Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 914183, 13 pages
doi:10.1155/2010/914183

Research Article

Extreme Points and Rotundity in Musielak-Orlicz-Bochner Function Spaces Endowed with Orlicz Norm

Shaoqiang Shang,[1] Yunan Cui,[2] and Yongqiang Fu[1]

[1] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China*
[2] *Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China*

Correspondence should be addressed to Shaoqiang Shang, sqshang@163.com

Received 17 June 2010; Accepted 27 August 2010

Academic Editor: Paul Eloe

The criteria for extreme point and rotundity of Musielak-Orlicz-Bochner function spaces equipped with Orlicz norm are given. Although criteria for extreme point of Musielak-Orlicz function spaces equipped with the Orlicz norm were known, we can easily deduce them from our main results.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and unit ball, respectively. By X^* denote the dual space of X. Let N, R, and R^+ denote the set natural number, reals, and nonnegative reals, respectively.

A point $x \in A$ is said to be extreme point of A if $2x = y + z$ and $y, z \in A$ imply $y = z$. The set of all extreme points of A is denoted by $\mathrm{Ext}A$. If $\mathrm{Ext}B(X) = S(X)$, then X is said to be rotund. A point $x \in S(X)$ is said to be strongly extreme point if for any $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \in X$ with $\|x_n\| \to 1, \|y_n\| \to 1$, and $x = (1/2)(x_n + y_n)$, there holds $\|x_n - y_n\| \to 0$ $(n \to \infty)$. If the set of all strongly extreme points of $B(X)$ is equal to $S(X)$, then X is said to be midpoint local uniform rotund.

The notion of extreme point plays an important role in some branches of mathematics. For example, the Krein-Milman theorem, Choquet integral representation theorem, Rainwater theorem on convergence in weak topology, Bessaga-Pelczynski theorem, and Elton test unconditional convergence are strongly connected with this notion. In [1], using the principle of locally reflexivity, a remarkable theorem describing connections between extreme points of $S(X)$ and strongly extreme points of $S(X)$ is proved. Namely, a Banach space X is midpoint local uniformly rotund if and only if every point of $S(X)$ is an extreme point in X^{**}. Another proof of this theorem based on Goldstein's theorem is given in [2]. Analyzing the proof of this fact one can easily see its local version, namely, if $x \in S(X)$ is a strongly extreme point in X, then $\kappa(x)$ is an extreme point in X^{**}, where κ is the mapping of canonical embedding of X into X^{**}.

The criteria for extreme point and rotundity in the classical Musielak-Orlicz function spaces have been given in [3] already. However, because of the complication of Musielak-Orlicz-Bochner function spaces equipped with Orlicz norm, at present, the criteria for extreme point and rotundity have not been discussed yet. The aim of this paper is to give criteria for extreme point and rotundity of Musielak-Orlicz-Bochner function spaces equipped with Orlicz norm. By the result of this paper, it is easy to see that the result of [3] is true.

Let $(T, \sum, \mu)$ be nonatomic measurable space. Suppose that a function $M : T\times[0,\infty) \to [0,\infty]$ satisfies the following conditions:

(1) for μ-$a.e$, $t \in T$, $M(t,0) = 0$, $\lim_{u\to\infty} M(t,u) = \infty$ and $M(t,u') < \infty$ for some $u' > 0$;

(2) for μ-$a.e$, $t \in T$, $M(t,u)$ is convex on $[0,\infty)$ with respect to u;

(3) for each $u \in [0,\infty)$, $M(t,u)$ is a μ-measurable function of t on T.

Let $p(t,u)$ denote the right derivative of $M(t,\cdot)$ at $u \in R^+$ (where if $M(t,u) = \infty$, let $p(t,u) = \infty$) and let $q(t,\cdot)$ be the generalized inverse function of $p(t,\cdot)$ defined on R^+ by

$$q(t,v) = \sup_{u\geq 0}\{u \geq 0 : p(t,u) \leq v\}. \tag{1.1}$$

Then $N(t,v) = \int_0^v q(t,s)ds$ for any $v \in R$ and μ-a.e. $t \in T$. It is well known that there holds the Young inequality $uv \leq M(t,u) + N(t,v)$ for μ-a.e. $t \in T$. And $uv = M(t,u) + N(t,u) \Leftrightarrow u = q(t,v)$ or $v = q(t,u)$. Let

$$e(t) = \sup\{u > 0 : M(t,u) = 0\}, \qquad E(t) = \sup\{u > 0 : M(t,u) < \infty\}. \tag{1.2}$$

For fixed $t \in T$ and $v \geq 0$, if there exists $\epsilon \in (0,1)$ such that

$$M(t,v) = \frac{1}{2}M(t,v+\varepsilon) + \frac{1}{2}M(t,v-\varepsilon) < \infty, \tag{1.3}$$

then we call v a nonstrictly convex points of M with respects to t. The set of all nonstrictly convex point of M with respect to t is denoted by K_t.

