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Research Article

An Efficient Family of Root-Finding Methods with Optimal Eighth-Order Convergence

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We derive a family of eighth-order multipoint methods for the solution of nonlinear equations. In terms of computational cost, the family requires evaluations of only three functions and one first derivative per iteration. This implies that the efficiency index of the present methods is 1.682. Kung and Traub (1974) conjectured that multipoint iteration methods without memory based on n evaluations have optimal order 2^{n-1}. Thus, the family agrees with Kung-Traub conjecture for the case $n = 4$. Computational results demonstrate that the developed methods are efficient and robust as compared with many well-known methods.

1. Introduction

Solving nonlinear equations is one of the most important problems in science and engineering [1, 2]. The boundary value problems arising in kinetic theory of gases, vibration analysis, design of electric circuits, and many applied fields are reduced to solving such equations. In the present era of advance computers, this problem has gained much importance than ever before.

In this paper, we consider iterative methods to find a simple root r of the nonlinear equation $f(x) = 0$, where $f : R \to R$ be the continuously differentiable real function. Newton's method [1] is probably the most widely used algorithm for solving such equations, which starts with an initial approximation x_0 closer to the root r and generates a sequence of successive iterates $\{x_i\}_0^\infty$ converging quadratically to the root. It is given by the following:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, 2, 3, \ldots. \tag{1.1}$$

In order to improve the local order of convergence, a number of ways are considered by many researchers, see [3–26] and references therein. In particular, King [3] developed a one-parameter family of fourth-order methods defined by

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$
$$x_{i+1} = w_i - \frac{f(x_i) + \beta f(w_i)}{f(x_i) + (\beta - 2) f(w_i)} \frac{f(w_i)}{f'(x_i)}, \tag{1.2}$$

where w_i is the Newton point and β is a constant.

This family requires two evaluations of the function f and one evaluation of first derivative f' per iteration. The famous Ostrowski's method [4, 5] is a member of this family for the case $\beta = 0$. From practical point of view, the methods (1.2) are important because of higher efficiency than Newton's method (1.1).

Traub [5] has divided iterative methods into two classes, namely, one-point methods and multipoint methods. Each class is further divided into two subclasses, namely, one-point methods with and without memory, and multipoint methods with and without memory. The important aspects related to these classes of methods are order of convergence and computational efficiency. Order of convergence shows the speed with which a given sequence of iterates converges to the root while the computational efficiency concerns with the economy of the entire process. Investigation of one-point methods with and without memory, has demonstrated theoretical restrictions on the order and efficiency of these two categories (see [5]). However, Kung and Traub [6] have conjectured that multipoint iteration methods without memory based on n evaluations have optimal order 2^{n-1}. In particular, with three evaluations a method of fourth-order can be constructed. The King's method (1.2) is a well-known example of fourth-order multipoint methods without memory.

Recently, based on Ostrowski's or King's methods some higher-order multipoint methods have been proposed and analyzed for solving nonlinear equations. For example, Grau and Díaz-Barrero [10], Sharma and Guha [11], and Chun and Ham [12] have developed sixth-order modified Ostrowski's methods each requires three f and one f' evaluations per iteration. Kou et al. [15] presented a family of variants of Ostrowski's method with seventh-order convergence requiring three f and one f' evaluations. With same number of evaluations, Bi et al. [18] developed a seventh-order family of modified King's methods. Bi et al. [19] also presented an eighth-order family of modified King's methods requiring four evaluations which agrees with the Kung-Traub conjecture.

In this paper, we present a new family of eighth-order methods without using second and higher derivatives. In terms of computational cost, it requires the evaluations of three functions and one first derivative per iteration. Thus the present methods provide a new example of multipoint methods without memory that with four evaluations a method of optimum order eight can be achieved as conjectured by Kung and Traub. The performance and effectiveness of the developed family of methods is tested and compared through some test functions.

Contents of the paper are summarized as follows. Some basic definitions relevant to the present work are presented in Section 2. In Section 3, we obtain new methods. Convergence analysis, for establishing eighth-order convergence, is carried out in Section 4. In Section 5, we provide some particular cases of the family. In Section 6, the method is tested and compared with other well-known methods on a number of problems. Concluding remarks are given in Section 7.

