Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 603968, 7 pages
doi:10.1155/2010/603968

Research Article

Carleson Measure in Bergman-Orlicz Space of Polydisc

An-Jian Xu[1, 2] and Zou Yang[3]

[1] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*
[2] *Institute of Applied Mathematics, Chongqing University of Post and Telecommunication, Chongqing 400065, China*
[3] *Department of Mathematics, Chongqing Education College, Chongqing 400067, China*

Correspondence should be addressed to An-Jian Xu, anjian.xu@gmail.com

Received 30 June 2010; Revised 26 August 2010; Accepted 2 September 2010

Academic Editor: Dumitru Baleanu

Let μ be a finite, positive measure on $\mathbb{D}^n$, the polydisc in $\mathbb{C}^n$, and let σ_n be 2n-dimensional Lebesgue volume measure on $\mathbb{D}^n$. For an Orlicz function φ, a necessary and sufficient condition on μ is given in order that the identity map $J : L_a^{\varphi}(\mathbb{D}^n, \sigma_n) \to L^{\varphi}(\mathbb{D}^n, \mu)$ is bounded.

1. Introduction

We denote by $\mathbb{D}^n$ the unit polydisc in $\mathbb{C}^n$ and by $\mathbb{T}^n$ the distinguished boundary of $\mathbb{D}^n$. We will use σ_n to denote the $2n$-dimensional Lebesgue volume measure on $\mathbb{D}^n$, normalized so that $\sigma_n(\mathbb{D}^n) = 1$. We use R to describe rectangles on $\mathbb{T}^n$, and we use $S(R)$ to denote the corona associated to these sets. In particular, if I is an interval on $\mathbb{T}$ of length $\delta \in (0,1)$ centered at $e^{i(\theta_0+\delta/2)}$,

$$S(I) = \{z \in \mathbb{D} \mid 1 - \delta < r < 1, \theta_0 < \theta < \theta_0 + \delta\}. \tag{1.1}$$

Then, if $R = I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{T}^n$, with I_j intervals having length δ_j and having centers $e^{i(\theta_j^0+\delta_j/2)}$, $S(R)$ is given by $S(R) = S(I_1) \times S(I_2) \times \cdots \times S(I_n)$, and let

$$\alpha_j = (1 - \delta_j) e^{i(\theta_j^0+\delta_j/2)}, \quad 1 \le j \le n. \tag{1.2}$$

If V is any open set in $\mathbb{T}^n$, we define $S(V) = \cup_{\gamma} S(R_{\gamma})$ where $\{R_{\gamma}\}$ runs through all rectangles in V.

An Orlicz function is a real-valued, nondecreasing, convex function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. To avoid pathologies, we will assume that we work with an Orlicz function φ having the following additional properties: φ is continuous and strictly convex (hence increasing), such that

$$\lim_{x \to \infty} \frac{\varphi(x)}{x} = \infty. \tag{1.3}$$

The Orlicz space $L^{\varphi}(\mu)$ is the space of all (equivalence classes of) measurable functions $f : \Omega \to \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_{\Omega} \varphi\left(\frac{|f(w)|}{C}\right) d\mu(w) < +\infty, \tag{1.4}$$

and then $\|f\|_{\mu}$ (the Luxemburg norm) is the infimum of all possible constant C such that this integral is ≤ 1. It is well known that $L^{\varphi}(\mu)$ is a Banach space under the Luxemburg norm $\|\cdot\|_{\mu}$. For $f \in L^{\varphi}$, let

$$M_{\mu}(f) := \int_{\Omega} \varphi(|f|) d\mu < +\infty. \tag{1.5}$$

The Bergman-Orlicz space $L_a^{\varphi}(\mathbb{D}^n, \sigma_n)$ consists of all analytic functions in $L^{\varphi}(\mathbb{D}^n, \sigma_n)$, which is a closed subspace of $L^{\varphi}(\mathbb{D}^n, \sigma_n)$, so it is an analytic Banach space also.

