Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 734126, 25 pages
doi:10.1155/2010/734126

Research Article

Convergence Theorem Based on a New Hybrid Projection Method for Finding a Common Solution of Generalized Equilibrium and Variational Inequality Problems in Banach Spaces

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Received 26 October 2009; Accepted 3 January 2010

Academic Editor: Simeon Reich

The purpose of this paper is to introduce a new hybrid projection method for finding a common element of the set of common fixed points of two relatively quasi-nonexpansive mappings, the set of the variational inequality for an α-inverse-strongly monotone, and the set of solutions of the generalized equilibrium problem in the framework of a real Banach space. We obtain a strong convergence theorem for the sequences generated by this process in a 2-uniformly convex and uniformly smooth Banach space. Base on this result, we also get some new and interesting results. The results in this paper generalize, extend, and unify some well-known strong convergence results in the literature.

1. Introduction

Let E be a real Banach space, E^* the dual space of E. A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in E$, with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. Let E be a Banach space. The *modulus of convexity* of E is the function

$\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x,y \in E,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}. \tag{1.2}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$; see [1, 2] for more details. Observe that every p-uniform convex is uniformly convex. One should note that no Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is 2-uniformly convex, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1}\right\} \tag{1.3}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a real Banach space with norm $\|\cdot\|$ and E^* denotes the dual space of E. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \forall x,y \in E. \tag{1.4}$$

Observe that, in a Hilbert space H, (1.4) reduces to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$. The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x); \tag{1.5}$$

existence and uniqueness of the mapping Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, e.g., [3–7]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \quad \forall x,y \in E. \tag{1.6}$$

Remark 1.1. If E is a reflexive, strictly convex, and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (2.13), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [5, 7] for more details.

Next, we give some examples which are closed relatively quasi-nonexpansive (see [8]).

Example 1.2. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then, Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Let E be a real Banach space and let C be a nonempty closed and convex subset of E, and $A : C \to E^*$ be a mapping. *The classical variational inequality problem* for a mapping A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions of (1.4) is denoted by $\mathrm{VI}(A, C)$. Recall that A is called

(i) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C, \tag{1.8}$$

(ii) an *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C. \tag{1.9}$$

Such a problem is connected with the convex minimization problem, the complementary problem, and the problem of finding a point $x^* \in E$ satisfying $Ax^* = 0$.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. *The equilibrium problem* (for short, EP) is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.10}$$

The set of solutions of (1.10) is denoted by $\mathrm{EP}(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $x^* \in \mathrm{EP}(f)$ if and only if $\langle Tx^*, y - x^* \rangle \geq 0$ for all $y \in C$; that is, x^* is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.10). Some methods have been proposed to solve the equilibrium problem; see, for instance, [9–11].

Let C be a closed convex subset of E; a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.11}$$

A point $x \in C$ is a *fixed point* of T provided that $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a point p in C is said to be an *asymptotic fixed point* of T [12] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F(T)}$. A mapping T from C into itself is said to be *relatively nonexpansive* [13–15] if $\widehat{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [16–18]. T is said to be *ϕ-nonexpansive*, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be *relatively quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(T)$. A mapping T in a Banach space E is *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Remark 1.3. The class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [16–19] which requires the strong restriction $F(T) = \widehat{F(T)}$.

In Hilbert spaces H, Iiduka et al. [20] proved that the sequence $\{x_n\}$ defined by: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n A x_n), \tag{1.12}$$

where P_C is the metric projection of H onto C, and $\{\lambda_n\}$ is a sequence of positive real numbers, and converges weakly to some element of $\mathrm{VI}(A, C)$.

It is well know that if C is a nonempty closed and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [4] recently introduced a generalized projection mapping Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

In 2008, Iiduka and Takahashi [21] introduced the following iterative scheme for finding a solution of the variational inequality problem for inverse-strongly monotone A in a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n) \tag{1.13}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^*, and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.13) converges weakly to some element of $\mathrm{VI}(A, C)$.

Matsushita and Takahashi [22] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \tag{1.14}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping, and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2005, Matsushita and Takahashi [19] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily,}\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n),\\
&C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\},\\
&Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\},\\
&x_{n+1} = \prod_{C_n \cap Q_n} x_0.
\end{aligned} \tag{1.15}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Recently, Takahashi and Zembayashi [23] proposed the following modification of iteration (1.15) for a relatively nonexpansive mapping:

$$\begin{aligned}
&x_0 = x \in C,\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n),\\
&u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \ge 0, \quad \forall y \in C,\\
&H_n = \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\},\\
&W_n = \{z \in C : \langle x_n - z, J x - J x_n \rangle \ge 0\},\\
&x_{n+1} = \prod_{H_n \cap W_n} x,
\end{aligned} \tag{1.16}$$

