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*Research Article*

# Functions on the Plane as Combinations of Powers of Distances to Points

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Some real functions on the plane can be expressed as a linear combination of powers of distances to certain points.

## 1. Introduction

In this paper we construct a function basis to express different powers of the distance between a point and the origin. Namely, for two given natural numbers $m$ and $n$ we find points in the plane such that the function $r^{n+2m}\cos(n\theta)$ can be expressed as linear combination of powers of distances to those points (see formula (3.15) below), both the linear combination and the given function having the same domain in the plane (for the notion of special function that we use, see Section 3.1).

One of the important observations in the theory of real functions is that series expansions lead to certain interesting numbers and functions (e.g., Fourier coefficients, $L$-functions). Namely, usually series representations can be constructed explicitly in some model spaces of sections of various function bundles over appropriate analytic manifolds. (By considering functions on the plane as maps $F$: $\mathbb{R}^2 \to \mathbb{R}$, the graphs of these functions are the sections described by the map $\psi$: $\mathbb{R}^2 \to \mathbb{R}^3$; $(x, y) \mapsto (x, y, F(x, y))$, which defines 2-sub-manifolds on $\mathbb{R}^3$.) (for analytic functions on the plane $\mathbb{R}^2$; see Section 2.2).

We denote by $\mathcal{A}(\mathbb{R}^2)$ the subspace of real analytic functions. It is well known that any polynomial of degree $n$ can be written exactly as a linear combination of powers of the right number of terms of the form $x \cdot v_k + b_k$. We realize a series representation in the space of analytic functions on the plane $\mathcal{A}(\mathbb{R}^2)$.

Consider a point $\mathbf{p}$ in the plane $\mathbb{R}^2$ with coordinates $(r\cos\theta, r\sin\theta)$.

The distance used here is the Euclidean distance. Denote

$$|\mathbf{p}| = r. \tag{1.1}$$

We prove the following theorem.

**Theorem 1.1.** *Consider two natural numbers $m$ and $n$. There are $N$ numbers $a_1, \ldots, a_N$ and $N$ points $\mathbf{p}_1, \ldots, \mathbf{p}_N$, such that*

$$r^{n+2m}\cos(n\theta) = \sum_{i=1}^{N} a_i |\mathbf{p} - \mathbf{p}_i|^{2(m+n)}. \tag{1.2}$$

*Remark 1.2.* The numbers $a_1, \ldots, a_N$ and the points $\mathbf{p}_1, \ldots, \mathbf{p}_N$ can be determined exactly.

**Corollary 1.3.** *The linear combinations of $r^{n+2m}\cos(n\theta)$, $m, n = 0, 1, \ldots$, can be expressed similarly.*

*Remark 1.4.* An analogous theorem holds for the sine function.

Denote by $\underline{x}$ the maximum integer less than or equal to $x$.

Consider the function

$$\delta(x,y) = \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases} \tag{1.3}$$

## 2. Proof of Theorem 1.1

We will denote by $p_d$ the Legendre polynomial of degree $d$.

The following formula is formula 8.921 of [1]:

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{d=0}^{\infty} t^d p_d(x), \quad |t| < \min\left| x \pm \sqrt{x^2 - 1} \right|. \tag{2.1}$$

Consider the set $|t| < \min|x \pm \sqrt{x^2 - 1}|$ of $\mathbb{R}^2$. Consider a subset $\mathscr{H}$ (not necessarily proper) of such set. We will work solely with analytic functions on $\mathscr{H}$.

Consider the series $\sum_{i=0}^{\infty}\sum_{j=0}^{i} a_{ij} t^{i+j} x^{i-j}$.

Divide this number by $\sqrt{1 - 2tx + t^2}$. By formula (2.1), the following formula holds:

$$\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{i} a_{ij} t^{i+j} x^{i-j}}{\sqrt{1 - 2tx + t^2}} = \sum_{i=0}^{\infty}\sum_{j=0}^{i} a_{ij} t^{i+j} x^{i-j} \cdot \sum_{d=0}^{\infty} t^d p_d(x). \tag{2.2}$$

We use the formula of the product of two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$:

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}. \tag{2.3}$$

The right side of formula (2.2) becomes

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{ij} t^{i+j} x^{i-j} \cdot \sum_{d=0}^{\infty} t^d p_d(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{i} \sum_{j=0}^{i-k} a_{i-k,j} p_k(x) t^{i+j} x^{i-k-j}. \tag{2.4}$$

Let us change the order of addition with respect to the indices $j$ and $k$:

$$\sum_{k=0}^{i} \sum_{j=0}^{i-k} a_{i-k,j} p_k(x) t^{i+j} x^{i-k-j} = \sum_{j=0}^{i} \sum_{k=0}^{i-j} a_{i-k,j} p_k(x) t^{i+j} x^{i-k-j}. \tag{2.5}$$

The following formula is formula (8.911) of [1]:

$$p_k(x) = \sum_{l=0}^{\overline{k/2}} b_{kl} x^{k-2l}, \tag{2.6}$$

where

$$b_{kl} = \frac{1}{2^k} \frac{(-1)^l (2k-2l)!}{l!(k-l)!(k-2l)!}. \tag{2.7}$$

Substituting (2.6) in (2.5),

$$\begin{aligned} \sum_{j=0}^{i} \sum_{k=0}^{i-j} a_{i-k,j} p_k(x) t^{i+j} x^{i-k-j} &= \sum_{j=0}^{i} \sum_{k=0}^{i-j} a_{i-k,j} \sum_{l=0}^{\overline{k/2}} b_{kl} x^{k-2l} t^{i+j} x^{i-k-j} \\ &= \sum_{j=0}^{i} \sum_{k=0}^{i-j} \sum_{l=0}^{\overline{k/2}} a_{i-k,j} b_{kl} t^{i+j} x^{i-j-2l}. \end{aligned} \tag{2.8}$$

Let us change the order of addition with respect to the indices $k$ and $l$:

$$\sum_{k=0}^{i-j} \sum_{l=0}^{\overline{k/2}} a_{i-k,j} b_{kl} t^{i+j} x^{i-j-2l} = \sum_{l=0}^{\overline{(i-j)/2}} \sum_{k=2l}^{i-j} a_{i-k,j} b_{kl} t^{i+j} x^{i-j-2l}. \tag{2.9}$$

