequence of closed convex sets and the corresponding sequence of projections plays an important role in this field [7–10]. In recent research, this concept is applied to convergence of an approximating scheme, which is called the shrinking projection method in Hilbert and Banach spaces; see [11, 12].

Motivated by these results, we propose a new concept of set convergence for a sequence of subsets in a Hadamard space, which follows the notion of Mosco convergence in a Banach space. We adopt Δ-convergence for weak convergence in a Hadamard space. In the main result, we obtain an equivalent condition for this convergence by using the notion of metric projections. In the final section, applying our main result, we prove a convergence theorem for an iterative scheme by the shrinking projection method in a real Hilbert ball.

2. Preliminaries

Let X be a metric space with a metric d. For a subset A of X, the closure of A is denoted by $\operatorname{cl} A$. For $x, y \in X$, a mapping $c : [0, l] \to X$, where $l \geq 0$, is called a geodesic with endpoints x, y if $c(0) = x$, $c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for $t, s \in [0, l]$. If, for every $x, y \in X$, a geodesic with endpoints x, y exists, then we call X a geodesic metric space. Furthermore, if a geodesic is unique for each $x, y \in X$, then X is said to be uniquely geodesic. To introduce some notations, we do not need to assume the uniqueness of geodesics. However, since CAT(0) spaces, which we mainly use in this paper, are always uniquely geodesic, we will assume that X is uniquely geodesic in what follows.

Let X be a uniquely geodesic metric space. For $x, y \in X$, the image of a geodesic c with endpoints x, y is called a geodesic segment joining x and y and is denoted by $[x, y]$. A geodesic triangle with vertices $x, y, z \in X$ is a union of geodesic segments $[x, y]$, $[y, z]$, and $[z, x]$, and we denote it by $\Delta(x, y, z)$. A comparison triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ in $\mathbb{E}^2$ for $\Delta(x, y, z)$ is a triangle in the 2-dimensional Euclidean space $\mathbb{E}^2$ with vertices $\overline{x}, \overline{y}, \overline{z} \in \mathbb{E}^2$ such that $d(x, y) = |\overline{x} - \overline{y}|_{\mathbb{E}^2}$, $d(y, z) = |\overline{y} - \overline{z}|_{\mathbb{E}^2}$, and $d(z, x) = |\overline{z} - \overline{x}|_{\mathbb{E}^2}$, where $|\cdot|_{\mathbb{E}^2}$ is the Euclidean norm on $\mathbb{E}^2$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = |\overline{x} - \overline{p}|_{\mathbb{E}^2}$. If, for any $p, q \in \Delta(x, y, z)$ and their comparison points $\overline{p}, \overline{q} \in \overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$, the inequality

$$d(p, q) \leq |\overline{p} - \overline{q}|_{\mathbb{E}^2} \tag{2.1}$$

holds for all triangles in X, then we call X a CAT(0) space. This inequality is called the CAT(0) inequality. Hadamard spaces are defined as complete CAT(0) spaces.

The CAT(0) space has been investigated in various fields in mathematics, and a great deal of results have been obtained. For more details, see [13].

For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$. We denote it by $tx \oplus (1 - t)y$. From the CAT(0) inequality, it is easy to see that

$$d(z, tx \oplus (1 - t)y)^2 \leq td(z, x)^2 + (1 - t)d(z, y)^2 - t(1 - t)d(x, y)^2 \tag{2.2}$$

for every $x, y, z \in X$ and $t \in [0, 1]$.

A subset C of X is said to be convex if, for every $x, y \in C$, a geodesic segment $[x, y]$ is included in C. For a subset A of X, a convex hull of A is defined as an intersection of all convex sets including A, and we denote it by $\operatorname{co} A$.

Let Y be a subset of X. A mapping $S : Y \to X$ is said to be nonexpansive if $d(Sx, Sy) \leq d(x, y)$ holds for every $x, y \in Y$. The set of all fixed points of S is denoted by $F(S)$; that is, $F(S) = \{z \in Y : Sz = z\}$. We know that $F(S)$ is closed and convex if S is nonexpansive. The following fixed point theorem for nonexpansive mappings on Hadamard spaces plays an important role in our results.

Theorem 2.1 (Kirk [1]). *Let U be a bounded open subset of a Hadamard space X and $S : \mathrm{cl}\,U \to X$ a nonexpansive mapping. Suppose that there exists $p \in U$ such that every x in the boundary of U does not belong to $[p, Sx] \setminus \{Sx\}$. Then, S has a fixed point in* $\mathrm{cl}\,U$.

