ON PERIODIC SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

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Let A be a closed linear operator on a Banach space E. We study periodic solutions of the differential equation $d^N u(t)/dt^N = Au(t)$ with an arbitrary integer $N \geq 1$.

Let E be a Banach space and let A be a closed linear operator on E with domain $D(A)$ (not necessarily dense in E). Given a number $T > 0$ and an integer $N \geq 1$, we consider the problem

$$\frac{d^N}{dt^N}u(t) = Au(t), \quad u(t+T) = u(t), \quad -\infty < t < \infty. \tag{1}$$

A function $u : \mathbb{R} \to E$ is a *classical solution* of (1) if $u \in C^N(\mathbb{R}, E)$, $u(t) \in D(A)$ for $-\infty < t < \infty$, and (1) is satisfied. Each classical solution of (1) is a smooth T-periodic function on $\mathbb{R}$. Therefore, (1) is called a *periodic problem*.

The equivalent boundary value problem on the finite interval $[0, T]$ is

$$\begin{gathered} \frac{d^N}{dt^N}u(t) = Au(t), \quad 0 \leq t \leq T, \\ u^{(j)}(0) = u^{(j)}(T), \quad j = 0, \dots, N-1, \end{gathered} \tag{2}$$

where classical solutions belong to $C^N([0, T], E)$. We deal mainly with problem (1) taking into account (2).

These problems and their modifications (cf. (30) below) have been treated often for the case $N = 1$ under some special assumptions. We refer to several examples in the literature:

(i) A is bounded [2].

(ii) A is the infinitesimal generator of a C_0 semigroup [5, 8].

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(iii) E is a Hilbert space and A is normal [11].
(iv) A is a differential operator of a prescribed structure [3, 9, 11].

There are more general problems of such kind, see for instance [4, 6, 9, 11]. Extensive bibliography of the subject can be found in [11]. Arguments of the periodicity are very useful also for the spectral theory of C_0 semigroups [1, 8].

The aim of this paper is to show that the periodic problem (1) may be studied by elementary approach, without loss of generality. It is clear that (1) has always the *trivial solution* $u(t) \equiv 0$. We first consider the question: are there other classical solutions? A result similar to Theorem 1 is given in [6, page 65] for the case $N = 1$ (see also [4, Theorem 4.3]).

THEOREM 1. *Let A be a closed linear operator on a Banach space E. Then the periodic problem (1) has exactly one classical solution $u(t) \equiv 0$ if and only if none of the numbers $\lambda_k = (2\pi ik/T)^N$, $k \in \mathbb{Z}$, is an eigenvalue of A.*

Proof. If $Af_k = \lambda_k f_k$ for some $\lambda_k = (2\pi ik/T)^N$ with eigenvector $f_k \neq 0$ then the function $u(t) = \exp(2\pi ikt/T) f_k$ is a nontrivial classical solution of (1). In the case of a real E and an eigenvalue λ_k with $k \neq 0$ the corresponding real solution of (1) is obtained by the real (or the imaginary) part of the function $u(t) = \exp(2\pi ikt/T) f_k$.

Assume now that none of the $\lambda_k = (2\pi ik/T)^N$, $k \in \mathbb{Z}$, is an eigenvalue of A. Let $u(t)$ be a classical solution of (1). Its Fourier coefficients are defined by

$$f_k = \frac{1}{T}\int_0^T u(t)e^{-2\pi ikt/T}\,dt. \tag{3}$$

Then $f_k \in D(A)$ and

$$\begin{aligned} Af_k &= \frac{1}{T}\int_0^T Au(t)e^{-2\pi ikt/T}\,dt = \frac{1}{T}\int_0^T u^{(N)}(t)e^{-2\pi ikt/T}\,dt \\ &= \frac{1}{T}\left(\frac{2\pi ik}{T}\right)^N \int_0^T u(t)e^{-2\pi ikt/T}\,dt = \lambda_k f_k. \end{aligned} \tag{4}$$