Denote by

$$a_{ijl} = \sum_{k=2l}^{i-j} a_{i-k,j} b_{kl}. \tag{2.10}$$

The sum (2.9) is $\sum_{l=0}^{(i-j)/2} a_{ijl} t^{i+j} x^{i-j-2l}$. We will substitute this sum in (2.8).

$$\sum_{j=0}^{i} \sum_{k=0}^{i-j} \sum_{l=0}^{k/2} a_{i-k,j} b_{kl} t^{i+j} x^{i-j-2l} = \sum_{j=0}^{i} \sum_{l=0}^{(i-j)/2} a_{ijl} t^{i+j} x^{i-j-2l}. \tag{2.11}$$

Let us change the order of addition with respect to the indices $j$ and $l$:

$$\sum_{j=0}^{i} \sum_{l=0}^{(i-j)/2} a_{ijl} t^{i+j} x^{i-j-2l} = \sum_{l=0}^{i/2} \sum_{j=0}^{i-2l} a_{ijl} t^{i+j} x^{i-j-2l}. \tag{2.12}$$

Let us add with respect to the index $i$ from zero to infinity

$$\sum_{i=0}^{\infty} \sum_{l=0}^{i/2} \sum_{j=0}^{i-2l} a_{ijl} t^{i+j} x^{i-j-2l}. \tag{2.13}$$

Let us change the order of addition with respect to the indices $i$ and $l$:

$$\sum_{i=0}^{\infty} \sum_{l=0}^{i/2} \sum_{j=0}^{i-2l} a_{ijl} t^{i+j} x^{i-j-2l} = \sum_{l=0}^{\infty} \sum_{i=2l}^{\infty} \sum_{j=0}^{i-2l} a_{ijl} t^{i+j} x^{i-j-2l}. \tag{2.14}$$

Change the index $i$ for index $i' = i - 2l$:

$$\sum_{l=0}^{\infty} \sum_{i=2l}^{\infty} \sum_{j=0}^{i-2l} a_{ijl} t^{i+j} x^{i-j-2l} = \sum_{l=0}^{\infty} \sum_{i'=0}^{\infty} \sum_{j'=0}^{i'} a_{i'+2l,j'l} t^{i'+2l+j'} x^{i'-j'}. \tag{2.15}$$

We change the indices $i'$ and $j'$ for the indices

$$\begin{gathered} i' = i - l, \\ j' = j - l, \\ \sum_{l=0}^{\infty} \sum_{i'=0}^{\infty} \sum_{j'=0}^{i'} a_{i'+2l,j'l} t^{i'+2l+j'} x^{i'-j'} = \sum_{l=0}^{\infty} \sum_{i=l}^{\infty} \sum_{j=l}^{i-l} a_{i+l,j-l,l} t^{i+j} x^{i-j}. \end{gathered} \tag{2.16}$$

Let us change the order of addition with respect to the indices $i$ and $l$:

$$\sum_{l=0}^{\infty} \sum_{i=l}^{\infty} \sum_{j=l}^{i-l} a_{i+l,j-l,l} t^{i+j} x^{i-j} = \sum_{i=0}^{\infty} \sum_{l=0}^{i} \sum_{j=l}^{i-l} a_{i+l,j-l,l} t^{i+j} x^{i-j}. \tag{2.17}$$

Let us change the order of addition with respect to the indices $j$ and $l$:

$$\sum_{l=0}^{i}\sum_{j=l}^{i-l} a_{i+l,j-l,l} t^{i+j} x^{i-j} = \sum_{j=0}^{i}\sum_{l=0}^{j} a_{i+l,j-l,l} t^{i+j} x^{i-j}. \tag{2.18}$$

Denote by

$$A_{ij} = \sum_{l=0}^{j} a_{i+l,j-l,l}. \tag{2.19}$$

The sum (2.18) becomes

$$\sum_{l=0}^{i}\sum_{j=l}^{i-l} a_{i+l,j-l,l} t^{i+j} x^{i-j} = \sum_{j=0}^{i} A_{ij} t^{i+j} x^{i-j}. \tag{2.20}$$

Let us add with respect to the index $i$ from zero to infinity.

The ratio (2.2) becomes

$$\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{i} a_{ij} t^{i+j} x^{i-j}}{\sqrt{1-2tx+t^2}} = \sum_{i=0}^{\infty}\sum_{j=0}^{i} A_{ij} t^{i+j} x^{i-j}. \tag{2.21}$$

### 2.1. *Coefficients with Three Indices*

Let us calculate the numbers $a_{i+l,j-l,l}$. By formula (2.10), the following formula holds:

$$a_{i+l,j-l,l} = \sum_{k=2l}^{i+2l-j} a_{i+l-k,j-l} b_{kl}. \tag{2.22}$$

Change the index $k$ for index $k-2l$:

$$\sum_{k=2l}^{i+2l-j} a_{i+l-k,j-l} b_{kl} = \sum_{k=0}^{i-j} a_{i-l-k,j-l} b_{2\,l+k,l}. \tag{2.23}$$

By formula (2.7), the following formula holds:

$$b_{2l+k,l} = \frac{(-1)^l (2l+2k)!}{2^{2l+k} l!(l+k)!k!}. \tag{2.24}$$

Let us use the formula

$$\frac{d^n}{dx^n}\frac{1}{\sqrt{x}} = \frac{(-1)^n (2n)! 4^{-n} x^{-n-1/2}}{n!} \tag{2.25}$$

to write

$$\frac{(-1)^l(2l+2k)!}{2^{2l+k}l!(l+k)!k!} = \frac{(-1)^k 2^k x^{l+k+1/2}}{l!k!}\frac{d^{l+k}}{dx^{l+k}}\frac{1}{\sqrt{x}}. \tag{2.26}$$

We will substitute this in (2.23):

$$\sum_{k=0}^{i-j} a_{i-l-k,j-l} b_{2l+k,l} = \sum_{k=0}^{i-j} a_{i-l-k,j-l}\frac{(-1)^k 2^k x^{l+k+1/2}}{l!k!}\frac{d^{l+k}}{dx^{l+k}}\frac{1}{\sqrt{x}}. \tag{2.27}$$

Formula (2.22) becomes

$$a_{i+l,j-l,l} = \sum_{k=0}^{i-j} a_{i-l-k,j-l}\frac{(-1)^k 2^k x^{l+k+1/2}}{l!k!}\frac{d^{l+k}}{dx^{l+k}}\frac{1}{\sqrt{x}}. \tag{2.28}$$

### *2.2. A Series of Half-Integer Powers*

Consider three numbers $a$, $b$, and $c$.

