EXTENSION OF THE BEST APPROXIMATION OPERATOR IN ORLICZ SPACES AND WEAK-TYPE INEQUALITIES

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We consider an extension of the best approximation operator from an Orlicz space L^{φ} to the space $L^{\varphi'}$, where φ' denotes the derivative of φ, and we prove a weak-type inequality in this space. Further, we obtain some strong inequalities for suitable L^{ψ} spaces.

1. Introduction

Let φ be a convex function from $[0,\infty)$ into itself such that $\varphi(0)=0$ and $\varphi(x)/x$ tends to zero or infinity when x tends to zero or infinity, respectively. Such a function is called an $\mathcal{N}$-function according to [2]. Given a probability space $(\Omega,\mathcal{A},P)$, let L^{φ} be the space of all $\mathcal{A}$-measurable functions f such that

$$\int \varphi\big(\lambda|f|\big)dP<\infty, \tag{1.1}$$

for some $\lambda>0$. Since we only deal with a Δ_2 function φ, that is, $\varphi(2x)\le K\varphi(x)$ for all $x\ge 0$ and for some constant K, the space L^{φ} can be defined as the space of all $\mathcal{A}$-measurable functions f where (1.1) holds for every positive number λ.

Set $L^{\varphi}(\mathcal{L})$ for the set of $\mathcal{L}$-measurable functions in L^{φ}, where $\mathcal{L}\subseteq\mathcal{A}$ is a σ-lattice, that is, a class of sets containing $\emptyset$ and Ω, which is closed under countable unions and intersections and where $\mathcal{L}$-measurable function means the class of functions $f:\Omega\to\mathbb{R}$ such that $\{f>a\}\in\mathcal{L}$, for all $a\in\mathbb{R}$.

It is well known that for every $f\in L^{\varphi}$ there exists an element $f^*\in L^{\varphi}(\mathcal{L})$ such that

$$\int \varphi\big(\big|f-f^*\big|\big)dP=\inf_{h\in L^{\varphi}(\mathcal{L})}\int \varphi\big(|f-h|\big)dP, \tag{1.2}$$

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we call f^* the best φ-approximation of the function f. If φ is a strictly convex function we have uniqueness for the best φ approximation of the function f.

In Section 2, we set some properties for the best approximation operator in an Orlicz space L^φ, most of them are obtained in a similar way as in the L^p case. Further, we extend the best approximation operator in a monotone continuous way to the space $L^{\varphi'}$, where the space $L^{\varphi'}$ is defined in an analogous way as the space L^φ. Note that the function φ' may be a nonconvex function. For the extended operator f^* we get similar properties to those of the original best approximation operator. In the case $L^\varphi = L^p$, where $1 < p < \infty$, the operator f^*, primarily defined in L^p is extended to L^{p-1} and it is proved that

$$\text{If } f \in L^r, \text{ then } f^* \in L^r, \quad \forall r \geq p-1, \tag{1.3}$$

(see [4, page 209]). A stronger result can be obtained using Theorem 3.4. Indeed, from this theorem we get

$$\left\| f^* \right\|_{L^r} \leq C_r \| f \|_{L^r}, \quad \forall r > p-1. \tag{1.4}$$

For the case $r = p-1$ we obtain, again, the weaker version given in (1.3).

To prove (1.3), the homogeneity property $(\lambda f)^* = \lambda f^*$, for every $\lambda \geq 0$, is used which holds in the L^p case but has no counterpart in the L^φ spaces. In order to obtain similar results for Orlicz spaces, we start with a weak-type inequality for the best approximation operator and we obtain inequalities of the type

$$\int \psi\left(\left|f^*\right|\right) dP \leq C_\psi \int \psi\left(|f|\right) dP, \tag{1.5}$$

for some class of functions ψ. Finally, the main new results of this paper are established in Section 3 where we get inequalities of the type (1.5) in an abstract setup, on some Orlicz spaces, which plays the role of (1.4) for the L^p case.

2. Extension of the best approximation operator and a weak-type inequality

We begin with some auxiliary results. The proof of the next lemma can be found in [2, Theorem 4.1, page 24].

Lemma 2.1. *A necessary and sufficient condition for a differentiable $\mathcal{N}$-function ψ to satisfy the Δ_2 condition is that there exists a constant $\alpha > 1$ such that*

$$u\psi'(u) \leq \alpha \psi(u), \quad \forall u \geq 0. \tag{2.1}$$

From now on, we will always consider a function φ which is a C^1 strictly convex $\mathcal{N}$-function fulfilling the Δ_2 condition. In the assumptions of our results, we just point out the necessary additional conditions on the function φ.

