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Research Article

On q-Euler Numbers Related to the Modified q-Bernstein Polynomials

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We consider q-Euler numbers, polynomials, and q-Stirling numbers of first and second kinds. Finally, we investigate some interesting properties of the modified q-Bernstein polynomials related to q-Euler numbers and q-Stirling numbers by using fermionic p-adic integrals on $\mathbb{Z}_p$.

1. Introduction

Let $C[0,1]$ be the set of continuous functions on $[0,1]$. The classical Bernstein polynomials of degree n for $f \in C[0,1]$ are defined by

$$\mathbb{B}_n(f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \quad 0 \le x \le 1, \tag{1.1}$$

where $\mathbb{B}_n(f)$ is called the Bernstein operator and

$$B_{k,n}(x) = \binom{n}{k} x^k (x-1)^{n-k} \tag{1.2}$$

are called the Bernstein basis polynomials (or the Bernstein polynomials of degree n) (see [1]). Recently, Acikgoz and Araci have studied the generating function for Bernstein polynomials (see [2, 3]). Their generating function for $B_{k,n}(x)$ is given by

$$F^{(k)}(t,x)=\frac{t^k e^{(1-x)t}x^k}{k!}=\sum_{n=0}^{\infty}B_{k,n}(x)\frac{t^n}{n!}, \tag{1.3}$$

where $k=0,1,\ldots$ and $x\in[0,1]$. Note that

$$B_{k,n}(x)=\begin{cases}\begin{pmatrix}n\\k\end{pmatrix}x^k(1-x)^{n-k}, & \text{if } n\geq k,\\ 0, & \text{if } n<k,\end{cases} \tag{1.4}$$

for $n=0,1,\ldots$ (see [2, 3]).

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p=p^{-1}$.

Throughout this paper, we use the following notation:

$$[x]_q=\frac{1-q^x}{1-q},\qquad [x]_{-q}=\frac{1-(-q)^x}{1+q} \tag{1.5}$$

(cf. [4–7]). Let $\mathbb{N}$ be the natural numbers and $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$.

Let $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-1/(p-1)}$ and $x\in\mathbb{Z}_p$. Then q-Bernstein type operator for $f\in\mathrm{UD}(\mathbb{Z}_p)$ is defined by (see [8, 9])

$$\mathbb{B}_{n,q}(f)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}[x]_q^k[1-x]_q^{n-k}=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_{k,n}(x,q), \tag{1.6}$$

for $k,n\in\mathbb{Z}_+$, where

$$B_{k,n}(x,q)=\binom{n}{k}[x]_q^k[1-x]_q^{n-k} \tag{1.7}$$

is called the modified q-Bernstein polynomials of degree n. When we put $q\to 1$ in (1.7), $[x]_q^k\to x^k$, $[1-x]_q^{n-k}\to(1-x)^{n-k}$, and we obtain the classical Bernstein polynomial, defined by (1.2). We can deduce very easily from (1.7) that

$$B_{k,n}(x,q)=[1-x]_qB_{k,n-1}(x,q)+[x]_qB_{k-1,n-1}(x,q) \tag{1.8}$$

(see [8]). For $0 \leq k \leq n$, derivatives of the nth degree modified q-Bernstein polynomials are polynomials of degree $n-1$:

$$\frac{\mathrm{d}}{\mathrm{d}x} B_{k,n}(x,q) = n\left(q^{x} B_{k-1,n-1}(x,q) - q^{1-x} B_{k,n-1}(x,q)\right) \frac{\ln q}{q-1} \tag{1.9}$$

(see [8]).

The Bernstein polynomials can also be defined in many different ways. Thus, recently, many applications of these polynomials have been looked for by many authors. In the recent years, the q-Bernstein polynomials have been investigated and studied by many authors in many different ways (see [1, 8, 9] and references therein [10, 11]). In [11], Phillips gave many results concerning the q-integers and an account of the properties of q-Bernstein polynomials. He gave many applications of these polynomials on approximation theory. In [2, 3], Acikgoz and Araci have introduced several type Bernstein polynomials. The Acikgoz and Araci paper to announced in the conference is actually motivated to write this paper. In [1], Simsek and Acikgoz constructed a new generating function of the q-Bernstein type polynomials and established elementary properties of this function. In [8], Kim et al. proposed the modified q-Bernstein polynomials of degree n, which are different q-Bernstein polynomials of Phillips. In [9], Kim et al. investigated some interesting properties of the modified q-Bernstein polynomials of degree n related to q-Stirling numbers and Carlitz's q-Bernoulli numbers.