We will develop the power of the sum of three numbers.

**Lemma 2.1.** *Consider a natural number $n$. Then, the following formula holds:*

$$(a+b+c)^n = \sum_{i=0}^{n}\sum_{j=0}^{i}\frac{n!a^{n-i}b^{i-j}c^j}{(n-i)!j!(i-j)!}. \tag{2.29}$$

Consider the series $\sum_{n=0}^{\infty} a_n(a^2-2abc+c^2)^n$.

By Lemma 2.1 the following formula holds:

$$\sum_{n=0}^{\infty} a_n\left(a^2-2abc+c^2\right)^n = \sum_{n=0}^{\infty} a_n \sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^{i-j}2^{i-j}n!a^{2n-i-j}b^{i-j}c^{i+j}}{(n-i)!j!(i-j)!}. \tag{2.30}$$

We will distribute this sum, and we will exchange the order of addition:

$$\sum_{n=0}^{\infty} a_n\left(a^2-2abc+c^2\right)^n = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{n=i}^{\infty} a_n\frac{(-1)^{i-j}2^{i-j}n!a^{2n-i-j}b^{i-j}c^{i+j}}{(n-i)!j!(i-j)!}. \tag{2.31}$$

Denote by

$$a_{ij} = \sum_{n=i}^{\infty}\frac{a_n(-1)^{i-j}2^{i-j}n!a^{2n-i-j}}{(n-i)!j!(i-j)!}. \tag{2.32}$$

The series becomes

$$\sum_{n=0}^{\infty} a_n \left(a^2 - 2abc + c^2\right)^n = \sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{ij} b^{i-j} c^{i+j}. \tag{2.33}$$

Consider the formula

$$\left.\frac{d^i x^n}{dx^i}\right|_{x=a^2} = \frac{a^{2n-2i} n!}{(n-i)!}. \tag{2.34}$$

*2.2.1. Formulas for Coefficients with Two Indices*

Let us use formula (2.34) to rewrite formula (2.32).

$$a_{ij} = \sum_{n=i}^{\infty} \frac{a_n (-1)^{i-j} 2^{i-j} a^{i-j}}{j!(i-j)!} \left.\frac{d^i x^n}{dx^i}\right|_{x=a^2}. \tag{2.35}$$

Consider the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{2.36}$$

Let us differentiate the series term by term:

$$\frac{d^i}{dx^i} f(x) = \sum_{n=0}^{\infty} a_n \frac{d^i x^n}{dx^i}. \tag{2.37}$$

The $i-1$ first terms in this series are equal to zero. Then, the following formula holds:

$$\frac{d^i}{dx^i} f(x) = \sum_{n=i}^{\infty} a_n \frac{d^i x^n}{dx^i}. \tag{2.38}$$

The number $a_{ij}$ becomes

$$a_{ij} = \frac{(-1)^{i-j} 2^{i-j} a^{i-j}}{j!(i-j)!} \left.\frac{d^i f(x)}{dx^i}\right|_{x=a^2}. \tag{2.39}$$

*(1) Coefficients with Two Indices*

We write the numbers $a_{i-l-k,j-l}$ substituting the numbers $a_{ij}$ for numbers $a^{i+j-1} a_{ij}$.

By formula (2.39), the following formula holds:

$$a^{i+j-k-2l-1} a_{i-l-k,j-l} = \frac{(-1)^{i-k-j} 2^{i-k-j} a^{2i-2k-2l-1}}{(j-l)!(i-k-j)!} \left.\frac{d^{i-k-l} f(x)}{dx^{i-k-l}}\right|_{x=a^2}. \tag{2.40}$$

Let us substitute in (2.28):

$$a_{i+l,j-l,l} = \sum_{k=0}^{i-j} \frac{(-1)^{i-k-j} 2^{i-k-j} a^{2i-2k-2l-1}}{(j-l)!(i-k-j)!} \frac{d^{i-k-l} f(x)}{dx^{i-k-l}}\bigg|_{x=a^2} \frac{(-1)^k 2^k x^{l+k+1/2}}{l!k!} \frac{d^{l+k}}{dx^{l+k}} \frac{1}{\sqrt{x}}. \tag{2.41}$$

Let

$$x = a^2. \tag{2.42}$$

Then, the following formula holds:

$$a_{i+l,j-l,l} = \sum_{k=0}^{i-j} \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{(j-l)!(i-k-j)!l!k!} \frac{d^{i-k-l} f(x)}{dx^{i-k-l}}\bigg|_{x=a^2} \frac{d^{l+k}}{dx^{l+k}} \frac{1}{\sqrt{x}}\bigg|_{x=a^2}. \tag{2.43}$$

The sum equals

$$a_{i+l,j-l,l} = \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{(i-j)!(j-l)!l!} \frac{d^{i-j}}{dx^{i-j}} \left( \frac{d^{j-l} f(x)}{dx^{j-l}} \frac{d^l}{dx^l} \frac{1}{\sqrt{x}} \right)\bigg|_{x=a^2}. \tag{2.44}$$

Let us add with respect of index $l$ from zero to $j$:

$$\begin{aligned} \sum_{l=0}^{j} a_{i+l,j-l,l} &= \sum_{l=0}^{j} \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{(i-j)!(j-l)!l!} \frac{d^{i-j}}{dx^{i-j}} \left( \frac{d^{j-l} f(x)}{dx^{j-l}} \frac{d^l}{dx^l} \frac{1}{\sqrt{x}} \right)\bigg|_{x=a^2} \\ &= \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{(i-j)!} \frac{d^{i-j}}{dx^{i-j}} \sum_{l=0}^{j} \frac{1}{(j-l)!l!} \frac{d^{j-l} f(x)}{dx^{j-l}} \frac{d^l}{dx^l} \frac{1}{\sqrt{x}}\bigg|_{x=a^2}. \end{aligned} \tag{2.45}$$

The sum equals

$$\sum_{l=0}^{j} \frac{1}{(j-l)!l!} \frac{d^{j-l} f(x)}{dx^{j-l}} \frac{d^l}{dx^l} \frac{1}{\sqrt{x}} = \frac{1}{j!} \frac{d^j}{dx^j} \frac{f(x)}{\sqrt{x}}. \tag{2.46}$$