A theorem of Carleson [1, 2] characterizes those positive measure μ on $\mathbb{D}$ for which the Hardy space H^p norm dominates the $L^p(\mu)$ norm of elements of H^p. Since then, there is a long history of the development and application of Carleson measures, see [3]. This rich area of research contains a large body of literature on characterizations of different classes of operators in different spaces and their applications. Chang [4] has characterized the bounded measures on $L^p(\mathbb{T}^2)$ using a two-line proof referring to a result of Stein. Characterization of the bounded identity operators on Hardy spaces is an immediate consequence of Chang's proof using standard arguments. Hastings [5] has given a similar result for unweighted Bergman spaces. MacCluer [6] has obtained a Carleson measure characterization of the identity operators on Hardy spaces of the unit ball in $\mathbb{C}^n$ using the well-known results of Hormander. Lefèvre et al. [7] have introduced an adapted version of Carleson measure in Hardy-Orlicz spaces. Xiao [8], Ortiz, and Fernandez [9] have got a characterization of the Carleson measure in Bergman-Orlicz spaces of the unit disc.

A finite, positive measure μ on $\mathbb{D}^n$ is called a φ Carleson measure if there is a constant C' such that

$$\mu(S(I)) \leq \frac{1}{\varphi\left(C' \varphi^{-1}\left(\prod_{j=1}^{n}\left(1/\delta_j^2\right)\right)\right)}', \tag{1.6}$$

for every rectangle $I \subset \mathbb{T}^n$.

In this paper, we proveTheorem 2.4.

2. Main Results and Proofs

Lemma 2.1. *For* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{D}^n$, *let* $u_\alpha(z_1, \ldots, z_n) = \prod_{j=1}^n (1-|\alpha_j|^2)^2/(1-\overline{\alpha}_j z_j)^4$. *Then* $u_\alpha(z_1, \ldots, z_n) \in L_a^\varphi(\mathbb{D}^n)$, *and*

$$\|u_\alpha(z)\|_{\sigma_n} \leq \frac{1}{\varphi^{-1}\left(\prod_{j=1}^n \left(1/\delta_j^2\right)\right)}. \tag{2.1}$$

Proof. It is easy to see that $\|u_\alpha(z)\|_\infty = \prod_{j=1}^n ((1+|\alpha_j|)/(1-|\alpha_j|))^2 = \prod_{j=1}^n ((2-\delta_j)/\delta_j)^2$. Since $\varphi(0) = 0$, the convexity of φ implies $\varphi(ax) \leq a\varphi(x)$ for $0 \leq a \leq 1$. Hence, for every $C > 0$, we have

$$\begin{aligned}\int_{\mathbb{D}^n} \varphi\left(\frac{\prod_{j=1}^n (\delta_j/(2-\delta_j))^2 |u_\alpha(z)|}{C}\right) d\sigma_n &\leq \prod_{j=1}^n \left(\frac{\delta_j}{2-\delta_j}\right)^2 \int_{\mathbb{D}^n} |u_\alpha(z)| \varphi\left(\frac{1}{C}\right) d\sigma_n \\ &= \prod_{j=1}^n \left(\frac{\delta_j}{2-\delta_j}\right)^2 \|u_{\alpha_1}(z)\|_1 \varphi\left(\frac{1}{C}\right),\end{aligned} \tag{2.2}$$

but $\prod_{j=1}^n (\delta_j/(2-\delta_j))^2 \|u_{\alpha_1}(z)\|_1 \varphi(1/C) \leq 1$ if and only if $C \geq 1/\varphi^{-1}(\prod_{j=1}^n (2-(\delta_j/\delta_j))^2 (1/\|u_\alpha(z)\|_1))$, that is,

$$\|u_\alpha(\mathrm{z})\|_{\sigma_n} \leq \frac{1}{\varphi^{-1}\left(\prod_{j=1}^n (2-\delta_j/\delta_j)^2 (1/\|u_\alpha(z)\|_1)\right)}. \tag{2.3}$$