In the present paper, we consider q-Euler numbers, polynomials, and q-Stirling numbers of first and second kinds. We also investigate some interesting properties of the modified q-Bernstein polynomials of degree n related to q-Euler numbers and q-Stirling numbers by using fermionic p-adic integrals on $\mathbb{Z}_p$.

2. q-Euler Numbers and Polynomials Related to the Fermionic p-Adic Integrals on $\mathbb{Z}_p$

For $N \geq 1$, the fermionic q-extension μ_q of the p-adic Haar distribution μ_{Haar},

$$\mu_{-q}\left(a + p^{N}\mathbb{Z}_p\right) = \frac{(-q)^{a}}{\left[p^{N}\right]_{-q}}, \tag{2.1}$$

is known as a measure on $\mathbb{Z}_p$, where $a + p^{N}\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x-a|_p \leq p^{-N}\}$ (cf. [4, 12]). We will write $d\mu_{-q}(x)$ to remind ourselves that x is the variable of integration. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. Then μ_{-q} yields the fermionic p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$:

$$I_{-q}(f) = \int_{Z_p} f(x)\mathrm{d}\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^{x} \tag{2.2}$$

(cf. [12–15]). Many interesting properties of (2.2) were studied by many authors (see [12, 13] and the references given there). Using (2.2), we have the fermionic p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$\lim_{q\to -1} I_q(f) = I_{-1}(f) = \int_{Z_p} f(a)\,\mathrm{d}\mu_{-1}(x). \tag{2.3}$$

For $n \in \mathbb{N}$, write $f_n(x) = f(x+n)$. We have

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1} (-1)^{n-l-1} f(l). \tag{2.4}$$

This identity is obtained by Kim in [12] to derive interesting properties and relationships involving q-Euler numbers and polynomials. For $n \in \mathbb{Z}_+$, we note that

$$I_{-1}\left([x]_q^n\right) = \int_{\mathbb{Z}_p} [x]_q^n \mathrm{d}\mu_{-1}(x) = E_{n,q}, \tag{2.5}$$

where $E_{n,q}$ are the q-Euler numbers (see [16]). It is easy to see that $E_{0,q} = 1$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{l=0}^{n} \binom{n}{l} q^l E_{l,q} &= \sum_{l=0}^{n} \binom{n}{l} q^l \lim_{N\to\infty} \sum_{x=0}^{p^N-1} [x]_q^l (-1)^x \\
&= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x \left(q[x]_q + 1\right)^n \\
&= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x [x+1]_q^n \\
&= -\lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x \left([x]_q^n + \left[p^N\right]_q^n\right) \\
&= -E_{n,q}.
\end{aligned} \tag{2.6}$$

From this formula, we have the following recurrence formula:

$$E_{0,q} = 1, \qquad (qE+1)^n + E_{n,q} = 0 \quad \text{if } n \in \mathrm{N}, \tag{2.7}$$

with the usual convention of replacing E^l by $E_{l,q}$. By the simple calculation of the fermionic p-adic invariant integral on $\mathbb{Z}_p$, we see that

$$E_{n,q} = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l}, \tag{2.8}$$

where $\binom{n}{l} = n!/l!(n-l)! = n(n-1)\cdots(n-l+1)/l!$. Now, by introducing the following equations:

$$[x]_{1/q}^{n} = q^{n} q^{-nx} [x]_{q}^{n}, \qquad q^{-nx} = \sum_{m=0}^{\infty} (1-q)^{m} \binom{n+m-1}{m} [x]_{q}^{m} \tag{2.9}$$

into (2.5), we find that

$$E_{n,1/q} = q^{n} \sum_{m=0}^{\infty} (1-q)^{m} \binom{n+m-1}{m} E_{n+m,q}. \tag{2.10}$$