We will substitute this sum in (2.45):

$$\begin{aligned} \sum_{l=0}^{j} a_{i+l,j-l,l} &= \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{(i-j)!} \frac{d^{i-j}}{dx^{i-j}} \frac{1}{j!} \frac{d^j}{dx^j} \frac{f(x)}{\sqrt{x}}\bigg|_{x=a^2} \\ &= \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{(i-j)!j!} \frac{d^i}{dx^i} \frac{f(x)}{\sqrt{x}}\bigg|_{x=a^2}. \end{aligned} \tag{2.47}$$

We will substitute this sum in (2.19):

$$A_{ij} = \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{j!(i-j)!} \frac{d^i}{dx^i} \frac{f(x)}{\sqrt{x}} \bigg|_{x=a^2}. \tag{2.48}$$

*2.2.2. A Series of Half-Integer Powers*

Divide both sides of (2.33) by $\sqrt{a^2 - 2abc + c^2}$:

$$\begin{aligned} \frac{\sum_{n=0}^{\infty} a_n \left(a^2 - 2abc + c^2\right)^n}{\sqrt{a^2 - 2abc + c^2}} &= \frac{\sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{ij} b^{i-j} c^{i+j}}{\sqrt{a^2 - 2abc + c^2}} \\ &= \frac{\sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{ij} b^{i-j} c^{i+j}}{a\sqrt{1 - 2(bc/a) + c^2/a^2}}. \end{aligned} \tag{2.49}$$

By formula (2.21), the following formula holds:

$$\frac{\sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{ij} b^{i-j} c^{i+j}}{a\sqrt{1 - 2(bc/a) + c^2/a^2}} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} A_{ij} c^{i+j} b^{i-j}, \tag{2.50}$$

where

(i) coefficients $a_{ij}$ in (2.21) are replaced by coefficients $a^{i+j-1} a_{ij}$,

(ii) coefficients $A_{ij}$ are given by formula (2.48).

The left side of (2.49) becomes

$$\frac{\sum_{n=0}^{\infty} a_n \left(a^2 - 2abc + c^2\right)^n}{\sqrt{a^2 - 2abc + c^2}} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} A_{ij} c^{i+j} b^{i-j}. \tag{2.51}$$

## 2.3. A Series of Half-Integer Powers

Consider the series $\sum_{n=0}^{\infty} a_n x^{n/2}$.

Let us separate this sum into

(i) a sum with respect to even indices,

(ii) a sum over odd indices.

Then, one has

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^{n/2} &= \sum_{n=0}^{\infty} a_{2n} x^n + \sum_{n=0}^{\infty} a_{1+2n} x^{1/2+n} \\ &= \sum_{n=0}^{\infty} a_{2n} x^n + \frac{\sum_{n=0}^{\infty} a_{1+2n} x^{n+1}}{\sqrt{x}}. \end{aligned} \tag{2.52}$$

In the second sum, we change the index $n$ for index $n' = n + 1$:

$$\sum_{n=0}^{\infty} a_{1+2n} x^{n+1} = \sum_{n'=1}^{\infty} a_{-1+2n'} x^{n'}. \tag{2.53}$$

Define

$$a_{-1} = 0. \tag{2.54}$$

Then, the following formula holds:

$$\sum_{n'=1}^{\infty} a_{-1+2n'} x^{n'} = \sum_{n=0}^{\infty} a_{-1+2n} x^{n}. \tag{2.55}$$

Formula (2.52) becomes

$$\sum_{n=0}^{\infty} a_n x^{n/2} = \sum_{n=0}^{\infty} a_{2n} x^n + \frac{\sum_{n=0}^{\infty} a_{-1+2n} x^n}{\sqrt{x}}. \tag{2.56}$$

We will denote

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_{2n} x^n, \\ g(x) &= \frac{\sum_{n=0}^{\infty} a_{-1+2n} x^n}{\sqrt{x}}. \end{aligned} \tag{2.57}$$

Let

$$x = a^2 - 2abc + c^2. \tag{2.58}$$

By formulas (2.33) and (2.51), the following formula holds:

$$\sum_{n=0}^{\infty} a_n \left(a^2 - 2abc + c^2\right)^{n/2} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{ij} b^{i-j} c^{i+j} + \sum_{i=0}^{\infty} \sum_{j=0}^{i} b_{ij} b^{i-j} c^{i+j}, \tag{2.59}$$

where (see (2.39) and (2.48))

$$\begin{aligned} a_{ij} &= \frac{(-1)^{i-j} 2^{i-j} a^{i-j}}{j!(i-j)!} \frac{d^i f(x)}{dx^i}\bigg|_{x=a^2}, \\ b_{ij} &= \frac{(-1)^{i-j} 2^{i-j} a^{2i}}{j!(i-j)!} \frac{d^i g(x)}{dx^i}\bigg|_{x=a^2}. \end{aligned} \tag{2.60}$$

We will denote

$$A_{ij} = \frac{(-1)^{i-j} 2^{i-j} a^{i-j}}{j!(i-j)!} \left( \left. \frac{d^i f(x)}{dx^i} \right|_{x=a^2} + a^{i+j} \left. \frac{d^i g(x)}{dx^i} \right|_{x=a^2} \right). \tag{2.61}$$

Formula (2.59) becomes

$$\sum_{n=0}^{\infty} a_n \left(a^2 - 2abc + c^2\right)^{n/2} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} A_{ij} b^{i-j} c^{i+j}. \tag{2.62}$$

*2.3.1. The Case of the Law of Cosines*

Consider a number $\theta$.