where J is the duality mapping on E. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap \mathrm{EP}(f)} x$, where $\Pi_{F(S)\cap \mathrm{EP}(f)}$ is the generalized projection of E onto $F(S) \cap \mathrm{EP}(f)$. Also, Takahashi and Zembayashi [24] proved the following iteration for a relatively nonexpansive mapping:

$$\begin{aligned}
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n),\\
&u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \ge 0, \quad \forall y \in C,\\
&C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\},\\
&x_{n+1} = \prod_{C_{n+1}} x,
\end{aligned} \tag{1.17}$$

where J is the duality mapping on E. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap \mathrm{EP}(f)}x$, where $\Pi_{F(S)\cap \mathrm{EP}(f)}$ is the generalized projection of E onto $F(S)\cap \mathrm{EP}(f)$. Qin and Su [25] proved the following iteration for relatively nonexpansive mappings T in a Banach space E:

$$\begin{gathered}
x_0 \in C, \quad \text{chosen arbitrarily}, \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\
z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T x_n), \\
C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\
Q_n = \{v \in C : \langle J x_0 - J x_n, x_n - v\rangle \ge 0\}, \\
x_{n+1} = \prod_{C_n \cap Q_n} x_0,
\end{gathered} \tag{1.18}$$

the sequence $\{x_n\}$ generated by (1.18) converges strongly to $\Pi_{F(T)}x_0$.

In 2009, Wei et al. [26] proved the following iteration for two relatively nonexpansive mappings in a Banach space E:

$$\begin{gathered}
x_0 \in C, \\
J z_n = \alpha_n J x_n + (1-\alpha_n) J T x_n, \\
J u_n = (\beta_n J x_n + (1-\beta_n) J S z_n), \\
H_n = \{v \in C : \phi(v, u_n) \le \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \le \phi(v, x_n)\}, \\
W_n = \{z \in C : \langle z - x_n, J x_0 - J x_n\rangle \le 0\}, \\
x_{n+1} = Q_{H_n \cap W_n} x_0;
\end{gathered} \tag{1.19}$$

if $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1)$ such that $\alpha_n \le 1-\delta_1$ and $\beta_n \le 1-\delta_2$ for some $\delta_1, \delta_2 \in (0,1)$, then $\{x_n\}$ generated by (1.19) converges strongly to a point $Q_{F(T)\cap F(S)}x_0$, where the mapping Q_C of E onto C is the generalized projection. Very recently, Cholamjiak [27] proved the following iteration:

$$\begin{gathered}
z_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\
y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S z_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \ge 0, \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0,
\end{gathered} \tag{1.20}$$

where J is the duality mapping on E. Assume that α_n, β_n, and γ_n are sequences in $[0,1]$. Then $\{x_n\}$ converges strongly to $q = \Pi_F x_0$, where $F := F(T)\cap F(S)\cap \mathrm{EP}(f)\cap \mathrm{VI}(A, C)$.

Motivated and inspired by Iiduka and Takahashi [21], Takahashi and Zembayashi [23, 24], Wei et al. [26], Cholamjiak [27], and Kumam and Wattanawitoon [28], we introduce a new hybrid projection iterative scheme which is difference from the algorithm (1.20) of Cholamjiak in [27, Theorem 3.1] for two relatively quasi-nonexpansive mappings in a Banach space. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$\begin{gathered}
w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\
z_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) JTw_n\right), \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JSz_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \left\{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\right\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{1.21}$$

where J is the duality mapping on E. Then, we prove that under certain appropriate conditions on the parameters, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.21) converge strongly to $\Pi_{F(S)\cap F(T)\cap \mathrm{EP}(f)\cap \mathrm{VI}(A,C)}$.

The results presented in this paper improve and extend the corresponding results announced by Iiduka and Takahashi [21], Wei et al. [26], Kumam and Wattanawitoon [28], and many other authors in the literature.

2. Preliminaries

We also need the following lemmas for the proof of our main results.

Lemma 2.1 (Beauzamy [29] and Xu [30]). *If E is a 2-uniformly convex Banach space, then, for all $x, y \in E$ we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \tag{2.1}$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $1/c$ in the Lemma is called the *p-uniformly convex constant* of E.