2. Basic Definitions

Definition 2.1. Let $f(x)$ be a real function with a simple root r and let $\{x_i\}_{i\in N}$ be a sequence of real numbers that converges towards r. Then, we say that the order of convergence of the sequence is p, if there exits a number $p \in \mathbb{R}^+$ such that

$$\lim_{i\to\infty} \frac{x_{i+1} - r}{(x_i - r)^p} = C, \tag{2.1}$$

for some $C \neq 0$, C is known as the asymptotic error constant.

If $p = 1, 2$ or 3, the sequence is said to have linear convergence, quadratic convergence or cubic convergence, respectively.

Definition 2.2. Let $e_i = x_i - r$ be the error in the ith iteration, we call the relation

$$e_{i+1} = Ce_i^p + O\left(e_i^{p+1}\right), \tag{2.2}$$

the error equation.

Definition 2.3. Let n be the number of new pieces of information required by a method. A "piece of information" typically is any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index [27] and is defined by the following:

$$E = p^{1/n}, \tag{2.3}$$

where p is the order of the method.

Definition 2.4. Suppose that x_{i+1}, x_i and x_{i-1} are three successive iterations closer to the root r. Then, the computational order of convergence ρ (see [24, 25, 28]) is approximated by using (2.2) as follows:

$$\rho \cong \frac{\ln|(x_{i+1} - r)/x_i - r|}{\ln|x_i - r/x_{i-1} - r|}. \tag{2.4}$$

3. The Method

We consider the iteration scheme of the form

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \omega(\lambda_i)\frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - W(\mu_i)\frac{f(z_i)}{f'(z_i)}, \end{aligned} \tag{3.1}$$

where $\lambda_i = f(w_i)/f(x_i)$, $\mu_i = f(z_i)/f(x_i)$, and $\omega(t)$ and $W(t)$ represent the real-valued functions (here onwards called weight functions). This scheme consists of three steps in which the first step represents Newton's method and last two are weighted-Newton steps. It is quite obvious that formula (3.1) requires five evaluations per iteration. However, we can reduce the number of evaluations to four by using some suitable approximation of the derivative $f'(z_i)$. We obtain this approximation by considering the approximation of $f(x)$ by a rational linear function of the form

$$y(x) - y(x_i) = \frac{(x - x_i) + a}{b(x - x_i) + c}, \tag{3.2}$$

where the parameters a, b, and c are determined by the condition that f and y coincide at x_i, w_i and z_i. That means $y(x)$ satisfies the conditions

$$y(x_i) = f(x_i), \qquad y(w_i) = f(w_i), \qquad y(z_i) = f(z_i). \tag{3.3}$$

From (3.2) and first condition of (3.3), it is easy to show that

$$a = 0. \tag{3.4}$$

Substituting the value of a into (3.2) then using the last two conditions of (3.3), after some simple calculations we obtain

$$\begin{aligned} b(w_i - x_i) + c &= \frac{1}{f[x_i, w_i]}, \\ b(z_i - x_i) + c &= \frac{1}{f[x_i, z_i]}, \end{aligned} \tag{3.5}$$

where $f[x_i, w_i] = (f(w_i) - f(x_i))/(w_i - x_i)$ and $f[x_i, z_i] = (f(z_i) - f(x_i))/(z_i - x_i)$ are first-divided differences.