For fixed $t \in T$, if $K_t = \Phi$, then we call that $M(t,u)$ is strictly convex with respect u for t.

Moreover, for a given Banach space $(X, \|\cdot\|)$, we denote by X_T the set of all strongly μ-measurable functions from T to X, and for each $u \in X_T$, define the modular of u by

$$\rho_M(u) = \int_T M(t, \|u(t)\|)dt. \tag{1.4}$$

Put

$$L_M^0(X) = \{u \in X_T : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\}. \tag{1.5}$$

Then the Musielak-Orlicz-Bochner function space

$$\|u\|^0 = \inf_{k>0} \frac{1}{k}\left[1 + \rho_M(ku)\right] \tag{1.6}$$

is Banach space. If $X = R$, $L_M^0(R)$ is said to be Musielak-Orlicz function space. Set

$$K(u) = \left\{k > 0 : \frac{1}{k}(1 + \rho_M(ku)) = \|u\|^0\right\}. \tag{1.7}$$

In particular, the set $K(u)$ can be nonempty. To show that, we give a proposition.

Proposition 1.1. *If* $\lim_{u\to\infty}(M(t,u)/u) = \infty$ *μ-a.e.* $t \in T$*, then* $K(u) \neq \phi$ *for any* $u \in L_M^0(X)$.

Proof. For any $u \in L_M^0(X)$, there exists $a > 0$ such that $\mu T_0 > 0$, where $T_0 = \{t \in T : \|u(t)\| \geq a\}$. It is easy to see that $T_0 = \cup_{n=1}^{\infty} G_n$, where

$$G_n = \left\{t \in T_0 : \frac{M(t,v)}{v} \geq \frac{3\|u\|^0}{a \cdot \mu T_0},\ v \geq n\right\}. \tag{1.8}$$

Noticing that $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$, then $\lim_{n\to\infty}\mu G_n = \mu T_0$. Hence there exists n_1 such that $\mu G_{n_1} > (1/2)\mu T_0$. This means that if $k > n_1/a$, we have

$$\begin{aligned}
\frac{1}{k}\left[1 + \int_T M(t,k\|u(t)\|)dt\right] &\geq \int_{G_{n_1}} \frac{M(t,k\|u(t)\|)}{k}dt \geq \int_{G_{n_1}} \frac{M(t,ka)}{k}dt \\
&= a\int_{G_{n_1}} \frac{M(t,ka)}{ka}dt \geq a\int_{G_{n_1}} \frac{M(t,n_1)}{n_1}dt \geq a\frac{3\|u\|^0}{a\cdot\mu T_0}\cdot\mu G_{n_1} > \frac{3}{2}\|u\|^0.
\end{aligned} \tag{1.9}$$

This implies that if $(1/k_n)(1+\rho_M(k_n u)) \to \|u\|^0 (n \to \infty)$, then sequence $\{k_n\}_{n=1}^{\infty}$ is bounded. Without loss of generality, we may assume that $k_n \to k_0$. Without loss of generality, we may assume that $k_1 \leq k_2 \leq \cdots k_n \leq \cdots \leq k_0$ or $k_1 \geq k_2 \geq \cdots k_n \geq \cdots \geq k_0$. If $k_1 \leq k_2 \leq \cdots k_n \leq \cdots \leq k_0$, by Levi theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{k_n}\left[1 + \rho_M(k_n u)\right] &= \lim_{n\to\infty}\frac{1}{k_n} + \lim_{n\to\infty}\int_T \frac{M(t,\|k_n u(t)\|)}{k_n}dt \\
&= \frac{1}{k_0} + \int_T \frac{M(t,\|k_0 u(t)\|)}{k_0}dt \\
&= \frac{1}{k_0}\left[1 + \rho_M(k_0 u)\right].
\end{aligned} \tag{1.10}$$

If $k_1 \geq k_2 \geq \cdots k_n \geq \cdots \geq k_0$, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{k_n}\left[1+\rho_M(k_n u)\right] &= \lim_{n\to\infty}\frac{1}{k_n} + \lim_{n\to\infty}\int_T \frac{M(t,\|k_n u(t)\|)}{k_n}dt \\
&= \frac{1}{k_0} + \int_T \frac{M(t,\|k_0 u(t)\|)}{k_0}dt \\
&= \frac{1}{k_0}\left[1+\rho_M(k_0 u)\right].
\end{aligned} \tag{1.11}$$

Therefore $(1/k_n)[1+\rho_M(k_n u)] \rightarrow (1/k_0)[1+\rho_M(k_0 u)]$ $(n \rightarrow \infty)$, namely, $(1/k_0)[1+\rho_M(k_0 u)] = \|u\|^0$. This implies $k_0 \in K(u)$. □

2. Main Results

In order to obtain the main theorems of this paper, we first give some lemmas.