Lemma 2.2 (Beauzamy [29] and Zălinescu [31]). *If E is a p-uniformly convex Banach space and p is a given real number with $p \geq 2$, then for all $x, y \in E$, $J_x \in J_p(x)$, and $J_y \in J_p(y)$,*

$$\langle x - y, Jx - Jy\rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p, \tag{2.2}$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Lemma 2.3 (Kamimura and Takahashi [6]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.4 (Alber [4]). *Let C be a nonempty closed and convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

Lemma 2.5 (Alber [4]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed and convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.4}$$

Lemma 2.6 (Qin et al. [8]). *Let E be a uniformly convex and smooth Banach space, let C be a closed and convex subset of E, and let T be a closed relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed and convex subset of C.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y); \tag{2.5}$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Lemma 2.7 (Blum and Oettli [9]). *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4), *and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{2.6}$$

Lemma 2.8 (Combettes and Hirstoaga [10]). *Let C be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}, \tag{2.7}$$

for all $x \in C$*. Then the following holds:*

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive-type mapping, for all* $x, y \in E$*,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \le \langle T_r x - T_r y, Jx - Jy \rangle; \tag{2.8}$$

(3) $F(T_r) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ *is closed and convex.*

Lemma 2.9 (Takahashi and Zembayashi [24]). *Let* C *be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space* E*, let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4)*, and let* $r > 0$*. Then, for* $x \in E$ *and* $q \in F(T_r)$*,*

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x). \tag{2.9}$$

Let E be a reflexive, strictly convex, and smooth Banach space and J the duality mapping from E into E^*. Then J^{-1} is also single value, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [4]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.10}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.10 (Alber [4]). *Let* E *be a reflexive, strictly convex, and smooth Banach space and let* V *be as in* (2.10) *. Then*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \le V\left(x, x^* + y^*\right) \tag{2.11}$$

for all $x \in E$ *and* $x^*, y^* \in E^*$*.*

Let A be an inverse-strongly monotone of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0,1]$ into E^*, defined by $F(t) = A(tx + (1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$*the normal cone* for C at a point $v \in C$; that is,

$$N_C(v) = \left\{x^* \in E^* : \langle v - y, x^* \rangle \ge 0,\ \forall y \in C\right\}. \tag{2.12}$$

Theorem 2.11 (Rockafellar [32]). *Let* C *be a nonempty, closed and convex subset of a Banach space* E*, and* A *a monotone, hemicontinuous mapping of* C *into* E^**. Let* $T \subset E \times E^*$ *be a mapping defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2.13}$$

Then T *is maximal monotone and* $T^{-1}0 = \mathrm{VI}(A, C)$*.*

3. Main Results

In this section, we establish a new hybrid iterative scheme for finding a common element of the set of solutions of an equilibrium problems, the common fixed point set of two relatively quasi-nonexpansive mappings, and the solution set of variational inequalities for α-inverse strongly monotone in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A,C) \neq \emptyset$. Let $T, S : C \to C$ be closed relatively quasi-nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered}
w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\
z_n = J^{-1}\left(\beta_n J x_n + \left(1-\beta_n\right) JTw_n\right), \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JSz_n), \\
u_n \in C \text{ such that } f\left(u_n, y\right) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \left\{z \in C_n : \phi(z,u_n) \leq \alpha_n \phi(z,x_n) + (1-\alpha_n)\phi(z,z_n) \leq \phi(z,x_n)\right\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{3.1}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1-\delta_1$ and $\beta_n \leq 1-\delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

Proof. We have several steps to prove this theorem as follows:

Step 1. We show that C_{n+1} is closed and convex.

Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know that

$$\phi(z,u_n) \leq \phi(z,x_n) \tag{3.2}$$

is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2. \tag{3.3}$$

So, C_{n+1} is closed and convex. Then, by induction, C_n is closed and convex for all $n \geq 1$.

Step 2. We show that $\{x_n\}$ is well defined.

Put $u_n = T_{r_n} y_n$ for all $n \geq 0$. On the other hand, from Lemma 2.8 one has T_{r_n} is relatively quasi-nonexpansive mappings and $\Omega \subset C_1 = C$. Supposing $\Omega \subset C_k$ for $k \in \mathbb{N}$, by the convexity of $\|\cdot\|^2$, for each $q \in \Omega \subset C_k$, we have

$$\begin{aligned}
\phi(q, u_k) &= \phi(q, T_{r_k} y_k) \\
&\leq \phi(q, y_k) \\
&= \phi\Big(q, J^{-1}(\alpha_k J x_k + (1-\alpha_k) J S z_k)\Big) \\
&= \|q\|^2 - 2\langle q, \alpha_k J x_k + (1-\alpha_k) J S z_k \rangle + \|\alpha_k J x_k + (1-\alpha_k) J S z_k\|^2 \\
&\leq \|q\|^2 - 2\alpha_k \langle q, J x_k \rangle - 2(1-\alpha_k)\langle q, J S z_k \rangle + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|S z_k\|^2 \\
&= \alpha_k \phi(q, x_k) + (1-\alpha_k)\phi(q, S z_k) \\
&\leq \alpha_k \phi(q, x_k) + (1-\alpha_k)\phi(q, z_k),
\end{aligned} \tag{3.4}$$

and so

$$\begin{aligned}
\phi(q, z_k) &= \phi\Big(q, J^{-1}(\beta_k J x_k + (1-\beta_k) J T w_k)\Big) \\
&= \|q\|^2 - 2\langle q, \beta_k J x_k + (1-\beta_k) J T w_k \rangle + \big\|\beta_k J x_k + (1-\beta_k) J T w_k\big\|^2 \\
&\leq \|q\|^2 - 2\beta_k \langle q, J x_k \rangle - 2(1-\beta_k)\langle q, J T w_k \rangle + \beta_k \|J x_k\|^2 + (1-\beta_k)\|J T w_k\|^2 \\
&= \beta_k \phi(q, x_k) + (1-\beta_k)\phi(q, T w_k) \\
&\leq \beta_k \phi(q, x_k) + (1-\beta_k)\phi(q, w_k).
\end{aligned} \tag{3.5}$$