Solving these equations, we can obtain b and c as follows:

$$b = \frac{1}{w_i - z_i}\left(\frac{1}{f[x_i, w_i]} - \frac{1}{f[x_i, z_i]}\right),$$
$$c = \frac{1}{w_i - z_i}\left(\frac{x_i - z_i}{f[x_i, w_i]} - \frac{x_i - w_i}{f[x_i, z_i]}\right). \tag{3.6}$$

Differentiation of (3.2) gives

$$y'(x) = \frac{c}{[b(x - x_i) + c]^2}. \tag{3.7}$$

We can now approximate the derivative $f'(x)$ with the derivative $y'(x)$ of rational function (3.2) and obtain

$$f'(z_i) \approx y'(z_i). \tag{3.8}$$

Substituting the values of b and c obtained in (3.6) into (3.7) then using (3.8), we get after simplifications

$$f'(z_i) = \frac{f[x_i, z_i] f[w_i, z_i]}{f[x_i, w_i]}. \tag{3.9}$$

Then the iteration scheme (3.1) in its final form is given by the following:

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$
$$z_i = w_i - \omega(\lambda_i)\,\frac{f(w_i)}{f'(x_i)}, \tag{3.10}$$
$$x_{i+1} = z_i - W(\mu_i)\frac{f[x_i, w_i] f(z_i)}{f[x_i, z_i] f[w_i, z_i]},$$

where $\lambda_i = f(w_i)/f(x_i)$, $\mu_i = f(z_i)/f(x_i)$, and $\omega(t)$ and $W(t)$ are the weight functions.

Thus the scheme (3.10) defines a new family of multipoint methods with two weight functions $\omega(t)$ and $W(t)$. In the next section, we will see that both of these functions play an important role in establishing eighth-order convergence of the methods.

4. Convergence of the Method

In order to examine the convergence property of the family (3.10), we prove the following theorem.

Theorem 4.1. *Let the function $f : R \to R$ be sufficiently smooth in R. If $f(x)$ has a simple root r in R and x_0 is sufficiently close to r, then the sequence $\{x_i\}$ generated by any method of the family* (3.10) *converges to r with convergence order eight, provided the weight functions $\omega(t)$ and $W(t)$ satisfy the conditions $\omega(0)=1, \omega'(0)=2, \omega''(0)=8, W(0)=1$, $W'(0)=1$, and $|\omega'''(0)|<\infty$.*

Proof. Let $e_i = x_i - r$ be the error in the iterate x_i. Using Taylor's series expansion, we get

$$\begin{aligned} f(x_i) &= f'(r)\left[e_i + A_2e_i^2 + A_3e_i^3 + A_4e_i^4 + A_5e_i^5 + A_6e_i^6 + A_7e_i^7 + A_8e_i^8 + O\left(e_i^9\right)\right], \\ f'(x_i) &= f'(r)\left[1 + 2A_2e_i + 3A_3e_i^2 + 4A_4e_i^3 + 5A_5e_i^4 + 6A_6e_i^5 + 7A_7e_i^6 + 8A_8e_i^7 + O\left(e_i^8\right)\right], \end{aligned} \tag{4.1}$$

where $A_k = f^{(k)}(r)/k!f'(r)$ for $k \in N$, N is the set of natural numbers.

Now,

$$\begin{aligned} \frac{f(x_i)}{f'(x_i)} = {} & e_i - A_2e_i^2 - 2\left(-A_2^2 + A_3\right)e_i^3 - \left(4A_2^3 - 7A_2A_3 + 3A_4\right)e_i^4 - K_1e_i^5 - K_2e_i^6 - K_3e_i^7 \\ & - K_4e_i^8 + O\left(e_i^9\right), \end{aligned} \tag{4.2}$$

Following are the expressions of K_n $(n = 1,2,3,4)$

$$\begin{aligned} K_1 &= -8A_2^4 + 20A_2^2A_3 - 6A_3^2 - 10A_2A_4 + 4A_5, \\ K_2 &= 16A_2^5 - 52A_2^3A_3 + 33A_2A_3^2 + 28A_2^2A_4 - 17A_3A_4 - 13A_2A_5 + 5A_6, \\ K_3 &= -32A_2^6 + 128A_2^4A_3 - 126A_2^2A_3^2 + 18A_3^3 - 72A_2^3A_4 + 92A_2A_3A_4 - 12A_4^2 + 36A_2^2A_5 \\ &\quad - 22A_3A_5 - 16A_2A_6 + 6A_7, \\ K_4 &= 64A_2^7 - 304A_2^5A_3 + 408A_2^3A_3^2 - 135A_2A_3^3 + 176A_2^4A_4 - 348A_2^2A_3A_4 + 75A_3^2A_4 + 64A_2A_4^2 \\ &\quad - 92A_2^3A_5 + 118A_2A_3A_5 - 31A_4A_5 + 44A_2^2A_6 - 27A_3A_6 - 19A_2A_7 + 7A_8. \end{aligned} \tag{4.3}$$

For the sake of brevity, we omit their specific forms. We will use the same means in the following.