LEMMA 2.2. *Let f, g be in L^φ. Then $\varphi'(|f|)g$ is an integrable function.*

Proof. Let ψ be the complementary function of φ. Then

$$\int |g|\varphi'\big(|f|\big)dP \le \int \varphi\big(|g|\big)dP + \int \psi\big(\varphi'\big(|f|\big)\big)dP. \tag{2.2}$$

But we always have $u\varphi'(u) = \varphi(u) + \psi(\varphi'(u))$, $u \ge 0$. Then, by Lemma 2.1, $u\varphi'(u) \le \alpha\varphi(u)$, for some $\alpha > 1$. Thus $\varphi(u) + \psi(\varphi'(u)) \le \alpha\varphi(u)$, which implies that $\psi(\varphi'(u)) \le (\alpha - 1)\varphi(u)$. Then we get

$$\int |g|\varphi'\big(|f|\big)dP \le \int \varphi\big(|g|\big)dP + (\alpha-1)\int \varphi\big(|f|\big)dP. \tag{2.3}$$

□

LEMMA 2.3. *Let $f \in L^\varphi$ and f^* be the best φ-approximation of f. Then, for every $g \in L^\varphi(\mathcal{L})$,*

$$\int \varphi'\big(\big|f - f^*\big|\big)\big(f^* - g\big)\,\mathrm{sign}\,\big(f - f^*\big)dP \ge 0. \tag{2.4}$$

Proof. In order to obtain (2.4), it is enough to prove that $0 \le dF(\varepsilon)/d\varepsilon\big|_{\varepsilon=0}$, with $F(\varepsilon) = \int_\Omega \varphi(|f - (\varepsilon g + (1-\varepsilon)f^*)|)dP$. By Lemma 2.2, and taking into account the next inequality we can differentiate inside the integral,

$$\begin{aligned}
&\big|\varphi\big(\big|f - \big(\varepsilon g + (1-\varepsilon)f^*\big)\big|\big) - \varphi\big(\big|f - f^*\big|\big)\big| \\
&\quad\le \big[\varphi'\big(\big|f - \big(\varepsilon g + (1-\varepsilon)f^*\big)\big|\big) + \varphi'\big(\big|f - f^*\big|\big)\big]\varepsilon\big|f^* - g\big| \\
&\quad\le \big[\varphi'\big(\big|f - f^*\big| + \big|g - f^*\big|\big)\big|f^* - g\big| + \varphi'\big(\big|f - f^*\big|\big)\big|f^* - g\big|\big]\varepsilon,
\end{aligned} \tag{2.5}$$

and the lemma follows. □

Now we list some properties for the best approximation operator f^* which can be proved in a similar way as done in [4],

$$\int \varphi'\big(\big|f - f^*\big|\big)f^*\,\mathrm{sgn}\,\big(f - f^*\big)dP = 0, \tag{2.6}$$

and using (2.4) we get

$$\int \varphi'\big(\big|f - f^*\big|\big)g\,\mathrm{sgn}\,\big(f - f^*\big)dP \le 0, \quad \forall g \in L^\varphi(\mathcal{L}), \tag{2.7}$$

and also

$$\int \varphi'\big(\big|f - f^*\big|\big)\,\mathrm{sgn}\,\big(f - f^*\big)dP = 0. \tag{2.8}$$

Since φ' is a nondecreasing function, we have

$$y\varphi'\big(|x|\big)\operatorname{sgn} x \geq y\varphi'\big(|x-y|\big)\operatorname{sgn}(x-y), \quad \forall x, y. \tag{2.9}$$

Further, for $y \neq 0$, we have strict inequality in (2.9) since φ' is a strictly increasing function (recall that φ is a strictly convex function).

By (2.6) and (2.9), we have

$$\int \varphi'(f) f^* \operatorname{sgn} f \, dP > 0, \tag{2.10}$$

if f and f^* are nonzero functions.

For a proof of the next statement see [3, Theorem 18, page 227],

$$\text{for } f, g \in L^{\varphi}, \text{ if } f \leq g \text{ then } f^* \leq g^*. \tag{2.11}$$

Using (2.11) and the uniqueness of best approximations, we get

$$\text{if } f_n, f \in L^{\varphi}, \ f_n \nearrow f\big(f_n \searrow f\big) \text{ then } f_n^* \nearrow f^*\big(f_n^* \searrow f^*\big). \tag{2.12}$$

Set $\bar{\mathscr{L}} = \{A/A^c \in \mathscr{L}\}$, the so-called dual σ-lattice of $\mathscr{L}$. Then

$$L^{\varphi}\big(\mathscr{L}\big) = -L^{\varphi}\big(\bar{\mathscr{L}}\big). \tag{2.13}$$

Let $\Psi : \mathbb{R} \to [0, \infty)$ be a Borel measurable function, then for any $f \in L^{\varphi}$ it holds that

$$\int \varphi'\big(\big|f - f^*\big|\big) g \Psi\big(f^*\big) \operatorname{sgn}\big(f - f^*\big) dP \leq 0, \tag{2.14}$$

for all $\mathscr{L}$-measurable function g, if the integral exists.