By the assumption we obtain $f_k = 0$ for all $k \in \mathbb{Z}$, that is,

$$\int_0^T u(t)e^{-2\pi ikt/T}\,dt = 0, \quad k \in \mathbb{Z}. \tag{5}$$

For a functional $f^* \in E^*$ the scalar T-periodic function $f^*(u(t))$ is continuous on $\mathbb{R}$ and orthogonal to all $\exp(2\pi ikt/T)$ on $[0, T]$. Hence $f^*(u(t)) \equiv 0$ for $-\infty < t < \infty$. By the Hahn-Banach theorem $u(t) \equiv 0$. □

Remark 2. The proof remains true under the assumptions that E is a sequentially complete locally convex space and A is a sequentially closed linear operator on

E (i.e., from $\{g_n\} \subset D(A)$, $g_n \to g$, and $Ag_n \to h$, it follows that $g \in D(A)$ and $Ag = h$).

Thus, if none of the $\lambda_k = (2\pi ik/T)^N$, $k \in \mathbb{Z}$, is an eigenvalue of A then a solution of the periodic problem (1) must be trivial. Suppose now that there exist eigenvalues of A among the numbers λ_k. It is clear that the function

$$u(t) = u\big(t; f_k\big) \equiv \exp\left(\frac{2\pi ikt}{T}\right) f_k, \quad \text{with } Af_k = \left(\frac{2\pi ik}{T}\right)^N f_k, \tag{6}$$

is a classical solution of (1) (nontrivial if $f_k \neq 0$). We will show that *every* classical solution of (1) can be expressed by linear combinations of the *elementary solutions* (6). The next result implies also Theorem 1.

THEOREM 3. *Let A be a closed linear operator on a Banach space E and let $u(t)$ be a classical solution of the periodic problem (1). Then $u(t)$ is represented by the Fourier series*

$$u(t) = \sum_{k=-\infty}^{\infty} e^{2\pi ikt/T} f_k \tag{7}$$

with elements $f_k \in D(A)$ such that $Af_k = (2\pi ik/T)^N f_k$. The series (7) converges to $u(t)$ in the norm of E uniformly on $\mathbb{R}$, that is, for every $\varepsilon > 0$ there exists an integer n_ε such that

$$\left\| u(t) - \sum_{k=-n}^{n} e^{2\pi ikt/T} f_k \right\| < \varepsilon \tag{8}$$

for $n > n_\varepsilon$ and $-\infty < t < \infty$.

Proof. For $k \in \mathbb{Z}$ we define f_k by (3). Since $u(t)$ is a smooth T-periodic function on $\mathbb{R}$, its Fourier series (7) converges in E to $u(t)$ uniformly on $\mathbb{R}$. The relations $Af_k = (2\pi ik/T)^N f_k$ follow from (4). □

Remark 4. The uniform convergence of the Fourier series can be shown in the usual way using Dirichlet integral. The proof in detail is given in [10] for the case of vector T-periodic Hölder continuous functions.

Remark 5. Let $N \geq 2$ and $u(t)$ be a classical solution of (1). Since

$$f_k = \frac{1}{T}\int_0^T u(t)e^{-2\pi ikt/T}\,dt = \frac{1}{T}\left(\frac{T}{2\pi ik}\right)^2 \int_0^T u''(t)e^{-2\pi ikt/T}\,dt, \tag{9}$$

we have $\|f_k\| = O(1/k^2)$, and the Fourier series (7) converges absolutely, that is,

$$\sum_{k=-\infty}^{\infty} \|f_k\| < \infty. \tag{10}$$

This fact yields an elementary proof of the uniform convergence of (7) in the case $N \geq 2$. However, for $N = 1$ the Fourier series (7) may be only conditionally convergent in contrast to the classical numerical case (cf. [12, Chapter VI.3, Theorem 3.8]).

Example 6. Let $E = c_0$ and $Ax = (ix_1, 2ix_2, 3ix_3, \dots)$ with domain

$$D(A) = \{(x_1, x_2, x_3, \dots) \in c_0 : kx_k \longrightarrow 0 \text{ as } k \longrightarrow \infty\}. \tag{11}$$

For the equation $du/dt = Au(t)$ we consider the 2π-periodic classical solution

$$u(t) = \left(0, \frac{e^{2it}}{2\log 2}, \frac{e^{3it}}{3\log 3}, \dots\right) = \sum_{k=2}^{\infty} e^{ikt} f_k, \tag{12}$$

where

$$f_k = \left(\dots 0, \frac{1}{k\log k}, 0 \dots\right). \tag{13}$$

The Fourier series converges in c_0 uniformly on $\mathbb{R}$, but $\sum \|f_k\| = \sum (k\log k)^{-1} = \infty$.