Let C be a nonempty closed convex subset of a Hadamard space X. Then, for each $x \in X$, there exists a unique point $y_x \in C$ such that $d(x, y_x) = \inf_{y \in C} d(x, y)$. The mapping $x \mapsto y_x$ is called a metric projection onto C and is denoted by P_C. We know that P_C is nonexpansive; see [13, pages 176-177].

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, let

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n),$$
$$r(\{x_n\}) = \inf_{x \in X} r(x, \{x_n\}). \tag{2.3}$$

The asymptotic center of $\{x_n\}$ is a set of points $x \in X$ satisfying that $r(x, \{x_n\}) = r(\{x_n\})$. It is known that the asymptotic center of $\{x_n\}$ consists of one point for every bounded sequence $\{x_n\}$ in a Hadamard space; see [3]. The following property of asymptotic centers is important for our results.

Theorem 2.2 (Dhompongsa et al. [3]). *Let C be a closed convex subset of a Hadamard space X and $\{x_n\}$ a bounded sequence in C. Then, the asymptotic center of $\{x_n\}$ is included in C.*

The notion of Δ-convergence was firstly introduced by Lim [6] in a general metric space setting. Following [5], we apply it to Hadamard spaces. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Δ-convergent to $x \in X$ if x is the unique asymptotic center of any subsequence of $\{x_n\}$. We know that every bounded sequence $\{x_n\}$ in a Hadamard space X has a Δ-convergent subsequence; see [5, 14].

3. Convergence of a Sequence of Sets

Let $\{C_n\}$ be a sequence of closed convex subsets of a Hadamard space X. As an analogy of Mosco convergence in Banach spaces [15], we introduce a new concept of set convergence. First let us define subsets $d\text{-}\mathrm{Li}_n C_n$ and $\Delta\text{-}\mathrm{Ls}_n C_n$ of X as follows: $x \in d\text{-}\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset X$ such that $\{d(x_n, x)\}$ converges to 0 and that $x_n \in C_n$ for all $n \in \mathbb{N}$. On the other hand, $y \in \Delta\text{-}\mathrm{Ls}_n C_n$ if and only if there exist a sequence $\{y_i\} \subset X$ and a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $\{y_i\}$ has an asymptotic center $\{y\}$ and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If a subset C_0 of X satisfies that $C_0 = d\text{-}\mathrm{Li}_n C_n = \Delta\text{-}\mathrm{Ls}_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco, and we write $C_0 = \Delta\mathrm{M}\text{-}\lim_{n \to \infty} C_n$. Since the inclusion $d\text{-}\mathrm{Li}_n C_n \subset \Delta\text{-}\mathrm{Ls}_n C_n$ is always true, to obtain C_0 is a limit of $\{C_n\}$ in the sense of Δ-Mosco, it suffices to show that $\Delta\text{-}\mathrm{Ls}_n C_n \subset C_0 \subset d\text{-}\mathrm{Li}_n C_n$.

It is easy to show that, if every C_n is convex, then so is $d\text{-}\mathrm{Li}_n C_n$. Moreover, we know that $d\text{-}\mathrm{Li}_n C_n$ is always closed. Therefore, $\Delta\mathrm{M}\text{-}\lim_{n \to \infty} C_n$ is closed and convex whenever $\{C_n\}$ is a sequence of closed convex subsets of X.

The following lemma is essentially obtained in [5] as the Kadec-Klee property in CAT(0) spaces. We modify it to a suitable form for our purpose. For the sake of completeness, we give the proof.

Lemma 3.1. *Let X be a Hadamard space and $\{x_n\}$ a sequence in X. Suppose that $\{x_n\}$ is Δ-convergent to $x \in X$ and $\{d(x_n, p)\}$ converges to $d(x, p)$ for some $p \in X$. Then, $\{x_n\}$ converges to x.*

Proof. Let $\{\overline{\Delta}(\overline{x}, \overline{p}, \overline{x}_n)\}$ be comparison triangles in $\mathbb{E}^2$ for $n \in \mathbb{N}$ with an identical geodesic segment $[\overline{p}, \overline{x}]$. Then, we have that $|\overline{x} - \overline{p}|_{\mathbb{E}^2} = d(x, p)$, $|\overline{x}_n - \overline{p}|_{\mathbb{E}^2} = d(x_n, p)$, and $|\overline{x}_n - x|_{\mathbb{E}^2} = d(x_n, x)$ for all $n \in \mathbb{N}$. We know that $\{\overline{x}_n\}$ is bounded in $\mathbb{E}^2$. Let $\{\overline{x}_{n_i}\}$ be an arbitrary subsequence of $\{\overline{x}_n\}$ converging to $\overline{y} \in \mathbb{E}^2$. Then, by assumption, we have that

$$\left|\overline{y} - \overline{p}\right|_{\mathbb{E}^2} = \lim_{i \to \infty} \left|\overline{x}_{n_i} - \overline{p}\right|_{\mathbb{E}^2} = \lim_{i \to \infty} d(x_{n_i}, p) = d(x, p) = \left|\overline{x} - \overline{p}\right|_{\mathbb{E}^2}. \tag{3.1}$$