This identity is a peculiarity of the p-adic q-Euler numbers, and the classical Euler numbers do not seem to have a similar relation. Let $F_q(t)$ be the generating function of the q-Euler numbers. Then we obtain that

$$\begin{aligned} F_q(t) &= \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{2}{(1-q)^{n}} \sum_{l=0}^{n} (-1)^{l} \binom{n}{l} \frac{1}{1+q^{l}} \frac{t^n}{n!} \\ &= 2e^{t/(1-q)} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(1-q)^{k}} \frac{1}{1+q^{k}} \frac{t^{k}}{k!}. \end{aligned} \tag{2.11}$$

From (2.11), we note that

$$F_q(t) = 2e^{t/(1-q)} \sum_{n=0}^{\infty} (-1)^{n} e^{(-q^{n}/(1-q))t} = 2\sum_{n=0}^{\infty} (-1)^{n} e^{[n]_q t}. \tag{2.12}$$

It is well known that

$$I_{-1}\left([x+y]^{n}\right) = \int_{\mathbb{Z}_p} [x+y]^{n} d\mu_{-1}(y) = E_{n,q}(x), \tag{2.13}$$

where $E_{n,q}(x)$ are the q-Euler polynomials (see [16]). In the special case $x = 0$, the numbers $E_{n,q}(0) = E_{n,q}$ are referred to as the q-Euler numbers. Thus, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} [x+y]^{n} d\mu_{-1}(y) &= \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \int_{\mathbb{Z}_p} [y]^{k} d\mu_{-1}(y) \\ &= \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} E_{k,q} \\ &= \left(q^{x} E + [x]_q\right)^{n}. \end{aligned} \tag{2.14}$$

It is easily verified, using (2.12) and (2.13), that the q-Euler polynomials $E_{n,q}(x)$ satisfy the following formula:

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y)\\ &= \sum_{n=0}^{\infty} \frac{2}{(1-q)^n}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\frac{q^{lx}}{1+q^l}\frac{t^n}{n!}\\ &= 2\sum_{n=0}^{\infty}(-1)^n e^{[n+x]_q t}.\end{aligned} \tag{2.15}$$

Using formula (2.15), when q tends to 1, we can readily derive the Euler polynomials, $E_n(x)$, namely,

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \tag{2.16}$$

(see [12]). Note that $E_n(0) = E_n$ are referred to as the nth Euler numbers. Comparing the coefficients of $t^n/n!$ on both sides of (2.15), we have

$$E_{n,q}(x) = 2\sum_{m=0}^{\infty}(-1)^m[m+x]_q^n = \frac{2}{(1-q)^n}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\frac{q^{lx}}{1+q^l}. \tag{2.17}$$

We refer to $[n]_q$ as a q-integer and note that $[n]_q$ is a continuous function of q. In an obvious way we also define a q-factorial,

$$[n]_q! = \begin{cases}[n]_q[n-1]_q\cdots[1]_q, & n\in\mathbb{N},\\ 1, & n=0,\end{cases} \tag{2.18}$$

and a q-analogue of binomial coefficient,

$$\binom{x}{n}_q = \frac{[x]_q!}{[x-n]_q![n]_q!} = \frac{[x]_q[x-1]_q\cdots[x-n+1]_q}{[n]_q!} \tag{2.19}$$

(cf. [14, 16]). Note that

$$\lim_{q\to 1}\binom{x}{n}_q = \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}. \tag{2.20}$$

It readily follows from (2.19) that

$$\binom{x}{n}_q = \frac{(1-q)^n q^{-\binom{n}{2}}}{[n]_q!}\sum_{i=0}^{n} q^{\binom{i}{2}}\binom{n}{i}_q(-1)^{n+i}q^{(n-i)x} \tag{2.21}$$

(cf. [7, 16]). It can be readily seen that

$$q^{lx} = \left([x]_q(q-1)+1\right)^l = \sum_{m=0}^{l}\binom{l}{m}(q-1)^m[x]_q^m. \tag{2.22}$$