Theorems 1 and 3 point out the main property of classical periodic solutions that their Fourier coefficients (3) satisfy the relations $Af_k = (2\pi ik/T)^N f_k$, $k \in \mathbb{Z}$. We now define a weak solution of the periodic problem (1) as follows.

Definition 7. A function $u : \mathbb{R} \to E$ is a *weak solution* of (1) if $u \in C(\mathbb{R}, E)$, $u(t+T) = u(t)$ for $-\infty < t < \infty$, and its Fourier coefficients (3) satisfy the relations $f_k \in D(A)$, $Af_k = (2\pi ik/T)^N f_k$ for all $k \in \mathbb{Z}$.

Clearly, if none of the numbers $\lambda_k = (2\pi ik/T)^N$, $k \in \mathbb{Z}$, is an eigenvalue of A then the periodic problem (1) has exactly one weak solution $u(t) \equiv 0$ (which is the unique classical solution, too). In the general case, a weak solution of (1) is a continuous T-periodic function $u : \mathbb{R} \to E$ with the Fourier series

$$u(t) \sim \sum_{k=-\infty}^{\infty} e^{2\pi ikt/T} f_k \tag{14}$$

consisting of the elementary solutions (6). Note that this Fourier series may be divergent in E for each $t \in \mathbb{R}$.

Example 8. Let $E = \mathrm{BU}(\mathbb{R})$ be the Banach space of bounded uniformly continuous complex-valued functions on $\mathbb{R}$ with the supremum norm. Define $Af = f'$ on $D(A) = \{f \in E : f' \text{ exists and } f' \in E\}$. It is well known that there exists a scalar 2π-periodic function $\varphi(\tau)$ which is continuous on $\mathbb{R}$ but its Fourier series diverges in some points $\tau \in \mathbb{R}$ (see [12, Chapter VIII.1]). Choosing such a function $\varphi(\tau)$ with the Fourier series $\sum_{-\infty}^{\infty} c_k e^{ik\tau}$ we set $U(x,t) = \varphi(x+t)$ for $-\infty < x, t < \infty$. Then $U(\cdot,t) \in E$ for each $t \in \mathbb{R}$ and $u(t) \equiv U(\cdot,t)$ is a vector function of $\mathbb{R}$ to E. Evidently, $u \in C(\mathbb{R}, E)$ and $u(t+T) = u(t)$ with $T = 2\pi$. Computing for $T = 2\pi$ the vector Fourier coefficients of $u(t)$ we have

$$\begin{aligned} f_k &= \frac{1}{2\pi}\int_0^{2\pi} u(t)e^{-ikt}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x+t)e^{-ikt}\,dt \\ &= \frac{1}{2\pi}\int_x^{2\pi+x} \varphi(\tau)e^{-ik(\tau-x)}d\tau = \frac{1}{2\pi}e^{ikx}\int_x^{2\pi+x} \varphi(\tau)e^{-ik\tau}d\tau \\ &= c_k e^{ikx}, \end{aligned} \tag{15}$$

whence $f_k \in D(A)$ and $Af_k = ikf_k$. Accordingly, the function $u(t)$ is a weak solution of 2π-periodic problem for the equation $du/dt = Au(t)$ in the space $E = \mathrm{BU}(\mathbb{R})$. The Fourier series of $u(t)$ has the form

$$u(t) \sim \sum_{k=-\infty}^{\infty} e^{ikt} f_k = \sum_{k=-\infty}^{\infty} e^{ikt}\big(c_k e^{ikx}\big) = \sum_{k=-\infty}^{\infty} c_k e^{ik(x+t)}. \tag{16}$$

This series diverges in $E = \mathrm{BU}(\mathbb{R})$ for each $t = t_0$ since $\sum_{-\infty}^{\infty} c_k e^{ik(x+t_0)}$ is the Fourier series of the function $\varphi(x+t_0)$.