For all $q \in \Omega$, we know from Lemma 2.10, that

$$\begin{aligned}
\phi(q, w_k) &= \phi\Big(q, \Pi_C J^{-1}(J x_k - \lambda_k A x_k)\Big) \\
&\leq \phi\Big(q, J^{-1}(J x_k - \lambda_k A x_k)\Big) \\
&= V(q, J x_k - \lambda_k A x_k) \\
&\leq V(q, (J x_k - \lambda_k A x_k) + \lambda_k A x_k) - 2\Big\langle J^{-1}(J x_k - \lambda_k A x_k) - q, \lambda_k A x_k \Big\rangle \\
&= V(q, J x_k) - 2\lambda_k \Big\langle J^{-1}(J x_k - \lambda_k A x_k) - q, A x_k \Big\rangle \\
&= \phi(q, x_k) - 2\lambda_k \langle x_k - q, A x_k \rangle + 2\Big\langle J^{-1}(J x_k - \lambda_k A x_k) - x_k, -\lambda_k A x_k \Big\rangle.
\end{aligned} \tag{3.6}$$

Since $q \in \mathrm{VI}(A, C)$ and from A being an α-inverse-strongly monotone, we get

$$\begin{aligned} -2\lambda_k \langle x_k - q, Ax_k \rangle &= -2\lambda_k \langle x_k - q, Ax_k - Aq \rangle - 2\lambda_k \langle x_k - q, Aq \rangle \\ &\le -2\lambda_k \langle x_k - q, Ax_k - Aq \rangle \\ &= -2\alpha\lambda_k \| Ax_k - Aq \|^2. \end{aligned} \tag{3.7}$$

From Lemma 2.1 and A being an α-inverse-strongly monotone, we obtain

$$\begin{aligned} 2\left\langle J^{-1}(Jx_k - \lambda_k Ax_k) - x_k, -\lambda_k Ax_k \right\rangle &= 2\left\langle J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k), -\lambda_k Ax_k \right\rangle \\ &\le 2\left\| J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k) \right\| \|\lambda_k Ax_k\| \\ &\le \frac{4}{c^2} \left\| JJ^{-1}(Jx_k - \lambda_k Ax_k) - JJ^{-1}(Jx_k) \right\| \|\lambda_k Ax_k\| \\ &= \frac{4}{c^2} \|Jx_k - \lambda_k Ax_k - Jx_k\| \|\lambda_k Ax_k\| \\ &= \frac{4}{c^2} \|\lambda_k Ax_k\|^2 \\ &= \frac{4}{c^2} \lambda_k^2 \|Ax_k\|^2 \\ &\le \frac{4}{c^2} \lambda_k^2 \|Ax_k - Aq\|^2. \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we have

$$\begin{aligned} \phi(q, w_k) &\le \phi(q, x_k) - 2\alpha\lambda_k \|Ax_k - Aq\|^2 + \frac{4}{c^2}\lambda_k^2 \|Ax_k - Aq\|^2 \\ &= \phi(q, x_k) + 2\lambda_k \left(\frac{2}{c^2}\lambda_k - \alpha \right) \|Ax_k - Aq\|^2 \\ &\le \phi(q, x_k). \end{aligned} \tag{3.9}$$

Replacing (3.9) into (3.5), we get

$$\phi(q, z_k) \le \phi(q, x_k). \tag{3.10}$$

Substituting (3.10) into (3.4), we also have

$$\begin{aligned} \phi(q, u_k) &\le \alpha_k \phi(q, x_k) + (1 - \alpha_k)\phi(q, x_k), \\ &= \phi(q, x_k). \end{aligned} \tag{3.11}$$

This shows that $q \in C_{k+1}$ and hence, $\Omega \subset C_{k+1}$. Hence, $\Omega \subset C_n$ for all $n \ge 1$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We show that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists and $\{x_n\}$ is bounded.

From $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1 , \tag{3.12}$$

and from Lemma 2.5, we have

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n}(x_0), x_0) \\ &\leq \phi(p, x_0) - \phi(p, x_n) \\ &\leq \phi(p, x_0), \quad \forall p \in \Omega.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), then $\{\phi(x_n, x_0)\}$ are nondecreasing and bounded. So, we obtain that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. In particular, by (1.6), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies that $\{x_n\}$ is also bounded.