A proof of (2.14) can be found in [4, page 205], for the L^p case, and the extension to the L^{φ} case is straightforward.

Now we extend the best approximation operator f^* to $L^{\varphi'}$ as in [4, Section 3]. First, observe that if φ satisfies the Δ_2 condition, then the function φ' also fulfill the Δ_2 condition. In fact, by Lemma 2.1 we have

$$\varphi'(2x) \leq \alpha \frac{\varphi(2x)}{2x} \leq \alpha K \frac{\varphi(x)}{x} \leq \alpha K \varphi'(x). \tag{2.15}$$

Now given two functions f and g, we denote $f \vee g$ $(f \wedge g)$ the pointwise maximum (minimum) of the functions. Let $f \in L^{\varphi'}$ and n be a fixed positive number. Thus, we define $(-n \vee f)^*$ as the increasing limit of $((-n \vee f) \wedge m)^*$ as m tends to infinity, (2.12) was used here. The decreasing limit of $(-n \vee f)^*$

as $n \to \infty$ will be, by definition, the extended best approximation operator of f, which will be denoted again by f^*. This operator satisfies the following properties:

$$\text{if } f \in L^{\varphi'}, \text{ then } f^* \in L^{\varphi'}, \tag{2.16}$$

$$\left(f^*\right)^* = f^*, \tag{2.17}$$

$$\text{if } f \le g, \text{ then } f^* \le g^*, \tag{2.18}$$

$$(f+a)^* = f^* + a, \quad a \in \mathbb{R}, \tag{2.19}$$

$$f_n \nearrow f\left(f_n \searrow f\right) \text{ then } f_n^* \nearrow f^*\left(f_n^* \searrow f^*\right). \tag{2.20}$$

Because of (2.20) we say that the operator is monotone continuous. We also have

$$\int \varphi'\left(\left|f - f^*\right|\right) \operatorname{sgn}\left(f - f^*\right) dP = 0, \tag{2.21}$$

$$\int \varphi'\left(\left|f - f^*\right|\right) g \Psi\left(f^*\right) \operatorname{sgn}\left(f - f^*\right) dP \le 0, \tag{2.22}$$

for g and Ψ as in (2.14),

$$\int \varphi'\left(\left|f - f^*\right|\right) \Psi\left(f^*\right) \operatorname{sgn}\left(f - f^*\right) dP = 0, \tag{2.23}$$

if the integral exists and Ψ is as in (2.14).

Theorem 2.4. *Let $f \in L^{\varphi'}$ and let m be a strictly increasing function such that $m(f^*)$ is bounded. Then $g = f^*$ if and only if $g \in L^{\varphi'}(\mathscr{L})$, $m(g)$ is bounded and*

(i) $\int \varphi'(|f-g|) h \operatorname{sgn}(f-g) dP \le 0$, *for all bounded $\mathscr{L}$-measurable functions h,*

(ii) $\int \varphi'(|f-g|) \operatorname{sgn}(f-g) m(g) dP = 0$.

Proof. The proof follows the same pattern as in [4, Theorem 3.4, page 207]. □

Again, as in [4], we can get the following properties.

Given $f \in L^{\varphi'}$, a function $g \in L^{\varphi'}(\mathscr{L})$ is the best approximation f^* if and only if the following two conditions hold:

$$\begin{gathered} \int_C \varphi'(|f-g|) \operatorname{sgn}(f-g) dP \le 0, \quad \forall C \in \mathscr{L}, \\ \int_{\{g \ge a\}} \varphi'(|f-g|) \operatorname{sgn}(f-g) dP = 0, \quad \forall a \in \mathbb{R}. \end{gathered} \tag{2.24}$$

For $D \in \bar{\mathscr{L}}$, $C \in (f^*)^{-1}(\mathscr{B})$, where $\mathscr{B}$ denotes the Borel σ-field in $\mathbb{R}$,

$$\int_{C \cap D} \varphi'\left(\left|f - f^*\right|\right) \operatorname{sgn}\left(f - f^*\right) dP \ge 0. \tag{2.25}$$