Let $P = P_{[\overline{p}, \overline{x}]}$ be a metric projection of $\mathbb{E}^2$ onto a closed convex set $[\overline{p}, \overline{x}]$. Since P is continuous, we have that $\{P\overline{x}_{n_i}\}$ converges to $P\overline{y} \in \mathbb{E}^2$. Let $z \in [p, x] \subset X$ be a point corresponding to $\overline{z} = P\overline{y} \in [\overline{p}, \overline{x}] \subset \mathbb{E}^2$. Using the CAT(0) inequality, we have that

$$\begin{aligned} r(\{x_{n_i}\}) &= \limsup_{i \to \infty} d(x, x_{n_i}) = \limsup_{i \to \infty} |\overline{x} - \overline{x}_{n_i}|_{\mathbb{E}^2} \\ &\geq \limsup_{i \to \infty} |P\overline{x}_{n_i} - \overline{x}_{n_i}|_{\mathbb{E}^2} = \limsup_{i \to \infty} |\overline{z} - \overline{x}_{n_i}|_{\mathbb{E}^2} \\ &\geq \limsup_{i \to \infty} d(z, x_{n_i}), \end{aligned} \tag{3.2}$$

and hence $r(z, \{x_{n_i}\}) \leq r(\{x_{n_i}\})$. By the uniqueness of the asymptotic center of $\{x_{n_i}\}$, we obtain that $z = x$, and thus $\overline{z} = \overline{x}$. Since

$$\left|\overline{x} - \overline{y}\right|_{\mathbb{E}^2} = \left|\overline{z} - \overline{y}\right|_{\mathbb{E}^2} = \left|P\overline{y} - \overline{y}\right|_{\mathbb{E}^2} \leq \left|(1 - t)\overline{x} + t\overline{p} - \overline{y}\right|_{\mathbb{E}^2} \tag{3.3}$$

for every $t \in]0, 1[\subset \mathbb{R}$, it follows that

$$\begin{aligned} \left|\overline{x} - \overline{y}\right|^2_{\mathbb{E}^2} &\leq \left|(1 - t)\overline{x} + t\overline{p} - \overline{y}\right|^2_{\mathbb{E}^2} \\ &= (1 - t)\left|\overline{x} - \overline{y}\right|^2_{\mathbb{E}^2} + t\left|\overline{p} - \overline{y}\right|^2_{\mathbb{E}^2} - t(1 - t)\left|\overline{x} - \overline{p}\right|^2_{\mathbb{E}^2} \\ &= (1 - t)\left|\overline{x} - \overline{y}\right|^2_{\mathbb{E}^2} + t^2\left|\overline{p} - \overline{x}\right|^2_{\mathbb{E}^2}, \end{aligned} \tag{3.4}$$

and thus $|\overline{x} - \overline{y}|^2_{\mathbb{E}^2} \leq t|\overline{p} - \overline{x}|^2_{\mathbb{E}^2}$. Tending $t \downarrow 0$, we obtain that $\overline{x} = \overline{y}$. Since any convergent subsequence $\{\overline{x}_{n_i}\}$ of a bounded sequence $\{\overline{x}_n\}$ in $\mathbb{E}^2$ has a limit $\overline{x}$, we have that $\{\overline{x}_n\}$ converges to $\overline{x}$. Thus we have that $d(x_n, x) = |\overline{x}_n - \overline{x}|_{\mathbb{E}^2} \to 0$ as $n \to \infty$, and hence $\{x_n\}$ converges to $x \in X$. □

Now we state the main theorem of this section. Using a sequence of metric projections corresponding to a sequence of closed convex subsets, we give a characterization of Δ-Mosco convergence in a Hadamard space.