Step 4. We show that $\{x_n\}$ is a Cauchy sequence in C.

Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$, for $m > n$, by Lemma 2.5, we have

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.14}$$

Taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. We have $\lim_{n\to\infty}\phi(x_{n+1}, x_0) = 0$. From Lemma 2.3, we get $\lim_{n\to\infty}\|x_{n+1} - x_0\| = 0$. Thus $\{x_n\}$ is a Cauchy sequence.

Step 5. We cliam that $\|Ju_n - Jx_n\| \to 0$, as $n \to \infty$.

By the completeness of E, the closedness of C and $\{x_n\}$ is a Cauchy sequence (from Step 4); we can assume that there exists $p \in C$ such that $\{x_n\} \to p$ as $n \to \infty$.

By definition of $\Pi_{C_n} x_0$, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.15}$$

Since $\lim_{n\to\infty}\phi(x_n, x_0)$ exists, we get

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{3.16}$$

It follow form Lemma 2.3, that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and from the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n), \quad \forall n \ge 1 \tag{3.18}$$

and so

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.19}$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.20}$$

By using the triangle inequality, we obtain

$$\begin{aligned} \|u_n - x_n\| &= \|u_n - x_{n+1} + x_{n+1} - x_n\| \\ &\le \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \end{aligned} \tag{3.21}$$

By (3.17), (3.20), we get

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{3.23}$$

Step 6. Show that $x_n \to p \in EP(f)$.

Applying (3.4) and (3.11), we get $\phi(p, y_n) \le \phi(p, x_n)$. From Lemma 2.9 and $u_n = T_{r_n} y_n$, we observe that

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\le \phi(p, y_n) - \phi(p, T_{r_n} y_n) \\ &\le \phi(p, x_n) - \phi(p, T_{r_n} y_n) \\ &= \phi(p, x_n) - \phi(p, u_n) \\ &= \|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2 - \left(\|p\|^2 - 2\langle p, Ju_n\rangle + \|u_n\|^2\right) \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|(\|x_n + u_n\|) + 2\|p\|\|Jx_n - Ju_n\|. \end{aligned} \tag{3.24}$$

From (3.22), (3.23) and Lemma 2.3, we get

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.25}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty}\|Ju_n - Jy_n\| = 0. \tag{3.26}$$

From $r_n > 0$, we have $\|Ju_n - Jy_n\|/r_n \to 0$ as $n \to \infty$ and

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C. \tag{3.27}$$

By (A2), that

$$\begin{aligned}\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -f(u_n, y) \\ &\geq f(y, u_n), \quad \forall y \in C\end{aligned} \tag{3.28}$$

and $u_n \to p$, we get $f(y,p) \leq 0$ for all $y \in C$. For $0 < t < 1$, define $y_t = ty + (1-t)p$. Then $y_t \in C$ which implies that $f(y_t, p) \leq 0$. From (A1), we obtain that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y). \tag{3.29}$$

Thus $f(y_t, y) \geq 0$. From (A3), we have $f(p, y) \geq 0$ for all $y \in C$. Hence $p \in \mathrm{EP}(f)$.

Step 7. We show that $x_n \to p \in F(T) \cap F(S)$.

From definition of C_n, we have

$$\alpha_n\phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n) \Longleftrightarrow \phi(z, z_n) \leq \phi(z, x_n). \tag{3.30}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n). \tag{3.31}$$

It follows from (3.16) that

$$\lim_{n\to\infty}\phi(x_{n+1}, z_n) = 0, \tag{3.32}$$

again from Lemma 2.3, we get

$$\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0. \tag{3.33}$$

By using the triangle inequality, we get

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \tag{3.34}$$

Again by (3.17) and (3.33), we also have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.35}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = 0. \tag{3.36}$$

Since

$$\|y_n - z_n\| \le \|y_n - u_n\| + \|u_n - x_n\| + \|x_n - z_n\|, \tag{3.37}$$

from (3.22), (3.25), and (3.35), we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.38}$$

Since J is uniformly norm-to-norm continuous, we also have

$$\lim_{n\to\infty} \|Jy_n - Jz_n\| = 0. \tag{3.39}$$

From (3.1), we get

$$\begin{aligned} \|Jy_n - Jz_n\| &= \|\alpha_n(Jx_n - Jz_n) + (1-\alpha_n)(JSz_n - Jz_n)\| \\ &= \|(1-\alpha_n)(JSz_n - Jz_n) - \alpha_n(Jz_n - Jx_n)\| \\ &\ge (1-\alpha_n)\|JSz_n - Jz_n\| - \alpha_n\|Jz_n - Jx_n\|; \end{aligned} \tag{3.40}$$

it follows that

$$(1-\alpha_n)\|JSz_n - Jz_n\| \le \|Jy_n - Jz_n\| + \alpha_n\|Jz_n - Jx_n\|, \tag{3.41}$$

and hence

$$\|JSz_n - Jz_n\| \le \frac{1}{1-\alpha_n}\left(\|Jy_n - Jz_n\| + \alpha_n\|Jz_n - Jx_n\|\right). \tag{3.42}$$