For

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)} \\ &= r + A_2e_i^2 + 2\left(-A_2^2 + A_3\right)e_i^3 + \left(4A_2^3 - 7A_2A_3 + 3A_4\right)e_i^4 + K_1e_i^5 + K_2e_i^6 + K_3e_i^7 \\ &\quad + K_4e_i^8 + O\left(e_i^9\right). \\ \widetilde{e}_i &= A_2e_i^2 + 2\left(-A_2^2 + A_3\right)e_i^3 + \left(4A_2^3 - 7A_2A_3 + 3A_4\right)e_i^4 + K_1e_i^5 + K_2e_i^6 + K_3e_i^7 \\ &\quad + K_4e_i^8 + O\left(e_i^9\right). \end{aligned} \tag{4.4}$$

Using Taylor's series expansion, we get

$$f(w_i) = f'(r)\left[\tilde{e}_i + A_2\tilde{e}_i^2 + A_3\tilde{e}_i^3 + A_4\tilde{e}_i^4 + O\left(e_i^9\right)\right], \tag{4.5}$$

therefore,

$$\begin{aligned}
\frac{f(w_i)}{f'(x_i)} &= \tilde{e}_i - 2A_2\tilde{e}_i e_i + \left[\left(4A_2^2 - 3A_3\right)\tilde{e}_i e_i^2 + A_2\tilde{e}_i^2\right] \\
&\quad + \left[-2A_2^2\tilde{e}_i^2 e_i + \left(-8A_2^3 + 12A_2A_3 - 4A_4\right)\tilde{e}_i e_i^3\right] + M_1 e_i^6 + M_2 e_i^7 + M_3 e_i^8 + O\left(e_i^9\right), \\
M_1 &= \left(4A_2^3 - 3A_2A_3\right)\frac{\tilde{e}_i^2}{e_i^4} + \left(16A_2^4 - 36A_2^2A_3 + 9A_3^2 + 16A_2A_4 - 5A_5\right)\frac{\tilde{e}_i}{e_i^2} + A_3\frac{\tilde{e}_i^3}{e_i^6}, \\
M_2 &= -2A_2A_3\frac{\tilde{e}_i^3}{e_i^6} + \left(-8A_2^4 + 12A_2^2A_3 - 4A_2A_4\right)\frac{\tilde{e}_i^2}{e_i^4} \\
&\quad + \left(-32A_2^5 + 96A_2^3A_3 - 54A_2A_3^2 - 48A_2^2A_4 + 24A_3A_4 + 20A_2A_5 - 6A_6\right)\frac{\tilde{e}_i}{e_i^2}, \\
M_3 &= \left(4A_2^2A_3 - 3A_3^2\right)\frac{\tilde{e}_i^3}{e_i^6} + \left(16A_2^5 - 36A_2^3A_3 + 9A_2A_3^2 + 16A_2^2A_4 - 5A_2A_5\right)\frac{\tilde{e}_i^2}{e_i^4} \\
&\quad + \left(64A_2^6 - 240A_2^4A_3 + 216A_2^2A_3^2 - 27A_3^3 + 128A_2^3A_4 - 144A_2A_3A_4\right. \\
&\qquad \left. + 16A_4^2 - 60A_2^2A_5 + 30A_3A_5 + 24A_2A_6 - 7A_7\right)\frac{\tilde{e}_i}{e_i^2} + A_3\frac{\tilde{e}_i^4}{e_i^8}.
\end{aligned} \tag{4.6}$$