Let

$$b = \cos\theta. \tag{2.63}$$

The following formula appears in [2]:

$$(\cos\theta)^n = 2^{1-n} n! \sum_{k=0}^{n/2} \frac{\cos((n-2k)\theta)}{k!(n-k)!(1+\delta(n,2k))}. \tag{2.64}$$

Let us use this formula to rewrite formula (2.62):

$$\sum_{n=0}^{\infty} a_n \left(a^2 - 2ac\cos\theta + c^2\right)^{n/2} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} A_{ij} 2^{1-i+j} (i-j)! \sum_{k=0}^{\overline{i/2-j/2}} \frac{\cos((i-j-2k)\theta)}{k!(i-j-k)!(1+\delta(i-j,2k))} c^{i+j}. \tag{2.65}$$

By Lemma A.2 (see the appendix), the following formula holds:

$$\begin{aligned} &\sum_{i=0}^{\infty} \sum_{j=0}^{i} A_{ij} 2^{1-i+j} (i-j)! \sum_{k=0}^{\overline{i/2-j/2}} \frac{\cos((i-j-2k)\theta)}{k!(i-j-k)!(1+\delta(i-j,2k))} c^{i+j} \\ &\quad = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \frac{2^{1-i+j} A_{i+k,j-k} 4^{-k} (i+2k-j)! \cos((i-j)\theta) c^{i+j}}{k!(i+k-j)!(1+\delta(i,j))}. \end{aligned} \tag{2.66}$$

By formula (2.61), the following formula holds:

$$A_{i+k,j-k} = \frac{(-1)^{i-j} 2^{i+2k-j} a^{i+2k-j} \left( d^{i+k} f(x)/dx^{i+k} \big|_{x=a^2} + a^{i+j} (d^{i+k} g(x)/dx^{i+k}) \big|_{x=a^2} \right)}{(j-k)!(i+2k-j)!}. \tag{2.67}$$

Multiply by $2^{1-i+j}4^{-k}(i+2k-j)!/k!(i+k-j)!(1+\delta(i,j))$:

$$\frac{2^{1-i+j}A_{i+k,j-k}4^{-k}(i+2k-j)!}{k!(i+k-j)!(1+\delta(i,j))} = 2\frac{(-1)^{i-j}\left(a^{i+2k-j}\left(d^{i+k}f(x)/dx^{i+k}\right)\big|_{x=a^2} + a^{2i+2k}\left(d^{i+k}g(x)/dx^{i+k}\right)\big|_{x=a^2}\right)}{(j-k)!k!(i+k-j)!(1+\delta(i,j))}. \tag{2.68}$$

Let us use formula (2.34) to rewrite

$$2\frac{(-1)^{i-j}\left(a^{i+2k-j}\left(d^{i+k}f(x)/dx^{i+k}\right)\big|_{x=a^2} + a^{2i+2k}\left(d^{i+k}g(x)/dx^{i+k}\right)\big|_{x=a^2}\right)}{(j-k)!k!(i+k-j)!(1+\delta(i,j))} = 2\frac{(-1)^{i-j}\left(d^{j-k}x^i/dx^{j-k}\right)\Big|_{x=a^2}\left(a^{-i+j}\left(d^{i+k}f(x)/dx^{i+k}\right)\big|_{x=a^2} + a^{2j}\left(d^{i+k}g(x)/dx^{i+k}\right)\big|_{x=a^2}\right)}{(j-k)!k!(1+\delta(i,j))i!}. \tag{2.69}$$

Let us add with respect to the index $k$ from zero to $j$:

$$\sum_{k=0}^{j}2\frac{(-1)^{i-j}\left(d^{j-k}x^i/dx^{j-k}\right)\Big|_{x=a^2}\left(a^{-i+j}\left(d^{i+k}f(x)/dx^{i+k}\right)\big|_{x=a^2} + a^{2j}d^{i+k}g(x)/dx^{i+k}\right)\big|_{x=a^2}\right)}{(j-k)!k!(1+\delta(i,j))i!} = 2\frac{(-1)^{i-j}\left(a^{-i+j}\left(d^j/dx^j\right)\left(x^i\left(d^if(x)/dx^i\right)\right)\big|_{x=a^2} + a^{2j}\left(d^j/dx^j\right)\left(x^i\left(d^ig(x)/dx^i\right)\right)\big|_{x=a^2}\right)}{i!j!(1+\delta(i,j))}. \tag{2.70}$$

Multiply by $\cos((i-j)\theta)c^{i+j}$.

Let us add with respect to the index $j$ from zero to $i$.

Let us add with respect to the index $i$ from zero to infinity.

By Lemma A.1 (see the appendix), the following formula holds:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}2\,\frac{(-1)^{i-j}a^{2j}\left(d^j/dx^j\right)\left(a^{-i-j}x^i\left(d^if(x)/dx^i\right)+x^i\left(d^ig(x)/dx^i\right)\right)\Big|_{x=a^2}}{i!j!(1+\delta(i,j))}\cos((i-j)\theta)c^{i+j} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}2\,\frac{(-1)^j\left(d^i/dx^i\left(x^{i+j}\left(d^{i+j}/dx^{i+j}\right)\left(a^{-j}f(x)+a^{2i}g(x)\right)\right)\right)\cos(j\theta)c^{2i+j}}{i!(i+j)!(1+\delta(0,j))}. \tag{2.71}$$

We will denote

$$B_{ij} = 2\frac{(-1)^j\left(d^i/dx^i\right)\left(x^{i+j}\left(d^{i+j}/dx^{i+j}\right)\left(a^{-j}f(x)+a^{2i}g(x)\right)\right)\Big|_{x=a^2}}{i!(i+j)!(1+\delta(0,j))}. \tag{2.72}$$

The left side of (2.65) becomes

$$\sum_{n=0}^{\infty} a_n \left(a^2 - 2ac\cos\theta + c^2\right)^{n/2} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} B_{ij}\cos(j\theta)c^{2i+j}. \tag{2.73}$$

*(1) A Formula for Calculating Coefficients*

Consider two natural numbers $m$ and $n$.

Let

$$f(x) = x^{m+n}, \qquad g(x) = 0. \tag{2.74}$$

Formula (2.72) becomes

$$B_{ij} = 2\,\frac{(-1)^j a^{-j+2\,m+2n-2i}((m+n)!)^2}{(m+n-i)!(m+n-i-j)!i!(i+j)!(1+\delta(0,j))}. \tag{2.75}$$

## 3. Functions on the Plane Expressed as a Linear Combination of Powers of Distances to Points

We deduce an identity regarding linear combinations of powers of distances from certain points, as well as similar new identities, from the formula of a power of the cosine. The formula of a power of the cosine is a powerful tool in mathematical analysis. In the space of analytic functions, we construct sums $\sum_{i=1}^{N} a_i|\mathbf{p}-\mathbf{p}_i|^p$ associated to them. We allow ourselves to use the name sum even if it is not finite.