Since $\alpha_n \leq 1 - \delta_1$ for some $\delta_1 \in (0,1)$, (3.36), and (3.39), one has $\lim_{n\to\infty}\|JSz_n - Jz_n\| = 0$. Since J^{-1} is uniformly norm-to-norm continuous, we get

$$\lim_{n\to\infty}\|Sz_n - z_n\| = 0. \tag{3.43}$$

Since

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|Sx_n - Sz_n\| + \|Sz_n - z_n\| + \|z_n - x_n\| \\ &\leq \|x_n - z_n\| + \|Sz_n - z_n\| + \|z_n - x_n\|,\end{aligned} \tag{3.44}$$

from (3.35) and (3.43), we obtain

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{3.45}$$

Since S is closed and $x_n \to p$, we have $p \in F(S)$.

On the other hand, we note that

$$\begin{aligned}\phi(q, x_n) - \phi(q, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle \\ &\leq \|x_n - u_n\|(\|x_n + u_n\|) + 2\|q\|\|Jx_n - Ju_n\|.\end{aligned} \tag{3.46}$$

It follows from $\|x_n - u_n\| \to 0$ and $\|Jx_n - Ju_n\| \to 0$, that

$$\phi(q, x_n) - \phi(q, u_n) \longrightarrow 0. \tag{3.47}$$

Furthermore, from (3.4) and (3.5),

$$\begin{aligned}\phi(q, u_n) &\leq \phi(q, y_n) \\ &\leq \alpha_n\phi(q, x_n) + (1-\alpha_n)\phi(q, z_n) \\ &\leq \alpha_n\phi(q, x_n) + (1-\alpha_n)\left[\beta_n\phi(q, x_n) + (1-\beta_n)\phi(q, w_n)\right] \\ &= \alpha_n\phi(q, x_n) + (1-\alpha_n)\beta_n\phi(q, x_n) + (1-\alpha_n)(1-\beta_n)\phi(q, w_n) \\ &\leq \alpha_n\phi(q, x_n) + (1-\alpha_n)\beta_n\phi(q, x_n) + (1-\alpha_n)(1-\beta_n) \\ &\quad \times \left[\phi(q, x_n) - 2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Aq\|^2\right] \\ &= \alpha_n\phi(q, x_n) + (1-\alpha_n)\beta_n\phi(q, x_n) + (1-\alpha_n)(1-\beta_n)\phi(q, x_n) \\ &\quad - (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Aq\|^2 \\ &= \phi(q, x_n) - (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Aq\|^2,\end{aligned} \tag{3.48}$$

and hence

$$\begin{aligned}\delta_1\delta_2 2a\left(\alpha-\frac{2a}{c^2}\right)\|Ax_n-Aq\|^2 &\le (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha-\frac{2}{c^2}\lambda_n\right)\|Ax_n-Aq\|^2\\ &\le \phi(q,x_n)-\phi(q,u_n).\end{aligned}\tag{3.49}$$

From (3.47) and (3.49), we have

$$\|Ax_n-Aq\|\longrightarrow 0.\tag{3.50}$$

From Lemma 2.5, Lemma 2.10, and (3.8), we compute

$$\begin{aligned}\phi(x_n,w_n)&=\phi\left(x_n,\Pi_C J^{-1}(Jx_n-\lambda_n Ax_n)\right)\\ &\le \phi\left(x_n,J^{-1}(Jx_n-\lambda_n Ax_n)\right)\\ &=V(x_n,Jx_n-\lambda_n Ax_n)\\ &\le V(x_n,(Jx_n-\lambda_n Ax_n)+\lambda_n Ax_n)-2\left\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,\lambda_n Ax_n\right\rangle\\ &=\phi(x_n,x_n)+2\left\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,-\lambda_n Ax_n\right\rangle\\ &=2\left\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,-\lambda_n Ax_n\right\rangle\\ &\le \frac{4\lambda_n^2}{c^2}\|Ax_n-Aq\|^2\\ &\le \frac{4b^2}{c^2}\|Ax_n-Aq\|^2.\end{aligned}\tag{3.51}$$