Thus, by (2.13) and (2.22), we have

$$\int_{\mathbb{Z}_p}\binom{x}{n}_q d\mu_{-1}(x) = \frac{(q-1)^n}{[n]_q! q^{\binom{n}{2}}}\sum_{i=0}^{n} q^{\binom{i}{2}}\binom{n}{i}_q(-1)^i\sum_{j=0}^{n-i}\binom{n-i}{j}(q-1)^j E_{j,q}. \tag{2.23}$$

From now on, we use the following notation:

$$\begin{gathered}\frac{[x]_q!}{[x-k]_q!} = q^{-\binom{k}{2}}\sum_{l=0}^{k} s_{1,q}(k,l)[x]_q^l, \quad k\in\mathbb{Z}_+,\\ [x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} s_{2,q}(n,k)\frac{[x]_q!}{[x-k]_q!}, \quad n\in\mathbb{Z}_+\end{gathered} \tag{2.24}$$

(see [7]). From (2.24), and (2.22), we calculate the following consequence:

$$\begin{aligned}[x]_q^n &= \sum_{k=0}^{n} q^{\binom{k}{2}} s_{2,q}(n,k)\frac{1}{(1-q)^k}\sum_{l=0}^{k}\binom{k}{l}_q q^{\binom{l}{2}}(-1)^l q^{l(x-k+1)}\\ &= \sum_{k=0}^{n} q^{\binom{k}{2}} s_{2,q}(n,k)\frac{1}{(1-q)^k}\sum_{l=0}^{k}\binom{k}{l}_q q^{\binom{l}{2}+l(1-k)}(-1)^l\\ &\quad\times\sum_{m=0}^{l}\binom{l}{m}(q-1)^m[x]_q^m\\ &= \sum_{k=0}^{n} q^{\binom{k}{2}} s_{2,q}(n,k)\frac{1}{(1-q)^k}\\ &\quad\times\sum_{m=0}^{k}(q-1)^m\left(\sum_{l=m}^{k}\binom{k}{l}_q q^{\binom{l}{2}+l(1-k)}\binom{l}{m}(-1)^l\right)[x]_q^m.\end{aligned} \tag{2.25}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For* $n\in\mathbb{Z}_+$,

$$E_{n,q} = \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=m}^{k} q^{\binom{k}{2}} s_{2,q}(n,k)(q-1)^{m-k}\binom{k}{l}_q q^{\binom{l}{2}+l(1-k)}\binom{l}{m}(-1)^{l+k}E_{m,q}. \tag{2.26}$$

By (2.22) and simple calculation, we find that

$$\begin{aligned}
\sum_{m=0}^{n}\binom{n}{m}(q-1)^{m}E_{m,q} &= \int_{\mathbb{Z}_p} q^{nx}\mathrm{d}\mu_{-1}(x) \\
&= \sum_{k=0}^{n}(q-1)^{k}q^{\binom{k}{2}}\binom{n}{k}_{q}\int_{\mathbb{Z}_p}\prod_{i=0}^{k-1}[x-i]_q\mathrm{d}\mu_{-1}(x) \\
&= \sum_{k=0}^{n}(q-1)^{k}\binom{n}{k}_{q}\sum_{m=0}^{k}s_{1,q}(k,m)\int_{\mathbb{Z}_p}[x]_q^m\mathrm{d}\mu_{-1}(x) \\
&= \sum_{m=0}^{n}\left(\sum_{k=m}^{n}(q-1)^{k}\binom{n}{k}_{q}s_{1,q}(k,m)\right)E_{m,q}.
\end{aligned} \tag{2.27}$$

Therefore, we deduce the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$,*

$$\sum_{m=0}^{n}\binom{n}{m}(q-1)^{m}E_{m,q} = \sum_{m=0}^{n}\sum_{k=m}^{n}(q-1)^{k}\binom{n}{k}_{q}s_{1,q}(k,m)E_{m,q}. \tag{2.28}$$

Corollary 2.3. *For $m, n \in \mathbb{Z}_+$ with $m \le n$,*

$$\binom{n}{m}(q-1)^{m} = \sum_{k=m}^{n}(q-1)^{k}\binom{n}{k}_{q}s_{1,q}(k,m). \tag{2.29}$$

By (2.17) and Corollary 2.3, we obtain the following corollary.