Theorem 3.2. *Let X be a Hadamard space and C_0 a nonempty closed convex subset of X. Then, for a sequence $\{C_n\}$ of nonempty closed convex subsets in X, the following are equivalent:*

(i) *$\{C_n\}$ converges to C_0 in the sense of Δ-Mosco;*

(ii) *$\{P_{C_n}x\}$ converges to $P_{C_0}x \in X$ for every $x \in X$.*

Proof. We first show that (i) implies (ii). Fix $x \in X$, and let $p_n = P_{C_n}x$ for $n \in \mathbb{N}$. Since $P_{C_0}x \in C_0 = d\text{-Li}_n C_n$, there exists $\{y_n\} \subset X$ such that $y_n \in C_n$ for all $n \in \mathbb{N}$ and that $\{y_n\}$ converges to $P_{C_0}x$. By the definition of metric projection, we have that $d(x, p_n) \leq d(x, y_n)$ for $n \in \mathbb{N}$. Thus, tending $n \to \infty$, we have that

$$\limsup_{n\to\infty} d(x, p_n) \leq \lim_{n\to\infty} d(x, y_n) = d(x, P_{C_0}x). \tag{3.5}$$

It also follows that $\{p_n\}$ is bounded. Let $\{p_{n_i}\}$ be an arbitrary subsequence of $\{p_n\}$ and p_0 an asymptotic center of $\{p_{n_i}\}$. Then, for fixed $\epsilon > 0$, it holds that

$$d(x, p_{n_i}) \leq d(x, P_{C_0}x) + \epsilon \tag{3.6}$$

for sufficiently large $i \in \mathbb{N}$. Since the closed ball with the center x and the radius $d(x, P_{C_0}x) + \epsilon$ is convex, by Theorem 2.2, we have that $d(x, p_0) \leq d(x, P_{C_0}x) + \epsilon$, and hence

$$d(x, p_0) \leq d(x, P_{C_0}x). \tag{3.7}$$

On the other hand, since $p_0 \in \Delta\text{-Ls}_n C_n = C_0$, we have that $d(x, P_{C_0}x) \leq d(x, p_0)$, and therefore we have that $d(x, P_{C_0}x) = d(x, p_0)$, which implies that $p_0 = P_{C_0}x$. Since all subsequences of $\{p_n\}$ have the same asymptotic center $P_{C_0}x$, $\{p_n\}$ is Δ-convergent to $P_{C_0}x$.

Let us show that $\liminf_{n\to\infty} d(x, p_n) \geq d(x, P_{C_0}x)$. If it were not true, then there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ satisfying that $\liminf_{n\to\infty} d(x, p_n) = \lim_{i\to\infty} d(x, p_{n_i}) < d(x, P_{C_0}x)$. Let $p \in X$ be an asymptotic center of $\{p_{n_i}\}$. For $\epsilon > 0$, we have that $d(x, p_{n_i}) \leq \delta + \epsilon$ for sufficiently large $i \in \mathbb{N}$, where $\delta = \lim_{i\to\infty} d(x, p_{n_i})$. Since the closed ball with the center x and the radius $\delta + \epsilon$ is convex, we have $d(x, p) \leq \delta + \epsilon$, and hence $d(x, p) \leq \delta = \lim_{i\to\infty} d(x, p_{n_i})$. Since $p \in \Delta\text{-Ls}_n C_n = C_0$, we get that

$$d(x, P_{C_0}x) > \lim_{i\to\infty} d(x, p_{n_i}) \geq d(x, p) \geq d(x, P_{C_0}x), \tag{3.8}$$

a contradiction. Therefore, we obtain that

$$d(x, P_{C_0}x) \leq \liminf_{n\to\infty} d(x, p_n) \leq \limsup_{n\to\infty} d(x, p_n) \leq d(x, P_{C_0}x), \tag{3.9}$$

and thus $\{d(x, p_n)\}$ converges to $d(x, P_{C_0}x)$. Using Lemma 3.1, we have that $\{p_n\}$ converges to $P_{C_0}x$. Hence (ii) holds.

Next we suppose (ii) and show that (i) holds. By assumption, for $y \in C_0$, a sequence $\{P_{C_n}y\}$ converges to $P_{C_0}y = y$. Since $P_{C_n}y \in C_n$ for all $n \in \mathbb{N}$, we have that $y \in d\text{-Li}_n C_n$, and hence $C_0 \subset d\text{-Li}_n C_n$. Let $z \in \Delta\text{-Ls}_n C_n$. Then, there exist $\{z_i\} \subset X$ and $\{n_i\} \subset \mathbb{N}$ such that