So, if $u(t)$ is a weak solution of (1) then the sequence of the partial sums

$$s_n(t) \equiv \sum_{k=-n}^{n} e^{2\pi ikt/T} f_k, \quad f_k = \frac{1}{T}\int_0^T u(\tau)e^{-2\pi ik\tau/T}d\tau, \tag{17}$$

may be divergent in E for each $t \in \mathbb{R}$. But we can always approximate a weak solution $u(t)$ by means of the Fejer sums

$$\sigma_n(t) \equiv \frac{1}{n}\big(s_0(t) + s_1(t) + \cdots + s_{n-1}(t)\big) = \frac{1}{T}\int_0^T K_n(t-\tau)u(\tau)d\tau, \tag{18}$$

where

$$K_n(t) \equiv \frac{1}{n}\left(\frac{\sin(\pi nt/T)}{\sin(\pi t/T)}\right)^2. \tag{19}$$

The positivity of the Fejer kernels $K_n(t)$ allows to adapt for a vector case the usual proof of the fact that $\sigma_n(t) \to u(t)$ as $n \to \infty$ uniformly on $\mathbb{R}$

(see [12, pages 84–92]). Thus for every $\varepsilon > 0$ there exists an integer $n_\varepsilon > 0$ such that $\|u(t)-\sigma_n(t)\| < \varepsilon$ for $n > n_\varepsilon$ and $-\infty < t < \infty$.

We now consider the question of the connection between weak and classical solutions of the periodic problem (1). This verifies our definition of a weak solution. Evidently, each classical solution of (1) is a weak solution, too. A more general assertion is the following.

THEOREM 9. *If $u(t)$ is a classical solution of the periodic problem (1) then the functions $u(t)$, $u'(t)$, $\ldots$, $u^{(N)}(t)$ are also weak solutions. Conversely, an arbitrary weak solution of (1), say $\tilde{u}(t)$, is represented in the form $\tilde{u}(t) = u^{(N)}(t) + \tilde{f}_0$, where $u(t)$ is a classical solution of (1) and $\tilde{f}_0$ is the zero Fourier coefficient of $\tilde{u}(t)$, that is, an element of $D(A)$ such that $A\tilde{f}_0 = 0$. In other words, each weak solution of (1) is the Nth derivative of a classical solution up to an addition of an element from the kernel of A. The Fourier coefficients of these solutions $\tilde{u}(t)$ and $u(t)$ are related as follows:*

$$\tilde{f}_k = \left(\frac{2\pi ik}{T}\right)^N f_k \quad \text{for } k \neq 0, \tag{20}$$

and for $k = 0$ the connection between $\tilde{f}_0$ and f_0 is missing (so it can be defined at will).

Proof. Let $u(t)$ be a classical solution of (1). Given $m \in \{1,\ldots,N\}$, we consider the function $u^{(m)}(t)$. Clearly, $u^{(m)} \in C(\mathbb{R},E)$ and $u^{(m)}(t+T) = u^{(m)}(t)$ for $-\infty < t < \infty$. Computing the Fourier coefficients of $u^{(m)}(t)$ we have

$$\begin{aligned} f_k^{(m)} &\equiv \frac{1}{T}\int_0^T u^{(m)}(t)e^{-2\pi ikt/T}\,dt \\ &= \frac{1}{T}\left(\frac{2\pi ik}{T}\right)^m \int_0^T u(t)e^{-2\pi ikt/T}\,dt = \left(\frac{2\pi ik}{T}\right)^m f_k. \end{aligned} \tag{21}$$

The elements f_k are the Fourier coefficients of the classical solution $u(t)$. Since $f_k \in D(A)$ and $Af_k = (2\pi ik/T)^N f_k$, the coefficients $f_k^{(m)}$ satisfy the same relations. Thus $u^{(m)}(t)$ is a weak solution of (1).