Setting $\overline{f^*}$ for the best φ-approximation operator (or its extension) when the approximation class is $L^\varphi(\bar{\mathcal{L}})$, we get the following:

$$
\begin{gathered}
\overline{f^*} = -(-f)^*, \;\; f \in L^{\varphi'}, \\
\left(f - af^*\right)^* = (1-a) f^*, \quad \text{for } a \leq 1, \;\; f \in L^{\varphi'}, \\
\left|f^*\right| \leq \max\left(|f|^*, \overline{|f|^*}\right), \\
\text{for each interval } I \subseteq \mathbb{R}, \;\; f \in I \text{ a.e. implies } f^* \in I \text{ a.e.}
\end{gathered}
\tag{2.26}
$$

The next result is the L^φ version of [4, Lemma 7.2(i) and (ii)].

LEMMA 2.5. *Let $\gamma : \mathbb{R} \to [0, \infty)$ be a nondecreasing function and assume that there exists $c \geq 1$ such that $\gamma(x+y) \leq c(\gamma(x) + \gamma(y))$, for all $x, y \geq 0$. Then*

$$
\gamma(a) + c\gamma\left(|x-a|\right) \operatorname{sgn}(x-a) \leq (c+1)\gamma(x), \quad \forall x, a \geq 0. \tag{2.27}
$$

Proof. We consider two cases. In the first one, let $x \geq a \geq 0$ then as $\gamma(a) \leq \gamma(x)$ and $\gamma(x-a) \leq \gamma(x)$, we get $\gamma(a) + c\gamma(x-a) \leq (c+1)\gamma(x)$.

In the second case, we have $0 \leq x \leq a$ and $\gamma(a) = \gamma(a - x + x) \leq c\gamma(a - x) + c\gamma(x)$. Thus $\gamma(a) - c\gamma(a-x) \leq (c+1)\gamma(x)$. □

If $f \in L^{\varphi'}$ then f^* is also in $L^{\varphi'}$ by (2.16), but we do not know an estimate of the type $\int \varphi'(|f^*|)dP \leq C \int \varphi'(|f|)dP$, with a constant C independent of f. However, the next theorem establishes a weak-type inequality for f^*.

THEOREM 2.6. *If $f \in L^{\varphi'}$ and $f \geq 0$, then*

$$
P\left\{f^* > a\right\} \leq \frac{c+1}{\varphi'(a)} \int_{\{f^* > a\}} \varphi'(f) dP, \quad \forall a > 0. \tag{2.28}
$$

Provided $\varphi'(x+y) \leq c(\varphi'(x) + \varphi'(y))$, $x, y \geq 0$, for some fixed constant $c > 0$.

Proof. For $C = \{f^* > a\}$ and $D = \Omega$ we get from (2.25),

$$
\int_{\{f^* > a\}} \varphi'\left(\left|f - f^*\right|\right) \operatorname{sgn}\left(f - f^*\right) dP \geq 0. \tag{2.29}
$$

Now since $\operatorname{sgn} t \varphi'(|t|)$ is a nondecreasing function, we have

$$
\int_{\{f^* - a > 0\}} \varphi'\left(|f - a|\right) \operatorname{sgn}(f - a) dP \geq 0. \tag{2.30}
$$

By Lemma 2.5, applied to φ', we have

$$\varphi'(a)+c\varphi'\big(|f-a|\big)\operatorname{sgn}(f-a)\le(c+1)\varphi'(f). \tag{2.31}$$

And now if we integrate (2.31), we get

$$\begin{aligned}(c+1)&\int_{\{f^*>a\}}\varphi'(f)dP\\ &\ge\varphi'(a)P\big(\{f^*>a\}\big)+c\int_{\{f^*>a\}}\varphi'\big(|f-a|\big)\operatorname{sgn}(f-a)dP,\end{aligned} \tag{2.32}$$

and using (2.30) the last integral is greater than $\varphi'(a)P\{f^*>a\}$. □

We stated Theorem 2.6 only for a nonnegative function f. Now it is easy to obtain a version of this theorem for general functions. In fact, given $f\in L^{\varphi'}$, by the third inequality in (2.26) we have

$$P\big\{|f|^*>2a\big\}\le P\big\{|f|^*+\overline{|f|^*}>2a\big\}\le P\big\{|f|^*>a\big\}+P\big\{\overline{|f|^*}>a\big\}. \tag{2.33}$$

Thus, using (2.28), we obtain

$$P\big\{\big|f^*\big|>2a\big\}\le\frac{c+1}{\varphi'(a)}\int_{\{|f|^*>a\}}\varphi'\big(|f|\big)dP+\frac{c+1}{\varphi'(a)}\int_{\{\overline{|f|^*}>a\}}\varphi'\big(|f|\big)dP. \tag{2.34}$$

We call weak-type inequalities to those given in (2.28) or more generally in (2.34). It is important to point out here the difference between this sort of weak type inequalities and the classical ones given for example in [5].