Lemma 2.1. *If $K(u) = \phi$, then $\|u\|^0 = \int_T A(t)\cdot\|u(t)\|dt$, where $A(t) = \lim_{u\to\infty}(M(t,u)/u)$.*

Proof. By proof of Proposition 1.1, we know that if $K(u) = \phi$, then there exists $\{k_n\}_{n=1}^{\infty}$ such that $(1/k_n)(1+\rho_M(k_n u)) \rightarrow \|u\|^0$ and $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality, we may assume that $k_1 \leq k_2 \leq \cdots k_n \leq \cdots$. By Levi theorem, we have

$$\begin{aligned}
\|u\|^0 &= \lim_{n\to\infty}\frac{1}{k_n}\left(1+\int_T M(t,k_n\|u(t)\|)dt\right) \\
&= \lim_{n\to\infty}\frac{1}{k_n}\left(1+\int_{\{t\in T:\|u(t)\|\neq 0\}} M(t,k_n\|u(t)\|)dt\right) \\
&= \lim_{n\to\infty}\frac{1}{k_n} + \lim_{n\to\infty}\int_{\{t\in T:\|u(t)\|\neq 0\}} \frac{M(t,k_n\|u(t)\|)}{k_n\|u(t)\|}\cdot\|u(t)\|dt \\
&= \int_{\{t\in T:\|u(t)\|\neq 0\}} \lim_{n\to\infty}\frac{M(t,k_n\|u(t)\|)}{k_n\|u(t)\|}\cdot\|u(t)\|dt \\
&= \int_{\{t\in T:\|u(t)\|\neq 0\}} A(t)\cdot\|u(t)\|dt \\
&= \int_T A(t)\cdot\|u(t)\|dt.
\end{aligned} \tag{2.1}$$

Hence the conclusion of the lemma is true. □

Lemma 2.2. *If the set $K(u)$ consists of one element from $(0,+\infty)$, then $\|u\|^0 < \int_T A(t)\cdot\|u(t)\|dt$.*

Proof. Pick $k_1 > k_2 > 0$; then we have

$$\begin{aligned}
\rho_M(k_1 u\chi_{E_n}) &\geq \int_T k_1\|u(t)\chi_{E_n}(t)\|\cdot p(t,\|k_2 u(t)\chi_{E_n}(t)\|)dt - \rho_N(p(k_2 u\chi_{E_n})),\\
\rho_M(k_2 u\chi_{E_n}) &= \int_T k_2\|u(t)\chi_{E_n}(t)\|\cdot p(t,\|k_2 u(t)\chi_{E_n}(t)\|)dt - \rho_N(p(k_2 u\chi_{E_n})),
\end{aligned} \tag{2.2}$$

where

$$E_n = \{t\in T : k_2\|u(t)\| \leq n, p(t,\|k_2 u(t)\|) \leq n\}. \tag{2.3}$$

It follows that

$$\begin{aligned}
&\frac{1}{k_1}[1+\rho_M(k_1 u\chi_{E_n})] - \frac{1}{k_2}[1+\rho_M(k_2 u\chi_{E_n})]\\
&\quad= \frac{k_1-k_2}{k_1k_2}\left(-1+\frac{k_2}{k_1-k_2}[\rho_M(k_1 u\chi_{E_n})-\rho_M(k_2 u\chi_{E_n})]-\rho_M(k_2 u\chi_{E_n})\right)\\
&\quad\geq \frac{k_1-k_2}{k_1k_2}\left(-1+\frac{k_2}{k_1-k_2}\int_T (k_1-k_2)\|u(t)\chi_{E_n}(t)\|\cdot p(t,\|k_2 u(t)\chi_{E_n}(t)\|)dt\right.\\
&\qquad\qquad\left.-\rho_M(k_2 u\chi_{E_n})\right)\\
&\quad= \frac{k_1-k_2}{k_1k_2}(\rho_N(p(k_2 u\chi_{E_n}))-1).
\end{aligned} \tag{2.4}$$

Let $n \to \infty$; then we obtain

$$\frac{1}{k_1}[1+\rho_M(k_1 u\chi_E)] \geq \frac{k_1-k_2}{k_1k_2}(\rho_N(p(k_2 u\chi_E))-1) + \frac{1}{k_2}[1+\rho_M(k_2 u\chi_E)], \tag{2.5}$$

where

$$E = \{t\in T : p(t,\|k_2 u(t)\|) < \infty\}. \tag{2.6}$$

If $p(t,\|k_2u(t)\|)=\infty$, then $M(t,k_1\|u(t)\|)\geq M(t,k_2\|u(t)\|)=\infty$. Hence we have

$$\frac{1}{k_1}\left[1+\rho_M(k_1u)\right]\geq\frac{k_1-k_2}{k_1k_2}\left(\rho_N(p(k_2u))-1\right)+\frac{1}{k_2}\left[1+\rho_M(k_2u)\right]. \tag{2.7}$$