Consider

(i) two numbers $r$ and $\theta$,

(ii) two natural numbers $m$ and $n$.

Assume that there are (i) numbers $\theta_1, \theta_2, \ldots$, (ii) positive numbers $p_1, p_2, \ldots$, and (iii) series $\sum_{i=0}^{\infty} a_{ki}(p_k^2 - 2p_k r\cos(\theta-\theta_k) + r^2)^{i/2}$, $k = 1, 2, \ldots$, such that

$$r^{n+2m}\cos(n\theta) = \sum_k \sum_{i=0}^{\infty} a_{ki}\left(p_k^2 - 2p_k r\cos(\theta-\theta_k) + r^2\right)^{i/2}. \tag{3.1}$$

We will denote

$$f_k(x) = \sum_{i=0}^{\infty} a_{k,2i}x^i, \qquad g_k(x) = \frac{\sum_{i=0}^{\infty} a_{k,-1+2i}x^i}{\sqrt{x}}. \tag{3.2}$$

By formula (2.73), the following formula holds:

$$\sum_{i=0}^{\infty} a_{ki}\left(p_k^2 - 2p_k r\cos(\theta-\theta_k) + r^2\right)^{i/2} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} B_{kij}\cos\left(j(\theta-\theta_k)\right) r^{2i+j}, \tag{3.3}$$

where the coefficients $B_{kij}$ are given by formula (2.72):

$$B_{ijk} = 2\,\frac{(-1)^j\left(d^i/dx^i\right)\left(x^{i+j}\left(d^{i+j}/dx^{i+j}\right)\left(a^{-j}f_k(x) + a^{2i}g_k(x)\right)\right)\Big|_{x=a^2}}{i!(i+j)!(1+\delta(0,j))}. \tag{3.4}$$

We use the formula

$$\cos(\theta-\phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi). \tag{3.5}$$

Formula (3.1) becomes

$$r^{n+2m}\cos(n\theta) = \sum_k \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} B_{kij}\left(\cos(j\theta)\cos(j\theta_k) + \sin(j\theta)\sin(j\theta_k)\right) r^{2i+j}. \tag{3.6}$$

The following formulas are sufficient conditions for the solution of the equation above:

$$\begin{gathered}\theta_k = \frac{k\pi}{n}, \quad k = 1, 2, \ldots, \\ \sum_k \cos\left(\frac{jk\pi}{n}\right) B_{kij} = \delta(i,m)\delta(j,n), \\ \sum_k \sin\left(\frac{jk\pi}{n}\right) B_{kij} = 0, \quad i, j = 0, 1, \ldots.\end{gathered} \tag{3.7}$$

### *3.1. The Product of a Power of the Distance to the Origin and Cosine*

Consider

(i) $1+m$ numbers $a_0, \ldots, a_m$

(ii) $1+m$ positive numbers $p_0, \ldots, p_m$.

We will denote

$$f_{kl}(x) = (-1)^k a_l x^{m+n}, \quad k = 1, \ldots, 2n;\ l = 0, \ldots, m. \tag{3.8}$$

In Section 2.3.1(1), we computed the numbers $B_{ij}$. Let us denote

$$B_{kl,ij} = 2\frac{(-1)^{k+j} a_l p_l^{-j+2m+2n-2i}\left((m+n)!\right)^2}{(m+n-i)!(m+n-i-j)!i!(i+j)!(1+\delta(0,j))}. \tag{3.9}$$

Multiply by $\cos(jk\pi/n)$ (($\sin(jk\pi/n)$, resp.), and add up over the index $k$ from one to $2n$, $\sum_{k=1}^{2n} B_{kl,ij}\cos(jk\pi/n)$ ($\sum_{l=1}^{2n} B_{kl,ij}\sin(jk\pi/n)$, resp.).

We use the formula

$$\sum_{k=1}^{2n}(-1)^k e^{\sqrt{-1}jk\pi/n} = 2n\delta(j,n). \tag{3.10}$$

The sums $\sum_{k=1}^{2n} B_{kl,ij}\cos(jk\pi/n)$ and $\sum_{k=1}^{2n} B_{kl,ij}\sin(jk\pi/n)$ are equal to

$$\begin{aligned}\sum_{k=1}^{2n} B_{kl,ij}\cos\left(\frac{jk\pi}{n}\right) &= \frac{4n\delta(j,n)(-1)^j a_l {p_l}^{-j+2m+2n-2i}((m+n)!)^2}{(m+n-i)!(m+n-i-j)!i!(i+j)!(1+\delta(0,j))} \\ &= \frac{4n\delta(j,n)(-1)^n a_l {p_l}^{n+2m-2i}((m+n)!)^2}{(m+n-i)!(m-i)!i!(i+n)!(1+\delta(0,n))}, \\ \sum_{k=1}^{2n} B_{kl,ij}\sin\left(\frac{jk\pi}{n}\right) &= 0.\end{aligned} \tag{3.11}$$

We want the numbers $a_l$ to satisfy the following equation:

$$\sum_{l=0}^{m}\sum_{k=1}^{2n} B_{kl,ij}\cos\left(\frac{jk\pi}{n}\right) = \delta(i,m)\delta(j,n), \quad i=0,\ldots,m;\ j=0,1,\ldots. \tag{3.12}$$

By formula (3.11), this sum is reduced to

$$\sum_{l=0}^{m} a_l {p_l}^{n+2m-2i} = \frac{\delta(i,m)(1+\delta(0,n))(-1)^n(n-1)!m!}{4(m+n)!}, \quad i=0,\ldots,m. \tag{3.13}$$

This is a system of linear equations with unknowns $a_0,\ldots,a_m$. The coefficient matrix for the system is ${p_l}^{n+2m-2i}$; $i,l=0,\ldots,m$.