Corollary 2.4. *For $n \in \mathbb{Z}_+$,*

$$E_{n,q}(x) = \frac{2}{(1-q)^{n}}\sum_{l=0}^{n}\sum_{k=l}^{n}(-1)^{l}(q-1)^{k-l}\binom{n}{k}_{q}s_{1,q}(k,l)\frac{q^{lx}}{1+q^{l}}. \tag{2.30}$$

It is easy to see that

$$\binom{n}{k}_{q} = \sum_{l_0+\cdots+l_k=n-k} q^{\sum_{i=0}^{k} il_i} \tag{2.31}$$

(cf. [7]). From (2.31) and Corollary 2.4, we can also derive the following interesting formula for q-Euler polynomials.

Theorem 2.5. *For $n \in \mathbb{Z}_+$,*

$$E_{n,q}(x) = 2\sum_{l=0}^{n}\sum_{k=l}^{n}\sum_{l_0+\cdots+l_k=n-k} q^{\sum_{i=0}^{k} il_i}\frac{1}{(1-q)^{n+l-k}}s_{1,q}(k,l)(-1)^{k}\frac{q^{lx}}{1+q^{l}}. \tag{2.32}$$

These polynomials are related to the many branches of Mathematics, for example, combinatorics, number theory, and discrete probability distributions for finding higher-order moments (cf. [14–16]). By substituting $x = 0$ into the above, we have

$$E_{n,q} = 2\sum_{l=0}^{n}\sum_{k=l}^{n}\sum_{l_0+\cdots+l_k=n-k} q^{\sum_{i=0}^{k} i l_i}\frac{1}{(1-q)^{n+l-k}}s_{1,q}(k,l)(-1)^k\frac{1}{1+q^l}, \tag{2.33}$$

where $E_{n,q}$ is the q-Euler numbers.

3. q-Euler Numbers, q-Stirling Numbers, and q-Bernstein Polynomials Related to the Fermionic p-Adic Integrals on $\mathbb{Z}_p$

First, we consider the q-extension of the generating function of Bernstein polynomials in (1.3).

For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$, we obtain that

$$\begin{aligned} F_q^{(k)}(t,x) &= \frac{t^k e^{[1-x]_q t}[x]_q^k}{k!} \\ &= [x]_q^k\sum_{n=0}^{\infty}\binom{n+k}{k}[1-x]_q^n\frac{t^{n+k}}{(n+k)!} \\ &= \sum_{n=k}^{\infty}\binom{n}{k}[x]_q^k[1-x]_q^{n-k}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}B_{k,n}(x,q)\frac{t^n}{n!}, \end{aligned} \tag{3.1}$$

which is the generating function of the modified q-Bernstein type polynomials (see [9]). Indeed, this generating function is also treated by Simsek and Acikgoz (see [1]). Note that $\lim_{q\to 1}F_q^{(k)}(t,x) = F^{(k)}(t,x)$. It is easy to show that

$$[1-x]_q^{n-k} = \sum_{m=0}^{\infty}\sum_{l=0}^{n-k}\binom{l+m-1}{m}\binom{n-k}{l}(-1)^{l+m}q^l[x]_q^{l+m}(q-1)^m. \tag{3.2}$$

From (1.6), (2.3), (2.15), and (3.2), we derive the following theorem.

Theorem 3.1. *For $k,n \in \mathbb{Z}_+$ with $n \geq k$,*

$$\int_{\mathbb{Z}_p}\frac{B_{k,n}(x,q)}{\binom{n}{k}}d\mu_{-1}(y) = \sum_{m=0}^{\infty}\sum_{l=0}^{n-k}\binom{l+m-1}{m}\binom{n-k}{l}(-1)^{l+m}q^l(q-1)^m E_{l+m+k,q}, \tag{3.3}$$

where $E_{n,q}$ are the q-Euler numbers.

