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Research Article
Hyers-Ulam Stability of Polynomial Equations

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We prove the Hyers-Ulam stability of the polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$. We give an affirmative answer to a problem posed by Li and Hua (2009).

1. Introduction and Preliminaries

A classical question in the theory of functional equations is that "when is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$". Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. It gave rise to the *stability theory* for functional equations. The result of Hyers was generalized by Rassias [3]. The topic of the Hyers-Ulam stability of functional equations and its applications has been studied by a number of mathematicians; see [3–40] and references therein.

Recently, Li and Hua [41] discussed and proved the Hyers-Ulam stability of the polynomial equation

$$x^n + \alpha x + \beta = 0, \tag{1.1}$$

where $x \in [-1, 1]$ and proved the following.

Theorem 1.1. *If $|\alpha| > n$, $|\beta| < |\alpha| - 1$, and $y \in [-1, 1]$ satisfy the inequality*

$$\left|y^n + \alpha y + \beta\right| \le \varepsilon, \tag{1.2}$$

then there exists a solution $v \in [-1,1]$ *of* (1.1) *such that*

$$|y - v| \leq K\varepsilon, \tag{1.3}$$

where $K > 0$ *is constant.*

They also asked an open problem whether the real polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1.4}$$

has the Hyers-Ulam stability for the case that this real polynomial equation has some solutions in $[a,b]$. The aim of this paper is to give a positive answer to this problem. First of all, we give the definition of the Hyers-Ulam stability.

Definition 1.2. One says that (1.4) has the Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0, y \in [-1,1]$, if

$$\left| a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 \right| \leq \varepsilon \tag{1.5}$$

then there exists some $z \in [-1,1]$ satisfying

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \tag{1.6}$$

such that $|y - z| \leq K\varepsilon$. One calls such K a Hyers-Ulam stability constant for (1.4). For the complex polynomial equation, $[-1,1]$ is replaced by closed unit disc

$$D = \{z \in \mathbb{C}; |z| \leq 1\}. \tag{1.7}$$

2. Main Results

The aim of this work is to investigate the Hyers-Ulam stability for (1.4).

Theorem 2.1. *If*

$$|a_0| < |a_1| - (|a_2| + |a_3| + \cdots + |a_n|), \tag{2.1}$$

$$|a_1| > 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| + n|a_n|, \tag{2.2}$$

then there exists an exact solution $v \in [-1,1]$ *of* (1.4).

Proof. If we set

$$g(x) = \frac{1}{a_1}\left(-a_0 - a_2 x^2 - a_3 x^3 - \cdots - a_{n-1} x^{n-1} - a_n x^n\right), \tag{2.3}$$

for $x \in [-1,1]$, then we have

$$\begin{aligned} |g(x)| &= \frac{1}{|a_1|}\left|-a_0 - a_2x^2 - \cdots - a_{n-1}x^{n-1} - a_nx^n\right| \\ &\le \frac{1}{|a_1|}(|a_0| + |a_2| + \cdots + |a_{n-1}| + |a_n|) \\ &\le 1 \end{aligned} \tag{2.4}$$

by (2.1).

Let $X = [-1,1]$ and $d(x,y) = |x-y|$. Then (X,d) is a complete metric space and g maps X to X. Now, we will show that g is a contraction from X to X. For any $x, y \in X$, we have

$$\begin{aligned} d(g(x),g(y)) &= \left|\frac{1}{a_1}\left(-a_0 - a_2x^2 - \cdots - a_nx^n\right) - \frac{1}{a_1}(-a_0 - \cdots - a_ny^n)\right| \\ &\le \frac{1}{|a_1|}|x-y|\left\{|a_2||x+y| + \cdots + |a_n|\left|x^{n-1} + \cdots + y^{n-1}\right|\right\} \\ &\le \frac{1}{|a_1|}|x-y|\{2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| + n|a_n|\}. \end{aligned} \tag{2.5}$$

For $x, y \in [-1,1], x \neq y$, from (2.2), we obtain

$$d(g(x),g(y)) \le \lambda d(x,y). \tag{2.6}$$

Here

$$\lambda = \frac{2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| + n|a_n|}{|a_1|} < 1. \tag{2.7}$$

Thus g is a contraction from X to X. By the Banach contraction mapping theorem, there exists a unique $v \in X$ such that

$$g(v) = v. \tag{2.8}$$

Hence (1.4) has a solution on $[-1,1]$. □

As an application of Rouche's theorem, we prove the following theorem for complex polynomial equation

$$a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = 0, \tag{2.9}$$

which is much better than the above result. In fact, we prove the following theorem.