Moreover, there exists $k_0\in R^+$ such that $\rho_N(p(ku))\geq 1$, whenever $k\geq k_0$. This means that function $F(k)=(1/k)[1+\rho_M(ku)]$ is nondecreasing, when n is large enough. Pick sequence $\{k_n\}_{n=1}^{\infty}$ such that $0<k_1<k_2\cdots<k_n<\cdots$. By Levi theorem, we have

$$\begin{aligned}
\int_T A(t)\cdot\|u(t)\|dt&=\int_T\lim_{n\to\infty}\frac{M(t,k_n\|u(t)\|)}{k_n\|u(t)\|}\cdot\|u(t)\|dt\\
&=\lim_{n\to\infty}\int_{\mathrm{T}}\frac{M(t,k_n\|u(t)\|)}{k_n\|u(t)\|}\cdot\|u(t)\|dt\\
&=\lim_{n\to\infty}\frac{1}{k_n}\int_T M(t,k_n\|u(t)\|)dt\\
&=\lim_{n\to\infty}\frac{1}{k_n}\left[1+\int_T M(t,k_n\|u(t)\|)dt\right]\\
&>\frac{1}{l}\left[1+\int_T M(t,l\|u(t)\|)dt\right]=\|u\|^0,
\end{aligned} \tag{2.8}$$

where $\{l\}=K(u)$. Hence the conclusion of the lemma is true. □

Lemma 2.3 (see [3]). *Let $L_M^0(R)$ be rotund, then $M(t,u)$ is strictly convex with respect to u for almost all $t\in T$.*

Theorem 2.4. *Let $L_M^0(X)$ be Musielak-Orlicz-Bochner function spaces, then $u\in S(L_M^0(X))$ is an extreme point of $B(L_M^0(X))$ if and only if*

(a) *the set $K(u)$ consists of one element from $(0,+\infty)$;*

(b) *$v,w\in X_T$ with $u=\lambda v+(1-\lambda)w$ and $\|u(t)\|=\|v(t)\|=\|w(t)\|$$\mu$-a.e that on T implies $v=w$, where $\lambda\in(0,1)$;*

(c) *$\mu\{t\in T:k\|u(t)\|\in K_t\}=0$, where $k\in K(u)$.*

Proof. Necessity. (a1) Suppose that u is an extreme point of the unit ball $B(L_M^0)$ and $K(u)=\phi$, then $\|u\|^0=\int_T A(t)\cdot\|u(t)\|dt$ by Lemma 2.1. Decompose T into T_1 and T_2 such that $\int_{T_1}A(t)\cdot\|u(t)\|dt=\int_{T_2}A(t)\cdot\|u(t)\|dt$. Pick $\varepsilon\in(0,1)$. Put

$$\begin{aligned}
u_1(t)&=(u(t)+\varepsilon u(t))\chi_{T_1}+(u(t)-\varepsilon u(t))\chi_{T_2},\\
u_2(t)&=(u(t)-\varepsilon u(t))\chi_{T_1}+(u(t)+\varepsilon u(t))\chi_{T_2}.
\end{aligned} \tag{2.9}$$

Obviously, $u = (1/2)(u_1 + u_2)$ and $u_1 \neq u_2$. Moreover, we have

$$\begin{aligned}
\|u_1\|^0 &\leq \int_T A(t) \cdot \|u_1(t)\| dt \\
&= \int_{T_1} A(t) \cdot \|u(t) + \varepsilon u(t)\| dt + \int_{T_2} A(t) \cdot \|u(t) - \varepsilon u(t)\| dt \\
&= \int_{T_1} A(t) \cdot \|u(t)\| dt + \int_{T_1} \varepsilon A(t) \cdot \|u(t)\| dt + \int_{T_2} A(t) \cdot \|u(t)\| \, dt \\
&\quad - \int_{T_2} \varepsilon A(t) \cdot \|u(t)\| dt \\
&= \int_{T_1} A(t) \cdot \|u(t)\| dt + \int_{T_2} A(t) \cdot \|u(t)\| dt \\
&= \int_T A(t) \cdot \|u(t)\| dt \\
&= 1.
\end{aligned} \tag{2.10}$$

Similarly, we have $\|u_2\|^0 \leq 1$. Hence $u_1, u_1 \in B(L_M^0(X))$. Therefore $u \in S(L_M^0(X))$ is not an extreme point of $B(L_M^0(X))$, a contradiction. Hence $K(u) \neq \phi$. Suppose that $\|u\|^0 = \int_T A(t) \cdot \|u(t)\| dt$. Similarly, we get a contradiction.