3. A strong inequality for the extension of the best approximation operator

Lemma 3.1. *Let $(\Omega,\mathcal{A},P)$ be a probability space and let $\xi,\eta:\Omega\to[0,\infty)$ be two measurable functions such that*

$$P\{\eta>\varepsilon\}\le\frac{1}{\varepsilon}\int_{\{\eta>\varepsilon\}}\xi\,dP,\quad\forall\varepsilon>0. \tag{3.1}$$

Then

$$\|\eta\|_p\le\frac{p}{p-1}\|\xi\|_p, \tag{3.2}$$

for $1<p<\infty$, and $\|\eta\|_\infty\le\|\xi\|_\infty$.

A proof of Lemma 3.1 can be found in [1, Lemma 6.6.9, page 231].

In particular if the function η of Lemma 3.1 is given by an operator, say $\eta=T\xi$, the inequality (3.1) implies

$$P\big\{|T\xi|>\alpha\big\}\le\frac{1}{\alpha}\int|\xi|dP,\quad\forall\xi,\alpha>0. \tag{3.3}$$

Now by taking $p \to \infty$, we get, from Lemma 3.1, the next strong inequality

$$\|T\xi\|_\infty \le \|\xi\|_\infty, \quad \forall \xi. \tag{3.4}$$

Conversely, if subadditive operator T satisfies the weak inequality (3.3) and the strong inequality (3.4), then, as it is easily seen, we have an inequality of the type (3.1), indeed

$$P\big\{|T\xi| > \alpha\big\} \le \frac{2}{\alpha} \int_{\{|\xi|>\alpha/2\}} |\xi| dP, \quad \forall \xi, \alpha > 0. \tag{3.5}$$

In the classical analysis there are many sublinear operator T fulfilling $|T(\xi)| \ge |\xi|$, almost everywhere, for example, the Hardy-Littlewood maximal function (see [5]). In this case the last inequality implies a weak-type inequality as the one given in (2.28), with $\varphi'(t) = t$.

The best approximation operators, in general, are not subadditive, so in dealing with them it is important to prove inequalities of the type (3.1) rather than the standard weak inequality (3.3).

Now we consider some extensions of Lemma 3.1 to Orlicz spaces. Instead of (3.1) we assume the next weak-type inequality

$$P\{\eta > \varepsilon\} \le \frac{c}{\varphi'(\varepsilon)} \int_{\{\eta>\varepsilon\}} \varphi'(\xi) dP, \tag{3.6}$$

for every $\varepsilon > 0$ and ξ, η two nonnegative fixed functions. Then we search for inequalities of the type

$$\int \psi(\eta) dP \le \mathscr{C} \int \psi(\xi) dP, \tag{3.7}$$

for suitable functions ψ, with a constant $\mathscr{C}$ independent of ξ and η.

Note 3.2. It is enough to prove (3.7) in the case $\int \psi(\eta) dP < \infty$.

Otherwise set $\eta_k = \eta \wedge k$, then

$$\big\{\eta_k > \varepsilon\big\} = \{\eta > \varepsilon\}, \quad \text{if } k > \varepsilon, \qquad \big\{\eta_k > \varepsilon\big\} = \emptyset, \quad \text{if } k \le \varepsilon. \tag{3.8}$$

In any case, (3.6) holds for the pair of functions η_k, ξ, for every k. Thus if (3.7) is proved for the pair η_k, ξ we get the inequality for the functions η, ξ by the classical Fatou's lemma. We have to assume that ψ is a continuous function.

In the next theorem we obtain a version of (3.7) where the function ψ is equal to φ.

THEOREM 3.3. *Let φ be such that its complementary function N satisfies the Δ_2 condition. If the nonnegative functions η and ξ satisfy (3.6), then*

$$\int \varphi(\eta) dP \le \mathscr{C} \int \varphi(\xi) dP, \tag{3.9}$$

where the constant $\mathscr{C}$ depends on c of (3.6) and on the constants of the Δ_2 conditions of the functions φ and N.