Applying Lemmas 2.3 and (3.50), we obtain that

$$\|x_n-w_n\|\longrightarrow 0.\tag{3.52}$$

Again since J is uniformly norm-to-norm continuous on bounded set, we have

$$\|Jx_n-Jw_n\|\longrightarrow 0.\tag{3.53}$$

Since

$$\|z_n-w_n\|\le\|z_n-x_n\|+\|x_n-w_n\|,\tag{3.54}$$

by (3.35) and (3.52), we have

$$\lim_{n\to\infty}\|z_n-w_n\|=0,\tag{3.55}$$

and hence

$$\lim_{n\to\infty} \|Jz_n - Jw_n\| = 0. \tag{3.56}$$

From (3.1) we obtain that

$$\begin{aligned} \|Jz_n - Jw_n\| &= \left\|\beta_n Jx_n + (1-\beta_n) JTw_n - Jw_n\right\| \\ &\geq (1-\beta_n)\|JTw_n - Jw_n\| - \beta_n\|Jw_n - Jx_n\|, \end{aligned} \tag{3.57}$$

and hence

$$(1-\beta_n)\|JTw_n - Jw_n\| \leq \|Jz_n - Jw_n\| + \beta_n\|Jw_n - Jx_n\|, \tag{3.58}$$

so

$$\|JTw_n - Jw_n\| \leq \frac{1}{1-\beta_n}\|Jz_n - Jw_n\| + \beta_n\|Jw_n - Jx_n\|. \tag{3.59}$$

By (3.53), (3.56) and condition $\beta_n \leq 1 - \delta_2$ for some $\delta_2 \in (0,1)$, we obtain

$$\|JTw_n - Jw_n\| \longrightarrow 0. \tag{3.60}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded set, we obtain

$$\|Tw_n - w_n\| \longrightarrow 0. \tag{3.61}$$

Since $x_n \to w_n$, then $\|Tx_n - x_n\| \to 0$. Thus by the closedness of T and $x_n \to p$, we get $p \in F(T)$. Hence $p \in F(T) \cap F(S)$.

Step 8. We show that $x_n \to p \in \mathrm{VI}(A, C)$.

Define $T \subset E \times E^*$ by Theorem 2.11; T is maximal monotone and $T^{-1}0 = \mathrm{VI}(A, C)$. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.

From $w_n \in C$, we have

$$\langle v - w_n, w - Av\rangle \geq 0. \tag{3.62}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$, then by Lemma 2.4, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n)\rangle \geq 0, \tag{3.63}$$

and hence

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.64}$$

It follows from (3.62) and (3.64), that

$$\begin{aligned}\langle v-w_n, w\rangle &\geq \langle v-w_n, Av\rangle \\ &\geq \langle v-w_n, Av\rangle + \left\langle v-w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n\right\rangle \\ &= \langle v-w_n, Av - Ax_n\rangle + \left\langle v-w_n, \frac{Jx_n - Jw_n}{\lambda_n}\right\rangle \\ &= \langle v-w_n, Av - Aw_n\rangle + \langle v-w_n, Aw_n - Ax_n\rangle + \left\langle v-w_n, \frac{Jx_n - Jw_n}{\lambda_n}\right\rangle \\ &\geq -\|v-w_n\|\frac{\|w_n - x_n\|}{\alpha} - \|v-w_n\|\frac{\|Jx_n - Jw_n\|}{a} \\ &\geq -M\left(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{a}\right).\end{aligned} \tag{3.65}$$

Where $M = \sup_{n\geq 1}\|v-w_n\|$. Taking the limit as $n \to \infty$ and (3.53), we obtain $\langle v-p, w\rangle \geq 0$. By the maximality of T, we have $p \in T^{-1}0$; that is, $p \in \mathrm{VI}(A,C)$.

Step 9. We show that $p = \Pi_{\Omega}x_0$.

From $x_n = \Pi_{C_n}x_0$, we have $\langle Jx_0 - Jx_n, x_n - z\rangle \geq 0, \forall z \in C_n$. Since $\Omega \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - y\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.66}$$

By taking limit $n \to \infty$, we obtain that

$$\langle Jx_0 - Jp, p - y\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.67}$$

By Lemma 2.4, we can conclude that $p = \Pi_{\Omega}x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

Setting $S \equiv T$ in Theorem 3.1., so, we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A,C) \neq \emptyset$. Let $T : C \to C$ be closed relatively quasi-nonexpansive mappings such that*

$\Omega := F(T) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A,C) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$, *define a sequence* $\{x_n\}$ *as follows:*

$$\begin{aligned}
w_n &= \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\
z_n &= J^{-1}(\beta_n J x_n + (1-\beta_n) J T w_n), \\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\
u_n &\in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\
x_{n+1} &= \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{aligned} \tag{3.68}$$

where J *is the duality mapping on* E. *Assume that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\alpha_n \leq 1 - \delta_1$ *and* $\beta_n \leq 1 - \delta_2$, *for some* $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$, *and* $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* E. *Then* $\{x_n\}$ *converges strongly to* $p \in \Omega$, *where* $p = \Pi_\Omega x_0$.

If $A \equiv 0$ in Theorem 3.1, then we obtain the following corollary.