The necessity of (b) is obvious.

(c) Set

$$\begin{aligned}
H_1 &= \{t \in T : u(t) = 0, \ e(t) > 0\}, \\
H_2 &= \{t \in T : 2M(t, k\|u(t)\|) = M(t, k\|u(t)\| + \varepsilon) + M(t, k\|u(t)\| - \varepsilon), \quad \|u(t)\| \neq 0\},
\end{aligned} \tag{2.11}$$

where $\{k\} \in K(u)$. Suppose that (c) does not hold. Then $\mu H_1 > 0$ or $\mu H_2 > 0$.

If $\mu H_1 > 0$, then for any $x \in S(X)$, by setting

$$(v(t), w(t)) = \begin{cases} \left(\dfrac{1}{2k} e(t) x, -\dfrac{1}{2k} e(t) x\right), & t \in H_1, \\ (u(t), u(t)), & t \in T \setminus H_1 \end{cases} \tag{2.12}$$

we have $u \neq w$ and $u = (1/2)(v + w)$. Moreover, we have

$$\begin{aligned}
\|v\|^0 &\leq \frac{1}{k}(1 + \rho_M(ku)) \\
&= \frac{1}{k}\left(1 + \int_{H_1} M\left(t, k\left\|\frac{1}{2k} e(t)\right\|\right) dt + \int_{T \setminus H_1} M(t, k\|u(t)\|) dt\right) \\
&= \frac{1}{k}\left(1 + \int_{T \setminus H_1} M(t, k\|u(t)\|) dt\right) \\
&\leq \|u\|^0.
\end{aligned} \tag{2.13}$$

Similarly, we have $\|w\|^0 \le 1$. Hence $v, w \in B(L_M^0(X))$. Therefore $u \in S(L_M^0(X))$ is not an extreme point of $B(L_M^0(X))$, a contradiction.

If $\mu H_2 > 0$, it is easy to see that $H_2 \subset \cup_{n=1}^{\infty}\{t \in T : \|u(t)\| \neq 0, M(t, k\|u(t)\|) = (1/2)M(t, (1+1/n)k\|u(t)\|) + (1/2)M(t, (1-1/n)k\|u(t)\|)\}$, where $\{k\} \in K(u)$. Then there exists $n_0 \in N$ such that

$$\begin{aligned} H = \Big\{ & t \in T : u(t) \neq 0, \\ & M(t, k\|u(t)\|) = \frac{1}{2}M\left(t, \left(1+\frac{1}{n_0}\right)k\|u(t)\|\right) + \frac{1}{2}M\left(t, \left(1-\frac{1}{n_0}\right)k\|u(t)\|\right) < \infty \Big\} \end{aligned} \tag{2.14}$$

is not a noll set. Decompose H into E and F such that $\int_E p(t, (1/n_0)k\|u(t)\|)dt = \int_F p(t, (1/n_0)k\|u(t)\|)dt$. Define

$$(v(t), w(t)) = \begin{cases} \left(\left(1+\dfrac{1}{n_0}\right)u(t), \left(1-\dfrac{1}{n_0}\right)u(t)\right), & t \in E, \\ \left(\left(1-\dfrac{1}{n_0}\right)u(t), \left(1+\dfrac{1}{n_0}\right)u(t)\right), & t \in F, \\ (u(t), u(t)), & t \in T \setminus (E \cup F). \end{cases} \tag{2.15}$$

Then $u \neq w$ and $u = (1/2)(v+w)$. Furthermore, we have

$$\begin{aligned} \|v\|^0 &\le \frac{1}{k}(1+\rho_M(kv)) \\ &= \frac{1}{k}\Bigg(1 + \int_{T\setminus(E\cup F)} M(t, k\|u(t)\|)dt + \int_E M\left(t, \left(1+\frac{1}{n_0}\right)k\|u(t)\|\right)dt \\ &\qquad + \int_F M\left(t, \left(1-\frac{1}{n_0}\right)k\|u(t)\|\right)dt\Bigg) \\ &= \frac{1}{k}\Bigg(1 + \int_{T\setminus(E\cup F)} M(t, k\|u(t)\|)dt + \int_E M(t, k\|u(t)\|)dt \\ &\qquad + \int_E p\left(t, \frac{1}{n_0}k\|u(t)\|\right)dt + \int_F M(t, k\|u(t)\|)dt - \int_F p\left(t, \frac{1}{n_0}k\|u(t)\|\right)dt\Bigg) \\ &= \frac{1}{k}\left(1 + \int_{T\setminus(E\cup F)} M(t, k\|u(t)\|)dt + \int_E M(t, k\|u(t)\|)dt + \int_F M(t, k\|u(t)\|)dt\right) \\ &= \frac{1}{k}(1+\rho_M(ku)) \\ &= \|u\|^0 = 1. \end{aligned} \tag{2.16}$$

Similarly, we have $\|w\|^0 \leq 1$. Hence $v, w \in B(L_M^0(X))$. Therefore $u \in S(L_M^0(X))$ is not an extreme point, a contradiction. Hence (c) is true.