It is possible to write $[x]_q^k$ as a linear combination of the modified q-Bernstein polynomials by using the degree evaluation formulae and mathematical induction. Therefore, we obtain the following theorem.

Theorem 3.2 (see [8, Theorem 7]). *For $k, n \in \mathbb{Z}_+$, $i \in \mathbb{N}$, and $x \in [0,1]$,*

$$\sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x,q) = [x]_q^i \left([x]_q + [1-x]_q\right)^{n-i}. \tag{3.4}$$

Let $i-1 \le n$. Then from (1.7), (3.2), and Theorem 3.2, we have

$$\begin{aligned}
[x]_q^i &= \frac{\sum_{k=i-1}^{n}\left(\binom{k}{i}\binom{n}{k}/\binom{n}{i}\right)[x]_q^k[1-x]_q^{n-k}}{[x]_q^{n-i}\left(1+\left([1-x]_q/[x]_q\right)\right)^{n-k}} \\
&= \sum_{m=0}^{\infty}\sum_{k=i-1}^{n}\sum_{l=0}^{m+n-k}\sum_{p=0}^{\infty}\frac{\binom{k}{i}\binom{n}{k}}{\binom{n}{i}}\binom{l+p-1}{p}\binom{m+n-k}{l} \\
&\quad\times\binom{n-i+m-1}{m}(-1)^{l+p+m}q^l(q-1)^p[x]_q^{i-n-m+k+p+l}.
\end{aligned} \tag{3.5}$$

Using (2.13) and (3.5), we obtain the following theorem.

Theorem 3.3. *For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$ with $i-1 \le n$,*

$$\begin{aligned}
E_{i,q} &= \sum_{m=0}^{\infty}\sum_{k=i-1}^{n}\sum_{l=0}^{m+n-k}\sum_{p=0}^{\infty}\frac{\binom{k}{i}\binom{n}{k}}{\binom{n}{i}}\binom{l+p-1}{p}\binom{m+n-k}{l} \\
&\quad\times\binom{n-i+m-1}{m}(-1)^{l+p+m}q^l(q-1)^p E_{i-n-m+k+p+l,q}.
\end{aligned} \tag{3.6}$$

The q-String numbers of the first kind is defined by

$$\prod_{k=1}^{n}\left(1+[k]_q z\right) = \sum_{k=0}^{n} S_1(n,k;q) z^k, \tag{3.7}$$

and the q-String number of the second kind is also defined by

$$\prod_{k=1}^{n}\left(1+[k]_q z\right)^{-1} = \sum_{k=0}^{n} S_2(n,k;q) z^k \tag{3.8}$$

(see [9]). Therefore, we deduce the following theorem.

Theorem 3.4 (see [9, Theorem 4]). *For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$,*

$$\frac{\sum_{k=i-1}^{n}\left(\binom{k}{i}/\binom{n}{i}\right)B_{k,n}(x,q)}{\left([x]_q+[1-x]_q\right)^{n-i}} = \sum_{k=0}^{i}\sum_{l=0}^{k} S_1(k,l;q)S_2(k,i-k;q)[x]_q^l. \tag{3.9}$$

By Theorems 3.2 and 3.4 and the definition of fermionic p-adic integrals on $\mathbb{Z}_p$, we obtain the following theorem.

Theorem 3.5. *For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$,*

$$\begin{aligned} E_{i,q} &= \sum_{k=i-1}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}\int_{\mathbb{Z}_p}\frac{B_{k,n}(x,q)}{\left([x]_q+[1-x]_q\right)^{n-i}}d\mu_{-1}(x) \\ &= \sum_{k=0}^{i}\sum_{l=0}^{k} S_1(k,l;q)S_2(k,i-k;q)E_{l,q}, \end{aligned} \tag{3.10}$$

where $E_{i,q}$ is the q-Euler numbers.