Assume that the numbers $p_0,\ldots,p_m$ are different from each other. Then,

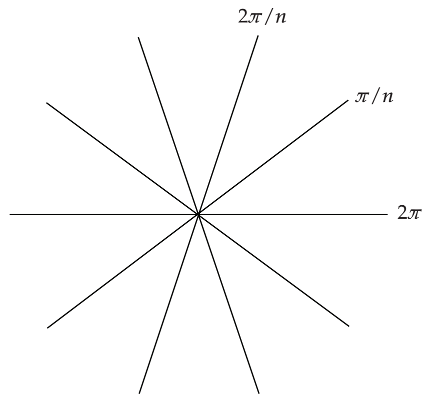
$$a_l = \frac{(1+\delta(0,n))(-1)^n(n-1)!m!}{4(m+n)!} p_l^{-n}\prod_{i\neq l}\frac{p_i^2}{p_i^2-p_l^2}, \quad l=0,\ldots,m. \tag{3.14}$$

Equations (3.7) are valid. Therefore,

$$r^{n+2m}\cos(n\theta) = \sum_{l=0}^{m}\sum_{k=1}^{2n}(-1)^l a_l\left(p_l^2 - 2p_l r\cos(\theta-\theta_k) + r^2\right)^{m+n}, \tag{3.15}$$

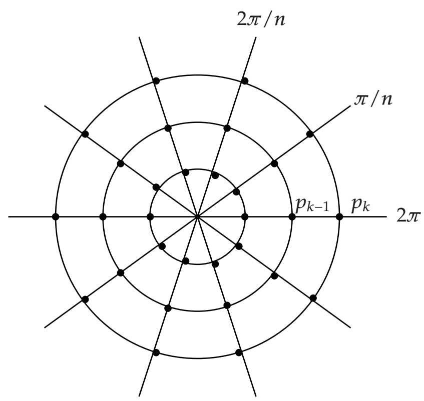
where, as in (3.7),

$$\theta_k = \frac{k\pi}{n}, \quad k=1,2\ldots. \tag{3.16}$$




Some angles $\theta_k$, $n = 5$

**Figure** 1




Points with coordinates $(p_l \cos\theta_k, p_l \sin\theta_k)$, $k = 1, \ldots, 2n; l = 0, \ldots, m$
End of proof

**Figure** 2

### *3.2. Example: The Difference between Two Paraboloids*

Write

$$r\cos(\theta) = \left(\frac{r\cos(\theta)}{2} + \frac{1}{16} + r^2\right) - \left(-\frac{r\cos(\theta)}{2} + \frac{1}{16} + r^2\right). \tag{3.17}$$

In the coordinates $(x, y)$,

$$r\cos(\theta) = x. \tag{3.18}$$

Figure 3 represents graphically Identity (3.17).

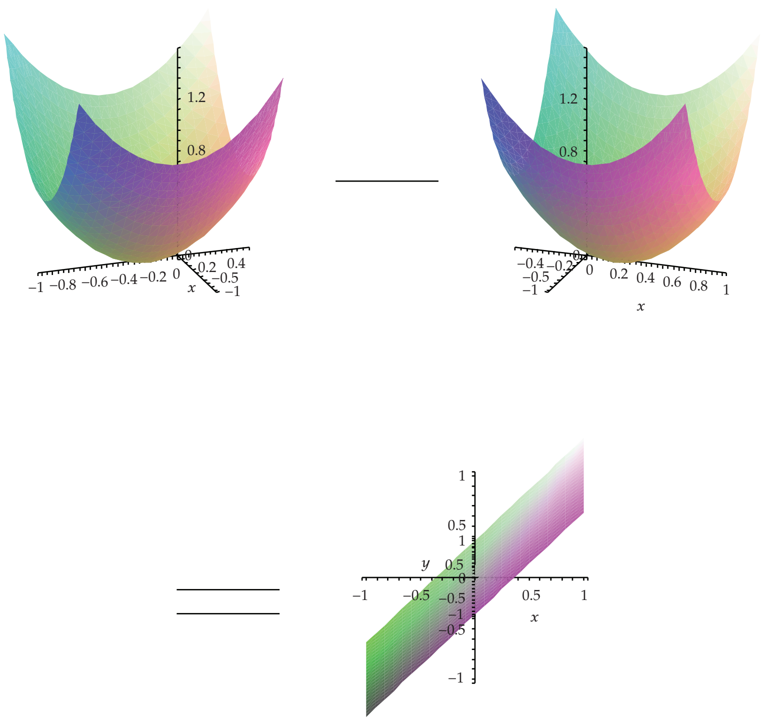



**Figure** 3

### 3.3. *Example: A Combination of Fourth-Degree Surfaces*

Write

$$\begin{aligned} r^2\cos(2\theta) = {} & -\left(\frac{1}{8}-\frac{1}{\sqrt{2}}r\sin(\theta)+r^2\right)^2+\left(\frac{1}{8}+\frac{1}{\sqrt{2}}r\cos(\theta)+r^2\right)^2 \\ & -\left(\frac{1}{8}+\frac{1}{\sqrt{2}}r\sin(\theta)+r^2\right)^2+\left(\frac{1}{8}-\frac{1}{\sqrt{2}}r\cos(\theta)+r^2\right)^2. \end{aligned} \tag{3.19}$$

In the coordinates $(x, y)$,

$$r^2\cos(2\theta) = x^2 - y^2. \tag{3.20}$$

Figure 4 represents graphically Identity (3.19).

### 3.4. *Example: The Product of a Bessel Function and Cosine*

Consider the product $r^n\sum_{m=0}^{\infty}(((-1)^m/m!(n+m)!)r^{2m})\cos n\theta$.