(a2) If $K(u) \neq \phi$ and $u \in S(L_M^0(X))$ is an extreme point, suppose that there exists $k_1, k_2 \in K(u)$ satisfying $k_1 \neq k_2$. Define $k = k_1k_2/(k_1+k_2)$,

$$\begin{aligned}
2 &= \|u\|^0 + \|u\|^0 \\
&= \frac{k_1+k_2}{k_1k_2}\left[1 + \frac{k_2}{k_1+k_2}\rho_M(k_1u) + \frac{k_1}{k_1+k_2}\rho_M(k_2u)\right] \\
&= \frac{k_1+k_2}{k_1k_2}\left[1 + \frac{k_2}{k_1+k_2}\int_T M(t,\|k_1u(t)\|)dt + \frac{k_1}{k_1+k_2}\int_T M(t,\|k_2u(t)\|)dt\right] \\
&\geq \frac{k_1+k_2}{k_1k_2}\left[1 + \int_T M\left(t, \frac{k_2}{k_1+k_2}\|k_1u(t)\| + \frac{k_1}{k_1+k_2}\|k_2u(t)\|\right)dt\right] \\
&= \frac{k_1+k_2}{k_1k_2}\left[1 + \int_T M\left(t, \left\|\frac{2k_1k_2}{k_1+k_2}u(t)\right\|\right)dt\right] \\
&= 2\frac{1}{2k}\left[1 + \rho_M(2ku)\right] \\
&\geq 2\|u\|^0 \\
&= 2.
\end{aligned} \tag{2.17}$$

This implies that

$$\|u\|^0 = \frac{1}{2k}\left[1 + \rho_M(2ku)\right] \tag{2.18}$$

(i.e., $2k \in K(u)$),

$$\frac{k_2}{k_1+k_2}M(t, k_1\|u(t)\|) + \frac{k_1}{k_1+k_2}M(t, k_2\|u(t)\|) = M(t, 2k\|u(t)\|). \tag{2.19}$$

Since $k_1\|u(t)\| \neq k_2\|u(t)\|$ on $\{t \in T : \|u(t)\| \neq 0\}$, then $2k\|u(t)\| \in K_t$ on $\{t \in T : \|u(t)\| \neq 0\}$, a contradiction. Therefore (a) is true.

Sufficiency. We first prove that for $u, u_1, u_2 \in S(L_M^0(X))$ with $u = (1/2)(u_1 + u_2)$ at least one of the sets $K(u_1)$ or $K(u_2)$ is nonempty. Suppose that $K(u_1) = \phi$ and $K(u_2) = \phi$. Hence we have

$$\begin{aligned}
1 &= \left\| \frac{1}{2}(u_1 + u_2) \right\|^0 \\
&< \int_T A(t) \cdot \left\| \frac{1}{2}(u_1(t) + u_2(t)) \right\| dt \\
&\leq \frac{1}{2} \int_T A(t) \cdot \|u_1(t)\| dt + \frac{1}{2} \int_T A(t) \cdot \|u_2(t)\| dt \\
&= \frac{1}{2}\|u_1\|^0 + \frac{1}{2}\|u_2\|^0 \\
&= 1,
\end{aligned} \tag{2.20}$$

a contradiction. This contradiction shows that $K(u_1) \neq \phi$ or $K(u_2) \neq \phi$.

Now we will prove that $K(u_1) \neq \phi$ and $K(u_2) \neq \phi$. Otherwise, we can assume without loss of generality that $K(u_1) \neq \phi$ and $K(u_2) = \phi$. Put

$$[u_1, u) = \{(1 - \lambda)u_1 + \lambda u : 0 < \lambda < 1\}, \qquad (u, u_2] = \{(1 - \lambda)u + \lambda u_2 : 0 < \lambda < 1\}. \tag{2.21}$$

Next we will prove that $K(y) \neq \phi$ for all $y \in [u_1, u)$ and $K(y) = \phi$ for all $y \in (u, u_2]$. Assume first for the contrary that this is $u_3 \in [u_1, u)$ such that $K(u_3) = \phi$. Then there exists $\lambda_3 \in [0, 1)$ such that $u_3 = (1 - \lambda_3)u_1 + \lambda_3 u$. Since $u_1 = 2u - u_2$, we have

$$u_3 = (1 - \lambda_3)(2u - u_2) + \lambda_3 u = (2 - \lambda_3)u - (1 - \lambda_3)u_2. \tag{2.22}$$