Proof. Set N for the complementary function of φ and assume the Δ_2 condition on both functions N and φ. Then by (3.6)

$$\begin{aligned}\int \varphi(\eta)dP &= \int_0^\infty \varphi'(t)P\{\eta>t\}dt \le \int_0^\infty \varphi'(t)\frac{c}{\varphi'(t)}\int_{\{\eta>t\}}\varphi'(\xi)dP\,dt\\ &= c\int \eta\varphi'(\xi)dP \le c\varepsilon\int\varphi(\eta)dP + c\int N\left(\frac{1}{\varepsilon}\varphi'(\xi)\right)dP.\end{aligned} \tag{3.10}$$

In the last inequality we have used $uv \le \varphi(\varepsilon u)+N((1/\varepsilon)v)$.

If $\bar{K}$ is the Δ_2 constant for the function N, then

$$\int N\left(\frac{1}{\varepsilon}\varphi'(\xi)\right)dP \le \bar{K}^l\int N\big(\varphi'(\xi)\big)dP, \tag{3.11}$$

where $2^{l-1} \le 1/\varepsilon < 2^l$.

Since φ is a positive C^1 strictly convex function it follows that φ' has an inverse. Therefore by using [2, Theorem 4.3, page 27], it holds that the function φ satisfies a Δ_2 condition if and only if

$$N\big(\varphi'(u)\big) \le \frac{1}{\alpha_1}u\varphi'(u), \tag{3.12}$$

for every $u \ge 0$. Thus, choosing $\varepsilon = 1/2c$ and taking into account that $u\varphi'(u) \le \varphi(2u) \le K\varphi(u)$, we get

$$\int\varphi(\eta)dP \le \frac{2c\bar{K}^lK}{\alpha_1}\int\varphi(\xi)dP. \tag{3.13}$$

$\square$

For the next theorem the function ψ of (3.7) is $(\varphi')^p$ with $p>1$.

THEOREM 3.4. *Let φ be a C^2 function. If the nonnegative functions η and ξ satisfy (3.6), then*

$$\left(\int\left|\varphi'(\eta)\right|^p dP\right)^{1/p} \le c\frac{p}{p-1}\left(\int\left|\varphi'(\xi)\right|^p dP\right)^{1/p}, \quad \forall p>1. \tag{3.14}$$

Proof. By (3.6), we have

$$\int\psi(\eta)dP = \int_0^\infty \psi'(t)P\{\eta>t\}dt \le c\int\varphi'(\xi)\left(\int_0^\eta \frac{\psi'(t)}{\varphi'(t)}dt\right)dP, \tag{3.15}$$

but in this case

$$\int_0^\eta \frac{\psi'(t)}{\varphi'(t)}dt = p\int_0^\eta (\varphi')^{p-2}(t)\varphi''(t)dt = \frac{p}{p-1}(\varphi')^{p-1}(\eta), \tag{3.16}$$

thus we get

$$\int \psi(\eta)dP \le c\frac{p}{p-1}\int \varphi'(\xi)(\varphi')^{p-1}(\eta)dP. \tag{3.17}$$

Now, using the Hölder inequality, the last integral is less than or equal to

$$c\frac{p}{p-1}\left(\int \left|\varphi'(\xi)\right|^p dP\right)^{1/p}\left(\int \left|\varphi'(\eta)\right|^p dP\right)^{(p-1)/p}. \tag{3.18}$$

By Note 3.2 it is enough to consider $\int |\varphi'(\eta)|^p dP < \infty$. This completes the proof. □

In order to prove the next theorem, we need some auxiliary results. Moreover, from now on we set $C\varphi$ for the complementary function of φ. Recall that we are dealing with C^1 strictly convex functions. Then its complementary $C\varphi$ is also a C^1 strictly convex function.

When φ is a strictly convex function so is φ^p, for $p \ge 1$. Sometimes it will be needed to look for $p > 0$ such that φ^p is a strictly convex function.

Lemma 3.5. *Let $p > 0$, $M = (C\varphi)^p$ be a strictly convex function and set $N = CM$, then for any constant $k > 0$, it holds that*

$$\frac{k\varphi^p}{H^{-1}\big(k^{1/p}\varphi\big)} \le N^{-1}\big(k\varphi^p\big), \tag{3.19}$$

where $H = C\varphi$.

Proof. Because of [2, page 13], we know that $N(M(u)/u) \le M(u)$, for all u. Then, for $v = M(u)$, we have $N(v/M^{-1}(v)) \le v$. Thus $v \le M^{-1}(v)N^{-1}(v)$.