Moreover,

$$\begin{aligned}\|u_\alpha(z)\|_1 &= \int_{\mathbb{D}} \frac{\left(1-|\alpha_1|^2\right)^2}{|1-\overline{\alpha}_1 z_1|^4} d\sigma_1(z_1) \cdots \int_{\mathbb{D}} \frac{\left(1-|\alpha_n|^2\right)^2}{|1-\overline{\alpha}_n z_n|^4} d\sigma_1(z_n) \\ &= \prod_{j=1}^n \left(1-|\alpha_j|^2\right)^2 \int_{\mathbb{D}} \frac{1}{|1-\overline{\alpha}_j z_j|^4} d\sigma_1(z_j) \\ &= \prod_{j=1}^n \left(1-|\alpha_n|^2\right)^2 \int_{\mathbb{D}} \frac{1}{(1-\overline{\alpha}_j z_j)^2} \frac{1}{(1-\alpha_j \overline{z_j})^2} d\sigma_1(z_j) \\ &= \prod_{j=1}^n \frac{\left(1-|\alpha_n|^2\right)^2}{\left(1-|\alpha_j|^2\right)^2} = 1.\end{aligned} \tag{2.4}$$

So, we have

$$\|u_\alpha(z)\|_{\sigma_n} \leq \frac{1}{\varphi^{-1}\left(\prod_{j=1}^n ((2-\delta_j)/\delta_j)^2\right)} \leq \frac{1}{\varphi^{-1}\left(\prod_{j=1}^n \left(1/\delta_j^2\right)\right)}. \tag{2.5}$$

For $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n, k = (k_1, \dots, k_n)$ with $1 \le k_j \le 2^{m_j+4}$, $(1 \le j \le n)$, let

$$T_{mk} = \left\{ \left(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n} \right) \mid 1 - 2^{-m_j} \le r_j < 1 - 2^{-m_j-1}, \right.$$
$$\left. \frac{2k_j\pi}{2^{m_j+4}} \le \theta_j < \frac{(2k_j+1)\pi}{2^{m_j+4}}, 1 \le j \le n \right\}, \tag{2.6}$$

let $z^{mk} = (z_1^{mk}, \dots, z_n^{mk})$, where

$$z_j^{mk} = \left(1 - 2^{-m_j}\right) e^{2(k_j+1/2)\pi i/2^{m_j+4}}, \quad 1 \le j \le n, \tag{2.7}$$

let

$$U_{mk} = \left\{ (z_1, \dots, z_n) \mid \left| z - z_j^{mk} \right| \le \frac{7}{8} 2^{-m_j}, 1 \le j \le n \right\}. \tag{2.8}$$

□

Lemma 2.2. *For fixed $m^0 = (m_1^0, \dots, m_n^0)$ and the corresponding $k^0 = (k_1^0, \dots, k_n^0)$, $T_{m^0k^0}$ intersect U_{mk} for at most $N = (5.57)^n$ choices of the pair (m, k).*

Proof. See [5]. □

Lemma 2.3. *If $f \in L_a^{\varphi}(\mathbb{D}^n)$, then*

$$\varphi\left(\left|f(z_1, \dots, z_n)\right|\right) \le C_1 \prod_{j=1}^{n} \int_{U_{mk}} \varphi\left(\left|f\right|\right) d\sigma_n \tag{2.9}$$

for $(6/8)2^{-m_j} \le \rho_j \le (7/8)2^{-m_j}$ and any $z \in T_{mk}$.

Proof. It is clear that $\varphi(|f(z_1, \dots, z_n)|)$ is an n-subharmonic function in $\mathbb{D}^n$. Repeated application of Harnack's inequality yields

$$\begin{aligned} &\varphi\left(\left|f(z_1, \dots, z_n)\right|\right) \\ &\quad \le \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \frac{\rho_j + \left|z_j - z_j^{mk}\right|}{\rho_j - \left|z_j - z_j^{mk}\right|} \int_0^{2\pi} \cdots \int_0^{2\pi} \varphi\left(\left|f\left(z_1^{mk} + \rho_1 e^{i\theta_1}, \dots, z_n^{mk} + \rho_n e^{i\theta_n}\right)\right|\right) d\theta_1 \cdots d\theta_n \\ &\quad \le C_2 \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \varphi\left(\left|f\left(z_1^{mk} + \rho_1 e^{i\theta_1}, \dots, z_n^{mk} + \rho_n e^{i\theta_n}\right)\right|\right) d\theta_1 \cdots d\theta_n. \end{aligned} \tag{2.10}$$