Let now $\tilde{u}(t)$ be an arbitrary weak solution of (1), that is, a continuous T-periodic function of $\mathbb{R}$ to E with the Fourier coefficients $\tilde{f}_k \in D(A)$ such that $A\tilde{f}_k = (2\pi ik/T)^N\tilde{f}_k$. Setting

$$\begin{gathered} v_0(t) \equiv \tilde{u}(t) - \tilde{f}_0; \\ v_j(t) \equiv \int_0^t v_{j-1}(\tau)d\tau - \frac{1}{T}\int_0^T (T-\tau)v_{j-1}(\tau)d\tau, \quad j = 1,\ldots,N, \end{gathered} \tag{22}$$

we obtain

$$\int_0^T v_j(t)\,dt = 0, \qquad v_j(t+T) = v_j(t), \qquad v_j^{(j)}(t) = v_0(t) = \tilde{u}(t) - \tilde{f}_0. \tag{23}$$

Put $u(t) \equiv v_N(t)$. Then $u(t)$ is T-periodic on $\mathbb{R}$, $u \in C^N(\mathbb{R}, E)$, and $u^{(N)}(t) = \tilde{u}(t) - \tilde{f}_0$. Consider the Fourier coefficients f_k of $u(t)$. By construction, we have $f_0 = 0$ and for $k \neq 0$,

$$\begin{aligned} f_k &\equiv \frac{1}{T}\int_0^T u(t)e^{-2\pi ikt/T}\,dt \\ &= \frac{1}{T}\left(\frac{T}{2\pi ik}\right)^N \int_0^T u^{(N)}(t)e^{-2\pi ikt/T}\,dt \\ &= \frac{1}{T}\left(\frac{T}{2\pi ik}\right)^N \int_0^T \big(\tilde{u}(t) - \tilde{f}_0\big)e^{-2\pi ikt/T}\,dt \\ &= \left(\frac{T}{2\pi ik}\right)^N \tilde{f}_k, \end{aligned} \tag{24}$$

whence $Af_k = \tilde{f}_k$. This implies that $A\sigma_n(t) = \tilde{\sigma}_n(t) - \tilde{f}_0$ for the Fejer sums $\sigma_n(t)$, $\tilde{\sigma}_n(t)$ of the functions $u(t)$, $\tilde{u}(t)$, respectively. But $\sigma_n(t) \to u(t)$ and $\tilde{\sigma}_n(t) \to \tilde{u}(t)$ in every $t \in \mathbb{R}$ as $n \to \infty$. Since A is closed, $u(t) \in D(A)$ and $Au(t) = \tilde{u}(t) - \tilde{f}_0 = u^{(N)}(t)$ for $-\infty < t < \infty$. Thus $u(t)$ is a classical solution of (1) and $\tilde{u}(t) = u^{(N)}(t) + \tilde{f}_0$. The relations (20) were also shown. □

Remark 10. Theorem 9 shows that the space of all classical solutions of (1) is isomorphic to the space of all weak solutions. For example, such isomorphism may be defined in terms of the Fourier-series expansions

$$u(t) = \sum_{k=-\infty}^{\infty} e^{2\pi ikt/T} f_k \longleftrightarrow \tilde{u}(t) \sim \sum_{k=-\infty}^{\infty} e^{2\pi ikt/T} \tilde{f}_k \tag{25}$$

by the relations $\tilde{f}_k = (2\pi ik/T)^N f_k$ for $k \neq 0$ and $\tilde{f}_0 = f_0$. Here $u(t)$ is a classical solution of (1) and $\tilde{u}(t)$ is the corresponding weak solution.

We now examine the space of *all weak solutions* of the periodic problem (1).

THEOREM 11. *The collection of all weak solutions of (1) is a Banach space under the norm*

$$\big\|u(t)\big\|_0 \equiv \max_{-\infty<t<\infty} \big\|u(t)\big\|. \tag{26}$$

The set of all finite linear combinations of elementary solutions (6) is dense in this space with respect to the norm (26).

Proof. It is clear that the collection of all weak solutions of (1) is a linear normed space under the norm (26). Let $\{u_n(t)\}$ be a Cauchy sequence of weak solutions relative to this norm. Since E is complete, there is a limit function $u(t)$ such that $|||u_n(t)-u(t)|||_0 \to 0$ as $n \to \infty$. Evidently, $u(t)$ is continuous and T-periodic on $\mathbb{R}$. For the Fourier coefficients we have

$$f_{k,n} \equiv \frac{1}{T}\int_0^T u_n(t)e^{-2\pi ikt/T}\,dt \longrightarrow \frac{1}{T}\int_0^T u(t)e^{-2\pi ikt/T}\,dt \equiv f_k \quad \text{as } n \longrightarrow \infty, \tag{27}$$

and $Af_{k,n} = (2\pi ik/T)^N f_{k,n} \to (2\pi ik/T)^N f_k$ as $n \to \infty$. Since A is closed, $f_k \in D(A)$ and $Af_k = (2\pi ik/T)^N f_k$. This implies that the limit function $u(t)$ is a weak solution of (1) and the space of all weak solutions is complete with respect to the norm (26).