Taking into account that $M^{-1}(v) = H^{-1}(v^{1/p})$, we get

$$k\varphi^p \le M^{-1}\big(k\varphi^p\big)N^{-1}\big(k\varphi^p\big) = H^{-1}\big(k^{1/p}\varphi\big)N^{-1}\big(k\varphi^p\big). \tag{3.20}$$

□

Lemma 3.6. *Let (M,N) be a pair of complementary $\mathcal{N}$-functions, where M is a C^1 strictly convex function. Given k and p, let s be such that $s-1 > k^{1/p}$, then*

$$k^{1/p}M(u) \le N\left(s\frac{M(u)}{u}\right), \quad \forall u. \tag{3.21}$$

Proof. In the inequality, $uv \le M(u) + N(v)$, we set $v = N^{-1}((k-r)x)$, $u = M^{-1}(rx)$, where $0 < r < k$, and we get $M^{-1}(rx)N^{-1}((k-r)x) \le kx$.

Then, for $x = M(u)/r$, the inequality (3.21) yields $uN^{-1}((k-r)(M(u)/r)) \le (k/r)M(u)$, thus, for $s = k/r$, it holds that $N^{-1}((s-1)M(u)) \le s(M(u)/u)$, which can be written as $(s-1)M(u) \le N(s(M(u)/u))$, and taking now into account that $k^{1/p} < s-1$, we get the lemma. □

LEMMA 3.7. *Let $M = (C\varphi)^p$ be a strictly convex function, $p > 0$ and let N be its complementary function. Then, if $k^{1/p} < s-1$, it holds that*

$$N\big(u\varphi^{p-1}(u)\big) \le N\left(\frac{s}{k}N^{-1}\big(k\varphi^p(u)\big)\right). \tag{3.22}$$

Proof. By Lemma 3.6 with $M = \varphi$ and $N = C\varphi$ we have $k^{1/p}\varphi(u) \le (C\varphi)(s(\varphi(u)/u))$. Thus, for $H = C\varphi$, we get $H^{-1}(k^{1/p}\varphi(u)) \le s(\varphi(u)/u)$, and so $u\varphi^{p-1}(u)H^{-1}(k^{1/p}\varphi(u)) \le s\varphi^p(u)$. Then the lemma follows using Lemma 3.5. □

In particular, with the hypothesis of Lemma 3.7 and setting $k = 1$ and $s = 3$, we have

$$N\big(u\varphi^{p-1}(u)\big) \le N\big(3N^{-1}\big(\varphi^p(u)\big)\big). \tag{3.23}$$

Now we get an inequality of the type (3.7) with $\psi = \varphi^p$, where the positive number p is not necessarily greater than 1.

THEOREM 3.8. *Let φ be such that its complementary function $C\varphi$ satisfies a Δ_2 condition with constant $\bar{K}$. If the nonnegative functions η and ξ satisfy (3.6), then*

$$\int \varphi^p(\eta)\,dP \le \mathscr{C}\int \varphi^p(\xi)\,dP, \tag{3.24}$$

where the constant $\mathscr{C}$ depends on K, $\bar{K}$, p, and c of (3.6). Furthermore, if $p < 1$ assume that $(C\varphi)^p$ is a strictly convex function and $K^{1-p} < 2$.

Proof. We have again

$$\int \varphi^p(\eta)\,dP = \int_0^\infty p\varphi^{p-1}(t)\varphi'(t)P\{\eta > t\}\,dt, \tag{3.25}$$

which is bounded, using (3.6), by

$$cp\int_0^\infty \frac{\varphi^{p-1}(t)\varphi'(t)}{\varphi'(t)}\left(\int_{\{\eta>t\}}\varphi'(\xi)\,dP\right)dt = cp\int \varphi'(\xi)\left(\int_0^\eta \varphi^{p-1}(t)\,dt\right)dP. \tag{3.26}$$

Thus,

$$\int \varphi^p(\eta)dP \le c\int \varphi'(\xi)\left(\int_0^\eta p\varphi^{p-1}(t)dt\right)dP. \tag{3.27}$$

If $p \ge 1$, the last integral is bounded by

$$cp\int \varphi'(\xi)\varphi^{p-1}(\eta)\eta\, dP. \tag{3.28}$$

For $p < 1$, we also obtain a similar inequality. In fact

$$\int_0^\eta p\varphi^{p-1}(t)dt = \sum_{l=0}^{\infty}\int_{\eta/2^{l+1}}^{\eta/2^l} p\varphi^{p-1}(t)dt \le p\sum_{l=0}^{\infty}\frac{\eta}{2^{l+1}}\varphi^{p-1}\left(\frac{\eta}{2^{l+1}}\right). \tag{3.29}$$

Since $\varphi(2t) \le K\varphi(t)$, for every t, also $\varphi^{p-1}(t) \le K^{(l+1)(1-p)}\varphi^{p-1}(2^{l+1}t)$, we have the last sum bounded by

$$p\sum_{l=0}^{\infty}\frac{\eta}{2^{l+1}}\left(K^{1-p}\right)^{l+1}\varphi^{p-1}(\eta) = p\eta\varphi^{p-1}(\eta)\sum_{l=0}^{\infty}\left(\frac{K^{1-p}}{2}\right)^{l+1}. \tag{3.30}$$

Then, given the constant K of the Δ_2 condition on φ we allow those p fulfilling $K^{1-p} < 2$.