$z_i \in C_{n_i}$ for all $i \in \mathbb{N}$ and z is an asymptotic center of $\{z_i\}$. Since each C_{n_i} is convex, from the definition of metric projection, it follows that

$$d\left(z, P_{C_{n_i}} z\right) \leq d\left(z, (1-t) P_{C_{n_i}} z \oplus t z_i\right) \tag{3.10}$$

for $t \in]0,1[$ and $i \in \mathbb{N}$. Then, we have that

$$\begin{aligned} d\left(z, P_{C_{n_i}} z\right)^2 &\leq d\left(z, (1-t) P_{C_{n_i}} z \oplus t z_i\right)^2 \\ &\leq (1-t) d\left(z, P_{C_{n_i}} z\right)^2 + t d(z, z_i)^2 - t(1-t) d\left(P_{C_{n_i}} z, z_i\right)^2, \end{aligned} \tag{3.11}$$

and thus

$$d\left(z, P_{C_{n_i}} z\right)^2 + (1-t) d\left(P_{C_{n_i}} z, z_i\right)^2 \leq d(z, z_i)^2. \tag{3.12}$$

Tending $t \downarrow 0$, we get that

$$d\left(z, P_{C_{n_i}} z\right)^2 + d\left(P_{C_{n_i}} z, z_i\right)^2 \leq d(z, z_i)^2 \tag{3.13}$$

for every $i \in \mathbb{N}$, and since $\{P_{C_{n_i}} z\}$ converges to $P_{C_0} z$ as $i \to \infty$, we have that

$$d(z, P_{C_0} z)^2 + \limsup_{i \to \infty} d(P_{C_0} z, z_i)^2 \leq \limsup_{i \to \infty} d(z, z_i)^2. \tag{3.14}$$

Since z is an asymptotic center of $\{z_i\}$, we have that

$$\begin{aligned} \limsup_{i \to \infty} d(z, z_i) &= r(z, \{z_i\}) = r(\{z_i\}) \\ &\leq r(P_{C_0} z, \{z_i\}) \\ &= \limsup_{i \to \infty} d(P_{C_0} z, z_i). \end{aligned} \tag{3.15}$$

It follows that

$$d(z, P_{C_0} z)^2 \leq \limsup_{i \to \infty} d(z, z_i)^2 - \limsup_{i \to \infty} d(P_{C_0} z, z_i)^2 \leq 0, \tag{3.16}$$

and therefore $z = P_{C_0} z \in C_0$, which implies that Δ-$\mathrm{Ls}_n C_n \subset C_0$. Consequently we have that $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco, and hence (ii) holds. □

Using the result in [8], we obtain the following characterization in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space and C_0 a nonempty closed convex subset of H. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets in H. Then, $\{C_n\}$ converges to C_0 in the sense of Mosco if and only if $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco.*

Proof. By [8, Theorems 4.1 and 4.2], $\{C_n\}$ converges to C_0 in the sense of Mosco if and only if $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$ for all $x \in H$. Therefore, using Theorem 3.2, we obtain the desired result. □

This result shows that Mosco convergence in Hilbert spaces is an example of Δ-Mosco convergence. Let us see other simple examples.

Example 3.4. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hadamard space X. Then, as a direct consequence of the definition, we obtain that

$$\operatorname{cl}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} C_n \subset d\text{-}\mathrm{Li}_n C_n \subset \Delta\text{-}\mathrm{Ls}_n C_n \subset \bigcap_{m=1}^{\infty}\operatorname{cl}\operatorname{co}\bigcup_{n=m}^{\infty} C_n. \tag{3.17}$$

In particular, if $\{C_n\}$ is a decreasing sequence with respect to inclusion, then $\{C_n\}$ is Δ-Mosco convergent to $\bigcap_{n=1}^{\infty} C_n$. Likewise, if $\{C_n\}$ is increasing, then the limit is $\operatorname{cl}\bigcup_{n=1}^{\infty} C_n$.

Example 3.5. Let $\{C_n\}$ be a sequence of nonempty bounded closed convex subsets of a Hadamard space X. If $\{C_n\}$ converges to a bounded closed convex subset $C_0 \subset X$ with respect to the Hausdorff metric, then $\{C_n\}$ also converges to C_0 in the sense of Δ-Mosco. The Hausdorff metric h between nonempty bounded closed subsets A, B of X is defined by

$$h(A,B) = \max\{e(A,B), e(B,A)\}, \tag{3.18}$$

where $e(A,B) = \sup_{x\in A} d(x,B)$ and $d(x,B) = \inf_{y\in B} d(x,y)$ for $x \in X$.