Hence $u = (1/(2 - \lambda_3))u_3 + ((1 - \lambda_3)/(2 - \lambda_3))u_2$. Therefore

$$\begin{aligned}
1 = \|u\|^0 &< \int_T A(t) \cdot \left\| \frac{1}{2 - \lambda_3} u_3(t) + \frac{1 - \lambda_3}{2 - \lambda_3} u_2(t) \right\| dt \\
&\leq \frac{1}{2 - \lambda_3} \int_T A(t) \cdot \|u_3(t)\| dt + \frac{1 - \lambda_3}{2 - \lambda_3} \int_T A(t) \cdot \|u_2(t)\| dt \\
&= \frac{1}{2 - \lambda_3}\|u_3\|^0 + \frac{1 - \lambda_3}{2 - \lambda_3}\|u_2\|^0 \\
&= 1,
\end{aligned} \tag{2.23}$$

a contradiction.

Assume now for the contrary that this is $u_4 \in (u, u_2]$ such that $K(u_4) \neq \phi$. We can find $u_5 \in (u, u_2]$ such that $u = (1/2)(u_4 + u_5)$ and $u_4 \neq u_5$. Therefore there are $k_4 \geq 1$ and $k_5 \geq 1$ such that

$$\|u_4\|^0 = \frac{1}{k_4}\left[1 + \rho_M(k_4 u_4)\right], \qquad \|u_5\|^0 = \frac{1}{k_5}\left[1 + \rho_M(k_5 u_5)\right]. \tag{2.24}$$

By the convexity of the modular ρ_M we have

$$\begin{aligned}
\rho_M\left(\frac{2k_4k_5}{k_4+k_5}u\right) &= \rho_M\left(\frac{k_4k_5}{k_4+k_5}(u_4+u_5)\right) \\
&= \rho_M\left(\frac{k_5}{k_4+k_5}k_4u_4 + \frac{k_4}{k_4+k_5}k_5u_5\right) \\
&= \int_T M\left(t, \left\|\frac{k_5}{k_4+k_5}k_4u_4(t) + \frac{k_4}{k_4+k_5}k_5u_5(t)\right\|\right)dt \\
&\leq \int_T M\left(t, \frac{k_5}{k_4+k_5}\|k_4u_4(t)\| + \frac{k_4}{k_4+k_5}\|k_5u_5(t)\|\right)dt \\
&\leq \frac{k_5}{k_4+k_5}\int_T M(t, \|k_4u_4(t)\|)dt + \frac{k_4}{k_4+k_5}\int_T M(t, \|k_5u_5(t)\|)dt \\
&= \frac{k_5}{k_4+k_5}\rho_M(k_4u_4) + \frac{k_4}{k_4+k_5}\rho_M(k_5u_5).
\end{aligned} \tag{2.25}$$

Hence

$$\begin{aligned}
2 &= 2\|u\|^0 \\
&\leq \frac{k_4+k_5}{k_4k_5}\left(1 + \rho_M\left(\frac{k_4k_5}{k_4+k_5}2u\right)\right) \\
&\leq \frac{k_4+k_5}{k_4k_5}\left(1 + \frac{k_5}{k_4+k_5}\rho_M(k_4u_4) + \frac{k_4}{k_4+k_5}\rho_M(k_5u_5)\right) \\
&\leq \frac{1}{k_4}(1 + \rho_M(k_4u_4)) + \frac{1}{k_5}(1 + \rho_M(k_5u_5)) \\
&= 2.
\end{aligned} \tag{2.26}$$

Consequently, all inequalities from the last three lines are equalities in fact. Therefore $2(k_4 k_5/(k_4 + k_5)) = k$ and

$$M(t, \|ku(t)\|) = \frac{k_5}{k_4+k_5}M(t, \|k_4u_4(t)\|) + \frac{k_4}{k_4+k_5}M(t, \|k_5u_5(t)\|) \tag{2.27}$$

for μ-$a.e$ $t \in T$. By $\mu\{t \in T : k\|u(t)\| \in K_t\} = 0$, it follows that $\|ku(t)\| = \|k_4u_4(t)\| = \|k_5u_5(t)\|$ for μ-$a.e$ $t \in T$. And we have $ku(t) = (k_5/(k_4+k_5))k_4u_4(t) + (k_4/(k_4+k_5))k_5u_5(t)$ for μ-$a.e$ $t \in T$. By (b), we have $ku = k_4u_4 = k_5u_5$. Since $u_4, u_5, u \in S(L_M^0(X))$, we get $k_4 = k_5 = k$, which gives