Theorem 2.2. *If*

$$|a_0| < |a_1| - (|a_2| + |a_3| + \cdots + |a_n|), \tag{2.10}$$

then there exists an exact solution in open unit disc for (2.9).

Proof. If we set

$$g(z) = \frac{1}{a_1}\left(-a_0 - a_2 z^2 - a_3 z^3 - \cdots - a_{n-1} z^{n-1} - a_n z^n\right), \tag{2.11}$$

then we have

$$\begin{aligned} \left|g(z)\right| &= \frac{1}{|a_1|}\left|-a_0 - a_2 z^2 - \cdots - a_{n-1} z^{n-1} - a_n z^n\right| \\ &\le \frac{1}{|a_1|}(|a_0| + |a_2| + \cdots + |a_{n-1}| + |a_n|), \quad \text{for } |z| \le 1 \\ &< 1 \end{aligned} \tag{2.12}$$

by (2.10).

Since $|g(z)| < 1$ for $|z| = 1$, then $|g(z)| < |-z| = 1$ and by Rouche's theorem, we observe that $g(z) - z$ has exactly one zero in $|z| < 1$ which implies that g has a unique fixed point in $|z| < 1$. □

Theorem 2.3. *If the conditions of Theorem 2.1 hold and $y \in [-1, 1]$ satisfies the inequality*

$$\left|a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0\right| \le \varepsilon, \tag{2.13}$$

then (1.4) *has the Hyers-Ulam stability.*

Proof. Let $\varepsilon > 0$ and $y \in [-1, 1]$ such that

$$\left|a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0\right| \le \varepsilon. \tag{2.14}$$

We will show that there exists a constant K independent of ε and v such that

$$\left|y - v\right| \le K\varepsilon \tag{2.15}$$

for some $v \in [-1, 1]$ satisfying (1.4).

Let us introduce the abbreviation $K = 1/|a_1|(1-\lambda)$. Then

$$\begin{aligned}|y-v| &= |y-g(y)+g(y)-g(v)| \le |y-g(y)| + |g(y)-g(v)| \\ &\le \left|y - \frac{1}{a_1}\left(-a_0 - a_2 y^2 - \cdots - a_n y^n\right)\right| + \lambda|y-v| \\ &= \frac{1}{|a_1|}\left|a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0\right| + \lambda|y-v|.\end{aligned} \tag{2.16}$$

Thus, we have

$$\begin{aligned}|y-v| &\le \frac{1}{|a_1|(1-\lambda)}\left|a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0\right| \\ &\le K\varepsilon\end{aligned} \tag{2.17}$$

by (2.13) and so the result follows. □

Corollary 2.4. *In Theorem 2.2, if there exists $y \in D$ satisfying the inequality* (2.13)*, then* (2.9) *has the Hyers-Ulam stability.*

Remark 2.5. For $a_n = 1$, $a_i = 0$, for $2 \le i \le n-1$, combining Theorems 2.1 and 2.3 gives Theorem 1.1.

Remark 2.6. By the similar way, one can easily prove the Hyers-Ulam stability of (1.4) on any finite interval $[a,b]$.

Remark 2.7. Let f be any complex function such that f is analytic in

$$\Delta = \{z \in \mathbb{C} : |z|\langle R, R\rangle 0\}. \tag{2.18}$$

It is an interesting open problem whether f has the Hyers-Ulam stability for the case that f has some zeros in Δ.

We note that there is an error in the proof of Theorem 2.2 of [41], when Li and Hua stated that if (X,d) is a complete metric linear space then metric d is invariant, more precisely

$$d(x,y) = d(x-y,0) \tag{2.19}$$

for all $x, y \in X$. We give a counterexample for this case. Suppose that $X = \mathbb{R}$, and we define metric d on X as follows:

$$d(x,y) = \left|x + [x] - (y + [y])\right|, \tag{2.20}$$

for all $x, y \in X(X, d)$ is a complete metric linear space, and d is not an invariant metric on X, that is, there are $x, y \in X$ such that

$$d(x, y) \neq d(x - y, 0). \tag{2.21}$$

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