Let us prove this fact. For $x \in C_0$, we have that $d(x,C_n) \le e(C_0,C_n) \le h(C_0,C_n)$ and since $h(C_n,C_0) \to 0$ as $n \to \infty$, there exists a sequence $\{x_n\} \subset X$ converging to x such that $x_n \in C_n$ for all $n \in \mathbb{N}$. It follows that $x \in d\text{-}\mathrm{Li}_n C_n$, and hence $C_0 \subset d\text{-}\mathrm{Li}_n C_n$.

To show $\Delta\text{-}\mathrm{Ls}_n C_n \subset C_0$, let $x \in \Delta\text{-}\mathrm{Ls}_n C_n$. Then, there exists a subsequence $\{n_i\}$ of $\mathbb{N}$ and a sequence $\{x_i\} \subset X$ whose asymptotic center is x and $x_i \in C_{n_i}$ for all $i \in \mathbb{N}$. Let $\epsilon > 0$ be arbitrary. Then, since $d(x_i,C_0) \le e(C_{n_i},C_0) \le h(C_{n_i},C_0) \to 0$ as $i \to \infty$, there exists $i_0 \in \mathbb{N}$ such that $d(x_i,C_0) < \epsilon$ for every $i \ge i_0$.

Let $D_\epsilon = \{y \in X : d(y,C_0) \le \epsilon\}$. Then, D_ϵ is closed and convex in X. Indeed, for $y_1, y_2 \in D_\epsilon$ and $t \in]0,1[$, there exist $z_1, z_2 \in C_0$ such that $d(y_1,z_1) < \epsilon$ and $d(y_2,z_2) < \epsilon$. Considering the comparison triangle of (y_1,z_1,z_2) and using the CAT(0) inequality, we have that

$$d\big(ty_1 \oplus (1-t)z_2, tz_1 \oplus (1-t)z_2\big) \le td\big(y_1,z_1\big). \tag{3.19}$$

In the same way, considering the comparison triangle of (y_1,y_2,z_2), we have that

$$d\big(ty_1 \oplus (1-t)y_2, ty_1 \oplus (1-t)z_2\big) \le (1-t)d\big(y_2,z_2\big). \tag{3.20}$$

Thus, we have that $d(ty_1 \oplus (1-t)y_2, tz_1 \oplus (1-t)z_2) \leq (1-t)d(y_2, z_2) + td(y_1, z_1) \leq \epsilon$. Since C_0 is convex, we have that $tz_1 \oplus (1-t)z_2 \in C_0$, and hence $ty_1 \oplus (1-t)y_2 \in D_\epsilon$. This shows that D_ϵ is convex. It is obvious that D_ϵ is closed since the function $d(\cdot, C_0)$ is continuous.

Since $x_i \in D_\epsilon$ for $i \geq i_0$, using Theorem 2.2, we have that $x \in D_\epsilon$; that is, $d(x, C_0) \leq \epsilon$. Since ϵ is arbitrary and C_0 is closed, we obtain that $x \in C_0$, and hence $\Delta\text{-}\mathrm{Ls}_n C_n \subset C_0$. Consequently we have that $\{C_n\}$ converges to C_0 in the sense of Δ-Mosco.

4. Shrinking Projection Method in a Real Hilbert Ball

As an example of Hadamard spaces, let us deal with a real Hilbert ball in this section. Let B_H be the open unit ball of a complex Hilbert space H with an inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. For an orthonormal basis $\{e_i : i \in I\}$ of H, let $H_R = \{z \in H : \mathrm{Im}\langle z, e_i \rangle = 0 \, \forall i \in I\}$. Then, a real Hilbert ball (B, ρ) is a metric space defined by $B = B_H \cap H_R$ and $\rho : B \times B \to \mathbb{R}$ by

$$\rho(x, y) = \operatorname{arctanh} \sqrt{1 - \frac{\left(1 - \|x\|^2\right)\left(1 - \|y\|^2\right)}{1 - |\langle x, y \rangle|^2}} \tag{4.1}$$

for $x, y \in B$. It is known that a real Hilbert ball is an example of Hadamard spaces. One of the most important properties for our results in this section is that a half space $C = \{z \in B : \rho(z, y) \leq \rho(z, x)\}$ is convex for any $x, y \in B$; see [16, 17].

Theorem 4.1. *Let B be a real Hilbert ball with the metric ρ. Let $\{T_i : i \in I\}$ be a family of nonexpansive mappings of B into itself with a nonempty set F of their common fixed points. Let $\{\alpha_n(i) : i \in I, \quad n \in \mathbb{N}\}$ be nonnegative real numbers in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n(i) < 1$ for each $i \in I$. For $x \in B$, generate an iterative sequence $\{x_n\}$ by $x_1 = x$, $C_0 = B$, and*

$$\begin{gathered} y_n(i) = \alpha_n(i)x_n \oplus (1 - \alpha_n(i))T_i x_n, \quad \text{for each } i \in I, \\ C_n = \left\{ z \in B : \sup_{i \in I} \rho(z, y_n(i)) \leq \rho(z, x_n) \right\} \cap C_{n-1}, \\ x_{n+1} = P_{C_n} x \end{gathered} \tag{4.2}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is well defined and converges to $P_F x \in B$.