Set

$$b = \begin{cases} c, & \text{if } p \ge 1, \\ \displaystyle\sum_{l=0}^{\infty}\left(\frac{K^{1-p}}{2}\right)^{l+1}, & \text{if } 0 < p < 1. \end{cases} \tag{3.31}$$

Thus, in any case we have

$$\int \varphi^p(\eta)dP \le bp\int \varphi'(\xi)\eta\varphi^{p-1}\, dP. \tag{3.32}$$

For $p < 1$ we assume $(C\varphi)^p$ is a strictly convex function and we set $M = (C\varphi)^p$ and $N = CM$. Since (M,N) is a pair of complementary functions, we have $uv \le M(u) + N(v)$, and so we obtain

$$\int \varphi^p(\eta)dP \le pb\int \left[M\left(\frac{1}{\varepsilon}\varphi'(\xi)\right) + N\left(\varepsilon\eta\varphi^{p-1}(\eta)\right)\right]dP = I_1 + I_2. \tag{3.33}$$

For the integral I_2 we have, for $0 < \varepsilon \le 1$,

$$I_2 = pb\int N\left(\varepsilon\eta\varphi^{p-1}(\eta)\right)dP \le \varepsilon pb\int N\left(\eta\varphi^{p-1}(\eta)\right)dP, \tag{3.34}$$

which is bounded, because of (3.23), by

$$\varepsilon pb\int N\left(3N^{-1}\left(\varphi^p(\eta)\right)\right)dP \le \varepsilon pbK_1^2\int N\left(N^{-1}\left(\varphi^p(\eta)\right)\right)dP, \tag{3.35}$$

where K_1 is the constant of the Δ_2 condition on N. Thus we get

$$I_2 \leq \varepsilon pbK_1^2 \int \varphi^p(\eta)dP. \tag{3.36}$$

On the other hand, we have

$$I_1 = pb\int M\Big(\frac{1}{\varepsilon}\varphi'(\xi)\Big)dP \leq \bar{K}^{pl}pb\int M\big(\varphi'(\xi)\big)dP, \tag{3.37}$$

where $\bar{K}$ is the constant that appears in the Δ_2 condition on M, and l is choosen in such a way that $2^{l-1} \leq 1/\varepsilon < 2^l$, so $M((1/\varepsilon)x) \leq \bar{K}^{pl}M(x)$.

Taking into account $v\varphi'(v) = (C\varphi)(\varphi'(v)) + \varphi(v)$, we have

$$(C\varphi)\big(\varphi'(v)\big) \leq v\varphi'(v) \leq \int_v^{2v} \varphi'(t)dt \leq \varphi(2v). \tag{3.38}$$

Since φ is Δ_2, we have

$$(C\varphi)\big(\varphi'(x)\big) \leq K\varphi(x). \tag{3.39}$$

Using (3.39) we get $I_1 \leq \bar{K}^{pl}pbK^p\int\varphi^p(\xi)dP$. Now, choosing ε such that $\varepsilon pbK_1^2 < 1/2$, we finally have

$$\int \varphi^p(\eta)dP \leq 2pbK^p\bar{K}^{pl}\int\varphi^p(\xi)dP. \tag{3.40}$$

□

Remark 3.9. The constant $\mathscr{C}$ in Theorem 3.8, or more explicitly, $2\bar{K}^{pl}pbK^p$, certainly is unbounded for large p. For small values of p the constant b may increase. In fact if we assume that there exists $p < 1$ such that φ^p is a convex function, then it is easy to see, at least, for φ a C^2 convex function, that the set $\{p \mid \varphi^p$ is a convex function and $K^{1-p} > 2\}$ is an infinite interval starting at $p_0 > 0$. Now the constant b tends to infinity as p tends to p_0 whenever $K^{1-p_0} = 2$.

Finally, note that, the abstract setting given by Theorems 3.4 and 3.8, can be applied to estimate $\int\psi(f^*)dP$ in terms of $\int\psi(f)dP$ by virtue of the weak-type inequality given by Theorem 2.6.

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