Let now $u(t)$ be an arbitrary weak solution of (1). Each Fejer sum $\sigma_n(t)$ of $u(t)$ is a finite linear combinations of elementary solutions (6) and $\sigma_n(t)$ tends to $u(t)$ in the norm (26) as $n \to \infty$. This completes the proof. □

It follows from Theorem 11 that the function $u : \mathbb{R} \to E$ is a weak solution of (1) if and only if there is a sequence of classical solutions of (1) which converges to $u(t)$ in the norm (26). A similar concept of weak solutions for the abstract *Cauchy problem* may be found in [7].

We now give a self-contained description for the space of *all classical solutions* of (1).

THEOREM 12. *The collection of all classical solutions of (1) is a Banach space under the norm*

$$|||u(t)|||_1 \equiv \max_{-\infty<t<\infty} \|u(t)\|_{D(A)} \equiv \max_{-\infty<t<\infty} \big(\|u(t)\| + \|Au(t)\|\big). \tag{28}$$

The set of all finite linear combinations of elementary solutions (6) is dense in this space with respect to the norm (28).

Proof. Let $\{u_n(t)\}$ be a sequence of classical solutions of (1) which is a Cauchy sequence relative to the norm (28). Then there exist two continuous T-periodic functions $u(t)$ and $\tilde{u}(t)$ such that $\|u_n(t)-u(t)\| \to 0$ and $\|Au_n(t)-\tilde{u}(t)\| \to 0$ uniformly on $\mathbb{R}$ as $n \to \infty$. Since A is closed, $u(t) \in D(A)$ and $Au(t) = \tilde{u}(t)$ for $-\infty < t < \infty$. So, $\|Au_n(t) - Au(t)\| \to 0$ uniformly on $\mathbb{R}$ and the Cauchy sequence $\{u_n(t)\}$ converges to $u(t)$ in the norm (28). We will show that $u(t)$ is a classical solution of (1). Note at first that $u(t)$ is a weak solution, since this function is a limit relative to the norm (26) of the sequence of classical solutions $u_n(t)$. This implies $Af_k = (2\pi ik/T)^N f_k$ for the Fourier coefficients of $u(t)$. Similarly, the function $\tilde{u}(t) = Au(t)$ is a limit with respect to the norm (26) of the sequence of weak solutions $u_n^{(N)}(t) = Au_n(t)$, whence $\tilde{u}(t)$ is a weak solution, too. Computing the Fourier coefficients of $\tilde{u}(t)$ we obtain

$$\begin{aligned}\tilde{f}_k &\equiv \frac{1}{T}\int_0^T \tilde{u}(t)e^{-2\pi ikt/T}\,dt = \frac{1}{T}\int_0^T Au(t)e^{-2\pi ikt/T}\,dt \\ &= Af_k = \left(\frac{2\pi ik}{T}\right)^N f_k, \quad k \in \mathbb{Z}.\end{aligned} \tag{29}$$

It follows from (20) that the Fourier coefficients of $u(t)$ are the same as the coefficients of the classical solution of (1) which was constructed in Theorem 9 for a weak solution $\tilde{u}(t)$. Thus $u(t)$ coincides with the classical solution and the space of all classical solutions is complete with respect to the norm (28).