Proof. Since each $F(T_i)$ is closed and convex, so is $F = \bigcap_{i \in I} F(T_i)$. For $z \in F$, it follows that

$$\begin{aligned} &\rho(z, y_n(i))^2 \\ &\quad \leq \alpha_n(i)\rho(z, x_n)^2 + (1 - \alpha_n(i))\rho(z, T_i x_n)^2 - \alpha_n(i)(1 - \alpha_n(i))\rho(x_n, T_i x_n)^2 \\ &\quad \leq \alpha_n(i)\rho(z, x_n)^2 + (1 - \alpha_n(i))\rho(T_i z, T_i x_n)^2 \\ &\quad \leq \rho(z, x_n)^2 \end{aligned} \tag{4.3}$$

for all $i \in I$, and thus $\sup_{i \in I} \rho(z, y_n(i)) \leq \rho(z, x_n)$ for every $n \in \mathbb{N}$. Therefore, we have $F \subset C_n$ and C_n is nonempty for $n \in \mathbb{N}$. Further, C_n is closed and convex by the property of a real Hilbert ball B. Hence, the metric projection P_{C_n} exists, and x_n is well defined for all $n \in \mathbb{N}$. Since $\{C_n\}$ is decreasing with respect to inclusion, as in Example 3.4, we have that $\{C_n\}$ converges to $C = \bigcap_{n=1}^{\infty} C_n$ in the sense of Δ-Mosco. By Theorem 3.2, we have that $\{x_n\}$ converges to $x_0 = P_C x$. Since $x_0 \in C_n$ for all $n \in \mathbb{N}$, we have that $\rho(x_0, y_n(i)) \leq \rho(x_0, x_n)$ for all $n \in \mathbb{N}$ and $i \in I$. Fix $i \in I$ arbitrarily, and let $\{\alpha_{n_k}(i)\}$ be a subsequence of $\{\alpha_n(i)\}$ converging to $\alpha_0(i) \in [0, 1[$. Then, since $\rho(x_n, y_n(i)) = (1 - \alpha_n(i))\rho(x_n, T_i x_n)$, we have that

$$
\begin{aligned}
\rho(x_0, T_i x_0) &\leq \rho(x_0, x_{n_k}) + \rho(x_{n_k}, T_i x_{n_k}) + \rho(T_i x_{n_k}, T_i x_0) \\
&\leq 2\rho(x_0, x_{n_k}) + \frac{1}{1 - \alpha_{n_k}(i)} \rho\big(x_{n_k}, y_{n_k}(i)\big) \\
&\leq 2\rho(x_0, x_{n_k}) + \frac{1}{1 - \alpha_{n_k}(i)} \big(\rho(x_{n_k}, x_0) + \rho\big(x_0, y_{n_k}(i)\big)\big) \\
&\leq 2\left(1 + \frac{1}{1 - \alpha_{n_k}(i)}\right) \rho(x_0, x_{n_k})
\end{aligned}
\tag{4.4}
$$

for $k \in \mathbb{N}$, and, as $k \to \infty$, we obtain that $x_0 = T_i x_0$; that is, $x_0 \in F(T_i)$. Since $i \in I$ is arbitrary, we have that $P_C x = x_0 \in F \subset C$, and therefore $x_0 = P_F x$, which is the desired result. □

Next, we consider the case of a single mapping. Motivated by [18], we obtain the following theorem. It shows that, without assuming the existence of fixed points, we may prove that the iterative sequence is well defined. Moreover, the boundedness of the sequence guarantees that the set of fixed points is nonempty.