Let $i-1 \le n$. It is easy to show that

$$\begin{aligned} &[x]_q^i\left([x]_q+[1-x]_q\right)^{n-i} \\ &\quad= \sum_{l=0}^{n-i}\binom{n-i}{l}[x]_q^{l+i}[1-x]_q^{n-i-l} \\ &\quad= \sum_{l=0}^{n-i}\sum_{m=0}^{n-i-l}\binom{n-i}{l}\binom{n-i-l}{m}(-1)^m q^m [x]_q^{m+i+l}q^{-mx} \\ &\quad= \sum_{l=0}^{n-i}\sum_{m=0}^{n-i-l}\sum_{s=0}^{\infty}\binom{n-i}{l}\binom{n-i-l}{m}\binom{m+s-1}{s}(-1)^m q^m (1-q)^s [x]_q^{m+i+l+s}. \end{aligned} \tag{3.11}$$

From (3.11) and Theorem 3.2, we have the following theorem.

Theorem 3.6. *For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$,*

$$\begin{aligned} \sum_{k=i-1}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_{-1}(x) = {} & \sum_{l=0}^{n-i}\sum_{m=0}^{n-i-l}\sum_{s=0}^{\infty}\binom{n-i}{l}\binom{n-i-l}{m}\binom{m+s-1}{s} \\ & \times (-1)^m q^m (1-q)^s E_{m+i+l+s,q}, \end{aligned} \tag{3.12}$$

where $E_{i,q}$ are the q-Euler numbers.

In the same manner, we can obtain the following theorem.

Theorem 3.7. *For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$,*

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)\,d\mu_{-1}(x) = \sum_{j=k}^{n}\sum_{m=0}^{\infty}\binom{j}{k}\binom{n}{j}\binom{j-k+m-1}{m}(-1)^{j-k+m}q^{j-k}(q-1)^m E_{m+j,q}, \tag{3.13}$$

where $E_{i,q}$ are the q-Euler numbers.

4. Further Remarks and Observations

The q-binomial formulas are known as

$$\begin{aligned}(a;q)_n &= (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-1)^i a^i,\\ \frac{1}{(a;q)_n} &= \frac{1}{(1-a)(1-aq)\cdots(1-aq^{n-1})} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q a^i.\end{aligned} \tag{4.1}$$

For $h \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, and $r \in \mathbb{N}$, we introduce the extended higher-order q-Euler polynomials as follows [16]:

$$E_{n,q}^{(h,r)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j)x_j}[x+x_1+\cdots+x_r]_q^n\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{4.2}$$

Then,

$$\begin{aligned}E_{n,q}^{(h,r)}(x) &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{lx}}{(-q^{h-1+l};q^{-1})_r}\\ &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{lx}}{(-q^{h-r+l};q)_r}.\end{aligned} \tag{4.3}$$

Let us now define the extended higher-order Nörlund type q-Euler polynomials as follows [16]:

$$E_{n,q}^{(h,-r)}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{lx}}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{l(x_1+\cdots+x_r)}q^{\sum_{j=1}^{r}(h-j)x_j}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)}. \tag{4.4}$$

In the special case $x = 0$, $E_{n,q}^{(h,-r)} = E_{n,q}^{(h,-r)}(0)$ are called the extended higher-order Nörlund type q-Euler numbers. From (4.4), we note that

$$E_{n,q}^{(h,-r)}(x) = \frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^{h-r+l};q\right)_r$$
$$= \frac{1}{2^r}\sum_{m=0}^{r} q^{\binom{m}{2}} q^{(h-r)m}\binom{r}{m}_q [m+x]_q^n. \tag{4.5}$$

A simple manipulation shows that

$$q^{\binom{m}{2}}\binom{r}{m}_q = \frac{q^{\binom{m}{2}}[r]_q\cdots[r-m+1]_q}{[m]_q!} = \frac{1}{[m]_q!}\prod_{k=0}^{m-1}\left([r]_q-[k]_q\right),$$
$$\prod_{k=0}^{n-1}\left(z-[k]_q\right) = z^n\prod_{k=0}^{n-1}\left(1-\frac{[k]_q}{z}\right) = \sum_{k=0}^{n} S_1(n-1,k;q)(-1)^k z^{n-k}. \tag{4.6}$$

Formula (4.5) implies the following lemma.