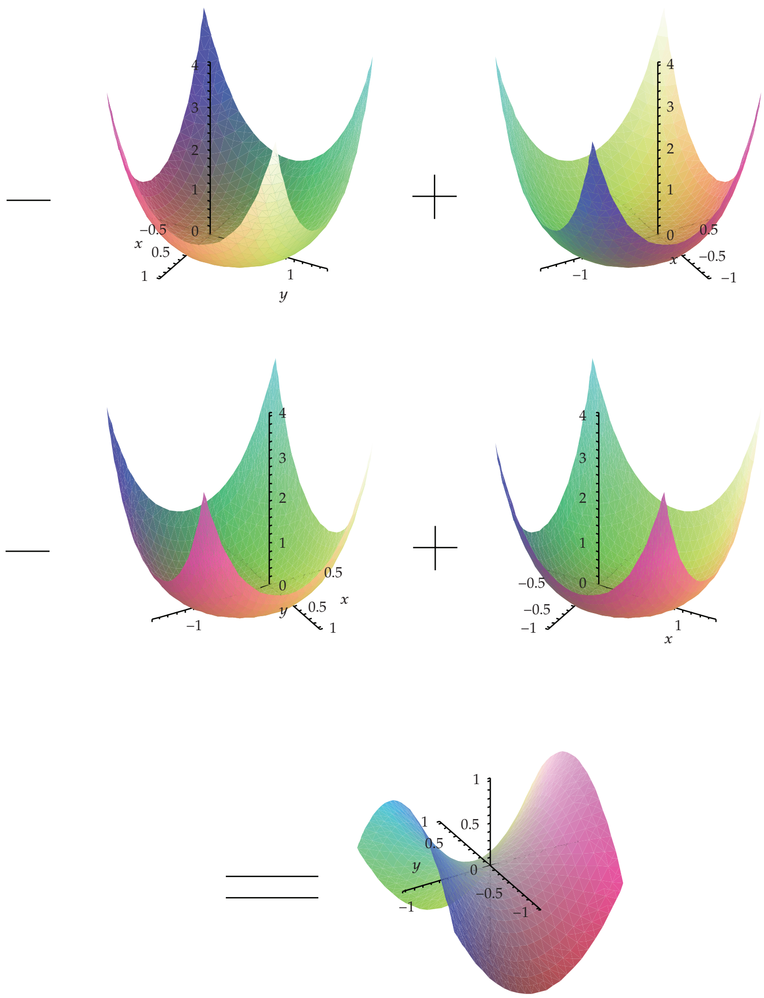


**Figure** 4

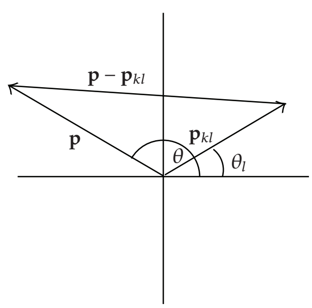



**Figure 5:** Representation of the difference $\mathbf{p} - \mathbf{p}_{kl}$.

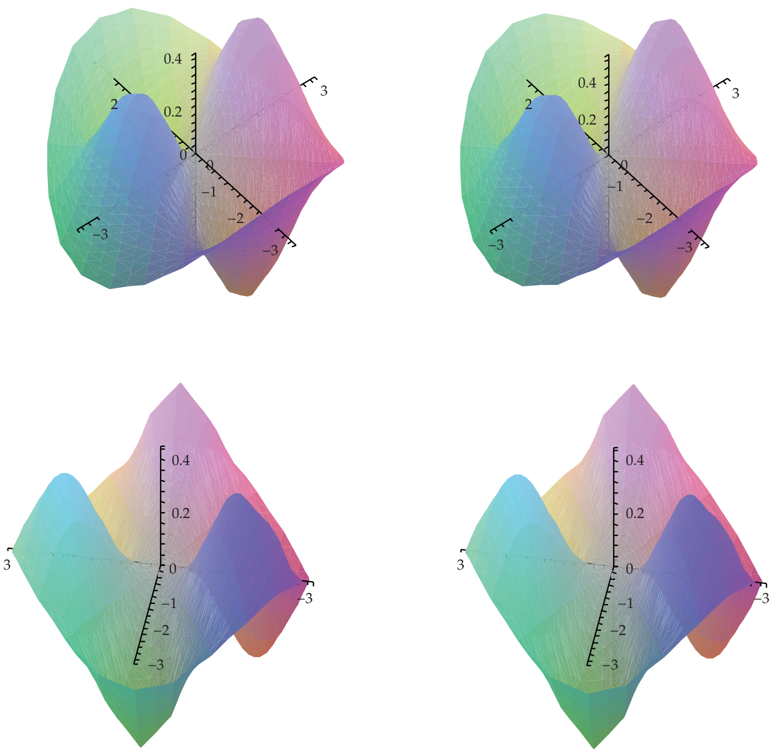



**Figure 6:** Graphics of both sides of formula (3.22), $n = 2, 3$. In the series on the right, we add up to $m = 6$.

We will denote

$$a_{klm} = \frac{1}{4}(-1)^{n+k+m}(1+\delta(0,n))p_l{}^{-n}n!((m+n)!)^{-2}\prod_{i\neq l}\frac{p_i{}^2}{p_i{}^2 - p_l{}^2}. \tag{3.21}$$

By formula (3.15),

$$r^n\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(n+m)!}r^{2m}\cos n\theta = \sum_{k=1}^{2n}\sum_{m=0}^{\infty}\sum_{l=0}^{m}a_{klm}\left(p_l^2 - 2p_l r\cos(\theta-\theta_k) + r^2\right)^{m+n}. \tag{3.22}$$

We will denote

$$\begin{aligned} &\mathbf{p} : \text{point with coordinates } (r\cos\theta, r\sin\theta),\\ &\mathbf{p}_{kl} : \text{point with coordinates } (p_l\cos\theta_k, p_l\sin\theta_k),\\ &|\mathbf{p}| = r, y \\ &s_{kl}(x) = x^n\sum_{m=l}^{\infty}a_{klm}x^m. \end{aligned} \tag{3.23}$$

Rewrite formula (3.22) as

$$r^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} r^{2m} \cos n\theta = \sum_{k=1}^{2n} \sum_{l=0}^{\infty} s_{kl} \left( |\mathbf{p} - \mathbf{p}_{kl}|^2 \right). \tag{3.24}$$

# Appendix

## A. Two Lemmas

We give two formulas for changing the order of the sums in double series.

**Lemma A.1.** *Consider the series* $\sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{i,i-j}$*. Assume that one can exchange the order of addition with respect to the indices* $i$ *and* $j$*. Then, the following formula holds:*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{ij} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{i+j,i}. \tag{A.1}$$

**Lemma A.2.** *Consider the series*

(i) $\sum_{i=0}^{\infty} \sum_{k=0}^{i/2} \sum_{j=0}^{i-2k} a_{ijk} y$,

(ii) $\sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \sum_{j=k}^{i} a_{i+k,j-k,k}$.

*Assume that one can exchange the order of addition with respect to the indices* $i$ *and* $k$*. Then, the following formula holds:*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{(i-j)/2} a_{ijk} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} a_{i+k,j-k,k}. \tag{A.2}$$

# References


[1] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, San Diego, Calif, USA, 6th edition, 2000, Translated from the Russian, Translation edited and with a preface by A. Jeffrey and D. Zwillinge.

[2] http://en.wikipedia.org/wiki/List_of_trigonometric_identities.