Theorem 4.2. *Let B be a real Hilbert ball and $T : B \to B$ a nonexpansive mapping. Let $\{\alpha_n\}$ be a nonnegative real sequence in $[0, 1]$ such that $\liminf_{n \to \infty} \alpha_n < 1$. For $x \in B$, generate an iterative sequence $\{x_n\}$ by $x_1 = x$, $C_0 = B$, and*

$$
\begin{aligned}
y_n &= \alpha_n x_n \oplus (1 - \alpha_n) T x_n, \quad \text{for each } i \in I, \\
C_n &= \big\{z \in B : \rho(z, y_n) \leq \rho(z, x_n)\big\} \cap C_{n-1}, \\
x_{n+1} &= P_{C_n} x
\end{aligned}
\tag{4.5}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is well defined and the following are equivalent:

(i) *$F(T)$ is nonempty;*

(ii) *$\{x_n\}$ is convergent;*

(iii) *$\{x_n\}$ is bounded;*

(iv) *$\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

Moreover, in this case the limit of $\{x_n\}$ is $P_{F(T)} x = P_{\bigcap_{n=1}^{\infty} C_n} x$.

Proof. First we show that $\{x_n\}$ is well defined. Since x_1 is given and $y_1 \in C_1$, C_1 is nonempty. Suppose that $C_1, C_2, \ldots, C_{n-1}$ are nonempty. Then, $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are defined.

Let $r = \max_{1\le k\le n}\rho(x, Tx_k)$ and $D = \{w \in B : \rho(x, w) \le r\}$. Then, since D is nonempty, bounded, closed, and convex, there exists a metric projection $P_D : B \to D$. Since P_D is nonexpansive, it follows that $P_D T|_D$ is also a nonexpansive mapping of D into itself. Moreover, D has a nonempty interior, and $[x, v] \setminus \{v\}$ does not intersect the boundary of D for every $v \in D$. Thus, by Theorem 2.1, there exists $u \in D$ such that $u = P_D Tu$. Since $Tx_k \in D$ for $k = 1, 2, \ldots, n$ and D is convex, it follows from the definition of the metric projection that

$$\rho(Tu, u) = \rho(Tu, P_D Tu) \le \rho(Tu, (1-t)u \oplus tTx_k) \tag{4.6}$$

for $t \in]0, 1[$. Thus, we have that

$$\begin{aligned}\rho(Tu, u)^2 &\le \rho(Tu, (1-t)u \oplus tTx_k)^2 \\ &\le (1-t)\rho(Tu, u)^2 + t\rho(Tu, Tx_k)^2 - t(1-t)\rho(u, Tx_k)^2,\end{aligned} \tag{4.7}$$

and thus

$$\begin{aligned}\rho(Tu, u)^2 &\le \rho(Tu, Tx_k)^2 - (1-t)\rho(u, Tx_k)^2 \\ &\le \rho(u, x_k)^2 - (1-t)\rho(u, Tx_k)^2.\end{aligned} \tag{4.8}$$

Tending $t \downarrow 0$, we have that $0 \le \rho(Tu, u)^2 \le \rho(u, x_k)^2 - \rho(u, Tx_k)^2$, and hence $\rho(u, Tx_k) \le \rho(u, x_k)$ for $k = 1, 2, \ldots, n$. It gives us that

$$\begin{aligned}\rho(u, y_k)^2 &= \rho(u, \alpha_k x_k \oplus (1-\alpha_k)Tx_k)^2 \\ &\le \alpha_k\rho(u, x_k)^2 + (1-\alpha_k)\rho(u, Tx_k)^2 \\ &\le \rho(u, x_k)^2\end{aligned} \tag{4.9}$$

for all $k = 1, 2, \ldots, n$, and hence $u \in C_n$. This shows that C_n is nonempty and obviously it is closed and convex. Therefore, $x_{n+1} = P_{C_n}x$ is defined. By induction, we obtain that $\{x_n\}$ is well defined.

Next, we show that (i)–(iv) are equivalent. We know from Theorem 4.1 for a single mapping that (i) implies (ii). We also have that $\{x_n\}$ converges to $P_{F(T)}x = P_{\bigcap_{n=1}^{\infty} C_n}x$. It is trivial that (ii) implies (iii). Let us suppose that (iii) holds and show (iv). Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ which is Δ-convergent to some $x_0 \in B$. From the definition of subsequence, for any $n \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $n_k > n$ for all $k \ge k_0$. Since $\{C_n\}$ is decreasing with respect to inclusion, we have $x_{n_k} \subset C_{n_k-1} \subset C_n$ for all $k \ge k_0$. By Theorem 2.2, we have that $x_0 \in C_n$ for every $n \in \mathbb{N}$, and hence (iv) holds. Lastly, we show that (iv) implies (i). Assume that $C = \bigcap_{n=1}^{\infty} C_n$ is nonempty. By Theorem 3.2, $\{x_n\}$ converges to $P_C x$. Then, as in the proof of Theorem 4.1, we have that $P_C x \in F(T)$, and thus (i) holds. Consequently, these four conditions are all equivalent. □

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