Hence, for $z \in T_{mk}$,

$$\varphi(|f(z_1,\dots,z_n)|) = C_2\left(\prod_{j=1}^{n}4^{m_j}\right)\int_{(6/8)2^{-m_1}}^{(7/8)2^{-m_1}}\cdots\int_{(6/8)2^{-m_n}}^{(7/8)2^{-m_n}}\varphi(|f(z)|)\rho_1\cdots\rho_n d\rho_1\cdots d\rho_n$$
$$\leq C_2C_3\left(\prod_{j=1}^{n}4^{m_j}\right)\int_{U_{mk}}\varphi(|f|)d\sigma_n. \tag{2.11}$$

□

Theorem 2.4 (Main theorem). *Let μ be a finite, positive measure on $\mathbb{D}^n$, and suppose that φ is an Orlicz function. Then, the identity map*

$$J : L_a^{\varphi}(\mathbb{D}^n,\sigma_n) \longrightarrow L^{\varphi}(\mathbb{D}^n,\mu) \tag{2.12}$$

is bounded if and only if μ is a φ Carleson measure.

Proof. Suppose that there exists a constant C such that

$$\|J(g)\|_{\sigma_n} \leq C\|g\|_{\mu}, \tag{2.13}$$

for all $g \in L_a^{\varphi}(\mathbb{D}^n)$. By Lemma 2.1,

$$u_{\alpha}(z) = \prod_{j=1}^{n}\frac{\left(1-|\alpha_j|^2\right)^2}{\left(1-\overline{\alpha}_j z_j\right)^4} \in L_a^{\varphi}(\mathbb{D}^n). \tag{2.14}$$

However, for $z_j \in S(I_j)$, we have

$$|1-\overline{\alpha}_j z| \leq \left|1-\overline{\alpha}_j e^{i(\theta_0+\delta_j/2)}\right| + \left|\overline{\alpha}_j e^{i(\theta_0+\delta_j/2)} - \overline{\alpha}_j z\right|$$
$$\leq \delta_j + (1-|\delta_j|)\left(\left|e^{i(\theta_0+\delta_j/2)}-\frac{z}{|z|}\right| + \left|\frac{z}{|z|}-z\right|\right)$$
$$\leq \delta_j + \left(1-\delta_j\right)\left(\delta_j + (1-|z|)\right)$$
$$\leq \delta_j + 2\delta_j\left(1-\delta_j\right) \leq 3\delta_j, \tag{2.15}$$

so,

$$|u_{\alpha}(z_1,\dots,z_n)| = \prod_{j=1}^{n}\frac{\left(1-|\alpha_j|^2\right)^2}{\left|1-\overline{\alpha}_j z_j\right|^4} \geq \frac{1}{3^n}\prod_{j=1}^{n}\frac{\left(1+\delta_j\right)^2}{\delta_j^2} \geq \frac{1}{3^n}. \tag{2.16}$$

Therefore,

$$\begin{aligned} 1 &\geq \int_{\mathbb{D}^n} \varphi\left(\frac{\varphi^{-1}\left(\prod_{j=1}^{n}\left(1/\delta_j^2\right)\right)|u_\alpha(z)|}{C}\right)d\mu \\ &\geq \int_{S(I)} \varphi\left(\frac{\varphi^{-1}\left(\prod_{j=1}^{n}\left(1/\delta_j^2\right)\right)}{3^n C}\right)d\mu \\ &= \varphi\left(\frac{\varphi^{-1}\left(\prod_{j=1}^{n}\left(1/\delta_j^2\right)\right)}{3^n C}\right)\mu(S(I)), \end{aligned} \tag{2.17}$$

that is,

$$\mu(S(I)) \leq \frac{1}{\varphi\left(C'\varphi^{-1}\left(\prod_{j=1}^{n}\left(1/\delta_j^2\right)\right)\right)}, \tag{2.18}$$

with $C' = 1/3^n C$.