Corollary 3.3. *Let* C *be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space* E. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4). *Let* $T, S : C \to C$ *is closed relatively quasi-nonexpansive mappings such that* $\Omega := F(T) \cap F(S) \cap \mathrm{EP}(f) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$, *define a sequence* $\{x_n\}$ *as follows:*

$$\begin{aligned}
z_n &= J^{-1}(\beta_n J x_n + (1-\beta_n) J T w_n), \\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S z_n), \\
u_n &\in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\
x_{n+1} &= \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{aligned} \tag{3.69}$$

where J *is the duality mapping on* E. *Assume that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\alpha_n \leq 1 - \delta_1$ *and* $\beta_n \leq 1 - \delta_2$, *for some* $\delta_1, \delta_2 \in (0,1)$ *and* $\{r_n\} \subseteq (0, 2\alpha)$. *Then* $\{x_n\}$ *converges strongly to* $p \in \Omega$, *where* $p = \Pi_\Omega x_0$.

4. Application

4.1. Complementarity Problem

Let K be a nonempty, closed and convex cone E, A a mapping of K into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0,\ \forall x \in K\}. \tag{4.1}$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \langle u, Au \rangle = 0. \tag{4.2}$$

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Theorem 4.1. *Let K be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be an α-inverse-strongly monotone of E into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, for all $y \in K$ and $u \in C(K, A) \neq \emptyset$. Let $T, S : K \to K$ be closed relatively quasi-nonexpansive mappings and $\Omega := F(T) \cap F(S) \cap \mathrm{EP}(f) \cap C(K, A) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{K_1}$ and $K_1 = K$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered}
w_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n), \\
z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) JTw_n), \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSz_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in K, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1 - \alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{4.3}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1 - \delta_1$ and $\beta_n \leq 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$, and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

Proof. As in the proof of Takahashi in [7, Lemma 7.11], we get that $\mathrm{VI}(K, A) = C(K, A)$. So, we obtain the result. □

4.2. Approximation of a Zero of a Maximal Monotone Operator

Let B be a multivalued mapping from E to E^* with domain $D(B) = \{z \in E : Az \neq \emptyset\}$ and range $R(B) = \cup\{Bz : z \in D(B)\}$. A mapping B is said to be a *monotone operator* if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(B)$ and $y_i \in Ax_i, i = 1, 2$. A monotone operator B is said to be *maximal* if its graph $G(B) = \{(x, y) : y \in Ax\}$ is not property contained in the graph of any other monotone operator. We know that if B is a maximal monotone operator, then $B^{-1}(0)$ is closed and convex. Let E be a reflexive, strictly convex, and smooth Banach space, and let B be a monotone operator from E to E^*, we know that B is maximal if and only if $R(J + rB) = E^*$ for all $r > 0$. Let $J_r : E \to D(B)$be defined by $J_r = (J + rB)^{-1}J$and such a J_r is called the resolvent of B . We know that J_r is a relatively nonexpansive (closed relatively quasi-nonexpansive for example; see [8]), and $B^{-1}(0) = F(J_r)$ for all $r > 0$ (see [7, 33–35] for more details).

Theorem 4.2. *Let C be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be α-inverse-strongly monotone of E into E^* satisfying $\|Ay\| \le \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A, C) \neq \emptyset$. Let B be a maximal monotone operator of E into E^* and let J_r be a resolvent of B and a closed mapping such that $\Omega := B^{-1}(0) \cap F(S) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_r w_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSz_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{gathered} \tag{4.4}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \le 1 - \delta_1$ and $\beta_n \le 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

Proof. Since J_r is a closed relatively nonexpansive mapping and $B^{-1}0 = F(J_r)$. So, we obtain the result. □

Corollary 4.3. *Let C be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let A be α-inverse-strongly monotone of E into E^* satisfying $\|Ay\| \le \|Ay - Au\|$, for all $y \in C$ and $u \in \mathrm{VI}(A, C) \neq \emptyset$. Let B be a maximal monotone operator of E into E^* and let J_r be a resolvent of B and closed such that $\Omega := B^{-1}(0) \cap \mathrm{EP}(f) \cap \mathrm{VI}(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered} w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_r w_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JJ_r z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{gathered} \tag{4.5}$$

where J is the duality mapping on E, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1 - \delta_1$ and $\beta_n \leq 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0,1)$, $\{r_n\} \subseteq (0, 2\alpha)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \Pi_\Omega x_0$.

Acknowledgments

The authors would like to thank the referee for the valuable suggestions on the manuscript. Siwaporn Saewan would like to thank the Office of the Higher Education Commission, Thailand for supporting by grant fund under the program Strategic Scholarships for Frontier Research Network for the Join Ph.D. Program Thai Doctoral degree for this research. Moreover, Poom Kumam was supported by the Thailand Research Fund and the Commission on Higher Education (MRG5180034).

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