$u_4 = u_5 = u$. This contradicts the inequality $u_4 \neq u_5$. Thus $K(y) = \phi$ for any $y \in (u, u_2]$. Take $u_n = (1-1/n)u + (1/n)u_2$ for all $n \in N$. Then $u_n \in (u, u_2]$ for all $n \in N$. Hence $K(u_n) = \phi$, and consequently $\|u_n\|^0 = \int_T A(t) \cdot \|u_n(t)\| dt$ for all $n \in N$. Note that $\|u_n - u\|^0 \to 0$ $(n \to \infty)$ and $\lim_{n\to\infty}\|u_n(t)\| = \|u(t)\|$ for μ-$a.e, t \in T$. Since $K(u) = \{k\}$, with $0 < k < \infty$, we have

$$\lim_{n\to\infty}\|u_n\|^0 = \|u\|^0 = \lim_{n\to\infty}\int_T A(t) \cdot \|u_n(t)\| dt \geq \int_T A(t) \cdot \|u(t)\| dt > \|u\|^0, \tag{2.28}$$

a contradiction. Therefore $K(u_1) \neq \phi$ and $K(u_2) \neq \phi$. Now repeating the same procedure as above, putting u_1 and u_2 instead of u_4 and u_5, respectively, we get

$$k_1 u_1(t) = k_2 u_2(t) = k u(t) \tag{2.29}$$

for μ-$a.e, t \in T$. Hence, by the fact that $u_1, u_2, u \in S(L_M^0(X))$, we have $k_1 = k_2 = k$, and consequently, $u_1 = u_2 = u$. Thus u is an extreme point of $B(L_M^0(X))$. □

Corollary 2.5 (see [3]). *$u \in S(L_M^0(R))$ is an extreme point of $B(L_M^0(R))$ if and only if*

(a) *the set $K(u)$ consists of one element from $(0, +\infty)$;*

(b) *$\mu\{t \in T : k|u(t)| \in K_t\} = 0$, where $k \in K(u)$.*

Finally, we investigate the rotundity of $L_M^0(X)$.

Theorem 2.6. *$L_M^0(X)$ is rotund if and only if:*

(a) *for any $u \in S(L_M^0(X))$, the set $K(u)$ consists of one element from $(0, +\infty)$;*

(b) *X is rotund;*

(c) *$M(t, u)$ is strictly convex with respect to u for almost all $t \in T$.*

Proof. Sufficiency is obvious by Theorem 2.4.

Necessity. (a) is obvious by (a) of Theorem 2.4. $L_M^0(R)$ is isometrically isomorphic to closed subspace of $L_M^0(X)$, thus $L_M^0(R)$ is rotund. By Lemma 2.3, (c) is obvious.

If (b) is not true, then there exist $x, y, z \in S(X)$ with $2x = y + z$ and $y \neq z$. Pick $h(t) \in S(L_M^0(X))$, then there exists $d > 0$ such that $\mu H > 0$, where $H = \{t \in T : \|h(t)\| \geq d\}$. Since $h(t) \in S(L_M^0(X))$, then there exists $k' > 0$ such that

$$\int_H M(t, k'\|h(t)\|) dt \leq \int_T M(t, k'\|h(t)\|) dt < \infty. \tag{2.30}$$

Set

$$u(t) = d \cdot x \cdot \chi_H(t), \qquad v(t) = d \cdot y \cdot \chi_H(t), \qquad w(t) = d \cdot z \cdot \chi_H(t). \tag{2.31}$$

We have

$$\int_T M(t, k'\|u(t)\|) dt = \int_H M(t, k'd) dt \leq \int_H M(t, k\|h(t)\|) dt < \infty. \tag{2.32}$$

This implies that $u(t) \in L_M^0(X)$. Similarly, we have $v(t), w(t) \in L_M^0(X)$. It is easy to see that $\|u\|^0 = \|v\|^0 = \|w\|^0$. Then

$$\left\| \frac{u}{\|u\|^0} \right\|^0 = \left\| \frac{v}{\|u\|^0} \right\|^0 = \left\| \frac{w}{\|u\|^0} \right\|^0 = 1, \qquad \frac{u}{\|u\|^0} = \frac{1}{2} \cdot \frac{v}{\|u\|^0} + \frac{1}{2} \cdot \frac{w}{\|u\|^0}. \tag{2.33}$$

However, $\|u(t)\| = \|v(t)\| = \|w(t)\|$ for $t \in T$. By (b) of Theorem 2.4, we have $u = w$. Hence $v/\|u\|^0 = w/\|u\|^0$. So u is not an extreme point of $B(L_M^0(X))$. Contradicting the rotundity of $L_M^0(X)$. □

Acknowledgment

The authors would like to thank the anonymous referees for some suggestions to improve the paper. This work was supported by Heilong Jiang Natural Science Fund A200902.

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