Conversely, suppose that $f(z) \in L_a^{\varphi}(\mathbb{D}^n)$, we have

$$\begin{aligned} \int_{\mathbb{D}^n} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 N\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\mu &= \sum_{m=(m_1,\ldots,m_n),m_j\geq 0}\ \sum_{k=(k_1,\ldots,k_n),1\leq k_j\leq 2^{m_j+4}} \int_{T_{mk}} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 N\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\mu \\ &\leq \sum_m \sum_k \mu(T_{mk})\left\{C_1 \prod_{j=1}^{n} \int_{U_{mk}} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 N\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\sigma_n\right\} \\ &\leq C_1\mu(\mathbb{D}^n)\sum_m \sum_k \left\{\int_{U_{mk}} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 N\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\sigma_n\right\} \\ &= C_1\mu(\mathbb{D}^n)\left\{\sum_{m,k}\sum_{m_0,k_0} \int_{T_{m^0 k^0}\cap U_{mk}} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 N\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\sigma_n\right\} \\ &= C_1\mu(\mathbb{D}^n)\left\{\sum_{m_0,k_0}\sum_{m,k} \int_{T_{m^0 k^0}\cap U_{mk}} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 N\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\sigma_n\right\} \\ &\leq C_1\mu(\mathbb{D}^n)\left\{N\sum_{m_0,k_0} \int_{T_{m^0 k^0}} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 \mathrm{N}\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\sigma_n\right\} \\ &= C_1 N\mu(\mathbb{D}^n)\left\{\int_{\mathbb{D}^n} \varphi\left(\frac{|f(z_1,\ldots,c_n)|}{C_1 N\mu(\mathbb{D}^n)\|f\|_{\sigma_n}}\right)d\sigma_n\right\} \leq 1, \end{aligned} \tag{2.19}$$

and the proof is complete. □

Corollary 2.5. *Let μ be a finite, positive measure on $\mathbb{D}^n$, and suppose that φ is an Orlicz function. Then μ is a φ Carleson measure if and only if there exists some $C > 1$ such that*

$$\|u_\alpha(z)\|_{\sigma_n} \leq \frac{C}{\varphi^{-1}\left(\prod_{j=1}^{n}\left(1/\delta_j^2\right)\right)}, \tag{2.20}$$

for every rectangle $I \subset \mathbb{D}^n$.

Proof. As a fact, for any measure μ and Orlicz function φ, we have

$$1 \geq \int_{\mathbb{D}^n} \varphi\left(\frac{|u_\alpha(z)|}{\|u_\alpha(z)\|_{\sigma_n}}\right) d\mu \geq \varphi\left(\frac{1}{3^n\|u_\alpha(z)\|_{\sigma_n}}\right)\mu(S(I)) \tag{2.21}$$

by the proof of the Main theorem. So,

$$\mu(S(I)) \leq \frac{1}{\varphi\left(1/3^n\|u_\alpha(z)\|_{\sigma_n}\right)}, \tag{2.22}$$

and the corollary follows. □

Acknowledgment

This paper was supported by the NNSF of China (10671083) and the Youth Foundation of CQUPT (A2007-28).

References

[1] L. Carleson, "An interpolation problem for bounded analytic functions," *American Journal of Mathematics*, vol. 80, pp. 921–930, 1958.

[2] L. Carleson, "Interpolations by bounded analytic functions and the corona problem," *Annals of Mathematics. Second Series*, vol. 76, pp. 547–559, 1962.

[3] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, NY, USA, 1970.

[4] S.-Y. A. Chang, "Carleson measure on the bi-disc," *Annals of Mathematics. Second Series*, vol. 109, no. 3, pp. 613–620, 1979.

[5] W. W. Hastings, "A Carleson measure theorem for Bergman spaces," *Proceedings of the American Mathematical Society*, vol. 52, pp. 237–241, 1975.

[6] B. D. MacCluer, "Compact composition operators on $H^p(B_N)$," *The Michigan Mathematical Journal*, vol. 32, no. 2, pp. 237–248, 1985.

[7] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, "Composition operators on Hardy-Orlicz spaces," *Memoirs of the American Mathematical Society*, vol. 207, no. 974, 2010.

[8] J. Xiao, "The Carleson measure on analytic Orlicz spaces," *Journal of Anhui Normal University. Natural Science*, vol. 16, no. 3, pp. 6–10, 1993 (Chinese).

[9] A. G. Ortiz and J. Fernandez, "Carleson condition in Bergman-Orlicz spaces," *Complex Analysis and Operator Theory*, vol. 4, no. 2, pp. 257–269, 2010.