It remains to show that linear combinations of the elementary solutions (6) is dense in this space with respect to (28). To this end suppose now $u(t)$ is an arbitrary classical solution of (1) with the Fourier series $\sum_{k=-\infty}^{\infty}\exp(2\pi ikt/T)f_k$. The Fourier coefficients of $Au(t)$ are equal to Af_k and each Fejer sum for the function $Au(t)$ has the form $A\sigma_n(t)$, where $\sigma_n(t)$ is the Fejer sum for the function $u(t)$. Since $\sigma_n(t) \to u(t)$ and $A\sigma_n(t) \to Au(t)$ as $n \to \infty$ uniformly on $\mathbb{R}$, the sequence $\{\sigma_n(t)\}$ converges to $u(t)$ in the norm (28). Every function $\sigma_n(t)$ is a linear combination of elementary solutions (6). This completes the proof. □

Theorems 11 and 12 may be treated as follows. In the space of all continuous T-periodic functions $u : \mathbb{R} \to E$ we select the linear subspace spanned by the elementary solutions (6). Then the closure of this subspace with respect to the norm (26) is the set of all weak solutions of (1), and the closure of this subspace with respect to the norm (28) is the set of all classical solutions of (1).

We supplement the paper with a uniqueness theorem for the boundary value problem on the finite interval $[0,T]$. Let $p :]0,T[\to E$ be a continuous function on the open interval $]0,T[$, and $u_0,\ldots,u_{N-1}$ any elements of E. Consider the problem (cf. (2))

$$\begin{aligned}\frac{d^N}{dt^N}u(t) &= Au(t)+p(t), \quad 0<t<T, \\ u^{(j)}(0)-u^{(j)}(T) &= u_j, \quad j=0,\ldots,N-1.\end{aligned} \tag{30}$$

A function $u : [0,T] \to E$ is a *strong solution* of (30) if $u \in C^N(]0,T[,E) \cap C^{N-1}([0,T],E)$, $u(t) \in D(A)$ for $0<t<T$, and (30) is satisfied. We stress that the differential equation of (30) holds in the *open interval* $]0,T[$.

THEOREM 13. *Let A be a closed linear operator on a Banach space E. Given $p(t)$ and $u_0,\ldots,u_{N-1} \in E$, suppose that problem (30) has a strong solution $u(t)$. This solution is unique if and only if none of the numbers $\lambda_k \equiv (2\pi ik/T)^N$, $k \in \mathbb{Z}$, is an eigenvalue of A.*

Proof. The necessary part can be shown as in Theorem 1 by means of the elementary solutions (6). For the proof of sufficiency, the immediate application of Theorem 1 is impossible because of the distinction between strong and classical solutions. The reasoning of Theorem 1 requires a small modification. Assume that none of the $\lambda_k \equiv (2\pi ik/T)^N$, $k \in \mathbb{Z}$ is an eigenvalue of A. Let $\tilde{u}(t)$ be another strong solution of (30) with the same $p(t)$, $u_0, \dots, u_{N-1}$. The function $v(t) \equiv u(t) - \tilde{u}(t)$ satisfies the equation $d^N v/dt^N = Av(t)$ for $0 < t < T$, and $v(0) = v(T), \dots, v^{(N-1)}(0) = v^{(N-1)}(T)$. Choosing $\varepsilon > 0$ and letting $\varepsilon \to 0+$ we obtain the relations

$$g_k(\varepsilon) \equiv \int_\varepsilon^{T-\varepsilon} v(t)e^{-2\pi ikt/T}\,dt \longrightarrow \int_0^T v(t)e^{-2\pi ikt/T}\,dt \equiv g_k, \tag{31}$$

$$\begin{aligned} Ag_k(\varepsilon) &= \int_\varepsilon^{T-\varepsilon} Av(t)e^{-2\pi ikt/T}\,dt \\ &= \int_\varepsilon^{T-\varepsilon} v^{(N)}(t)e^{-2\pi ikt/T}\,dt \\ &= \left(v^{N-1}(t) + \cdots + \left(\frac{2\pi ik}{T}\right)^{N-1} v(t)\right)e^{-2\pi ikt/T}\Big|_\varepsilon^{T-\varepsilon} \\ &\quad + \left(\frac{2\pi ik}{T}\right)^N g_k(\varepsilon) \longrightarrow \left(\frac{2\pi ik}{T}\right)^N g_k. \end{aligned} \tag{32}$$

Since A is closed, $g_k \in D(A)$ and $Ag_k = (2\pi ik/T)^N g_k$. By assumption, $g_k = 0$ for all $k \in \mathbb{Z}$ and hence $v(t) \equiv 0$ on $[0, T]$. Hence $u(t) \equiv \tilde{u}(t)$. □

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