Lemma 4.1. *For $h \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, and $r \in \mathbb{N}$,*

$$E_{n,q}^{(h,-r)}(x) = \frac{1}{2^r[m]_q!}\sum_{m=0}^{r}\sum_{k=0}^{m} q^{(h-r)m} S_1(m-1,k;q)(-1)^k [r]_q^{m-k}[x+m]_q^n. \tag{4.7}$$

From (2.22), we can easily see that

$$[x+m]_q^n = \frac{1}{(1-q)^n}\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}(-1)^{j+l}(1-q)^l q^{mj}[x]_q^l. \tag{4.8}$$

Using (2.13) and (4.8), we obtain the following lemma.

Lemma 4.2. *For $m, n \in \mathbb{Z}_+$,*

$$E_{n,q}(m) = \frac{1}{(1-q)^n}\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}(-1)^{j+l}(1-q)^l q^{mj} E_{l,q}. \tag{4.9}$$

By Lemma 4.2, and the definition of fermionic p-adic integrals on $\mathbb{Z}_p$, we obtain the following theorem.

Theorem 4.3. *For $h \in \mathbb{Z}, n \in \mathbb{Z}_+$, and $r \in \mathbb{N}$,*

$$\begin{aligned}\int_{\mathbb{Z}_p} E_{n,q}^{(h,-r)}(x)\mathrm{d}\mu_{-1}(x) &= \frac{2^{-r}}{[m]_q!}\sum_{m=0}^{r}\sum_{k=0}^{m} q^{(h-r)m} S_1(m-1,k;q)(-1)^k [r]_q^{m-k} E_{n,q}(m) \\ &= \frac{1}{2^r [m]_q!}\sum_{m=0}^{r}\sum_{k=0}^{m} q^{(h-r)m} S_1(m-1,k;q)(-1)^k [r]_q^{m-k} \\ &\quad \times \frac{1}{(1-q)^n}\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}(-1)^{j+l}(1-q)^l q^{mj} E_{l,q}.\end{aligned} \tag{4.10}$$

Put $h=0$ in (4.4). We consider the following polynomials $E_{n,q}^{(0,-r)}(x)$:

$$E_{n,q}^{(0,-r)}(x) = \sum_{l=0}^{n} \frac{(1-q)^{-n}\binom{n}{l}(-1)^l q^{lx}}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{l(x_1+\cdots+x_r)} q^{-\sum_{j=1}^{r} j x_j}\mathrm{d}\mu_{-1}(x_1)\cdots \mathrm{d}\mu_{-1}(x_r)}. \tag{4.11}$$

Then,

$$E_{n,q}^{(0,-r)}(x) = \frac{1}{2^r}\sum_{m=0}^{r}\binom{r}{m} q^{\binom{m}{2}-rm}[m+x]_q^n. \tag{4.12}$$

A simple calculation of the fermionic p-adic invariant integral on $\mathbb{Z}_p$ shows that

$$\int_{\mathbb{Z}_p} E_{n,q}^{(0,-r)}(x)\mathrm{d}\mu_{-1}(x) = \frac{1}{2^r}\sum_{m=0}^{r}\binom{r}{m} q^{\binom{m}{2}-rm} E_{n,q}(m). \tag{4.13}$$

Using Theorem 4.3, we can also prove that

$$\int_{\mathbb{Z}_p} E_{n,q}^{(0,-r)}(x)\mathrm{d}\mu_{-1}(x) = \frac{2^{-r}}{[m]_q!}\sum_{m=0}^{r}\sum_{k=0}^{m} q^{-rm} S_1(m-1,k;q)(-1)^k [r]_q^{m-k} E_{n,q}(m). \tag{4.14}$$

Therefore, we obtain the following theorem.

Theorem 4.4. *For $m \in \mathbb{Z}_+, r \in \mathbb{N}$ with $m \le r$,*

$$\binom{r}{m} q^{\binom{m}{2}-rm} = \frac{1}{[m]_q!}\sum_{k=0}^{m} q^{-rm} S_1(m-1,k;q)(-1)^k [r]_q^{m-k}. \tag{4.15}$$

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