Also

$$\begin{aligned}
\lambda_i = \frac{f(w_i)}{f(x_i)} &= \frac{\tilde{e}_i}{e_i} - A_2\tilde{e}_i + \left[A_2\frac{\tilde{e}_i^2}{e_i} + \left(A_2^2 - A_3\right)\tilde{e}_i e_i\right] + \left[\left(-A_2^3 + 2A_2A_3 - A_4\right)\tilde{e}_i e_i^2 - A_2^2\tilde{e}_i^2\right] \\
&\quad + \left[\left(A_2^4 - 3A_2^2A_3 + A_3^2 + 2A_2A_4 - A_5\right)\tilde{e}_i e_i^3 + \left(A_2^3 - A_2A_3\right)\tilde{e}_i^2 e_i + A_3\frac{\tilde{e}_i^3}{e_i}\right] \\
&\quad + \left[\left(-A_2^5 + 4A_2^3A_3 - 3A_2A_3^2 - 3A_2^2A_4 + 2A_3A_4 + 2A_2A_5 - A_6\right)\tilde{e}_i e_i^4\right. \\
&\qquad \left. + \left(-A_2^4 + 2A_2^2A_3 - A_2A_4\right)\tilde{e}_i^2 e_i^2 - A_2A_3\tilde{e}_i^3\right] + O\left(e_i^7\right).
\end{aligned} \tag{4.7}$$

Thus, using the Taylor expansion, we get

$$
\begin{aligned}
\omega(\lambda_i) &= \omega(0) + \omega'(0)\lambda_i + \frac{1}{2!}\omega''(0)\lambda_i^2 + \frac{1}{3!}\omega'''(0)\lambda_i^3 + O\left(\lambda_i^4\right) \\
&= \omega(0) + \omega'(0)\frac{\widetilde{e}_i}{e_i} + \left[\frac{1}{2}\omega''(0)\frac{\widetilde{e}_i^2}{e_i^2} - A_2\omega'(0)\widetilde{e}_i\right] \\
&\quad + \left[\frac{1}{6}\omega'''(0)\frac{\widetilde{e}_i^3}{e_i^3} + A_2\left(\omega'(0) - \omega''(0)\right)\frac{\widetilde{e}_i^2}{e_i} + \left(A_2^2 - A_3\right)\omega'(0)\widetilde{e}_i e_i\right] \\
&\quad + L_1 e_i^4 + L_2 e_i^5 + L_3 e_i^6 + O\left(e_i^7\right),
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
L_1 &= \frac{1}{24}\omega^{iv}(0)\frac{\widetilde{e}_i^4}{e_i^8} + \frac{1}{2}A_2\left(2\omega''(0) - \omega'''(0)\right)\frac{\widetilde{e}_i^3}{e_i^6} \\
&\quad + \left(-A_3\omega''(0) + A_2^2\left(-\omega'(0) + \frac{3}{2}\omega''(0)\right)\right)\frac{\widetilde{e}_i^2}{e_i^4} - \left(A_2^3 + 2A_2A_3 - A_4\right)\omega'(0)\frac{\widetilde{e}_i}{e_i^2}, \\
L_2 &= \frac{1}{120}\omega^{v}(0)\frac{\widetilde{e}_i^5}{e_i^{10}} + \frac{1}{6}A_2\left(3\omega'''(0) - \omega^{(iv)}(0)\right) \\
&\quad \times \left(A_3\left(\omega'(0) - \frac{1}{2}\omega'''(0)\right) + A_2^2\left(-2\omega''(0) + \omega'''(0)\right)\right)\frac{\widetilde{e}_i^3}{e_i^6} \\
&\quad + \left(-A_2A_3\left(\omega'(0) - 3\omega''(0)\right) + A_2^3\left(\omega'(0) - 2\omega''(0)\right) - A_4\,\omega''(0)\right)\frac{\widetilde{e}_i^2}{e_i^4} \\
&\quad + \left(A_2^4 - 3A_2^2A_3 + A_3^2 + 2A_2A_4 - A_5\right)\omega'(0)\frac{\widetilde{e}_i}{e_i^2}, \\
L_3 &= \frac{1}{720}\omega^{vi}(0)\frac{\widetilde{e}_i^6}{e_i^{12}} + \frac{1}{24}A_2\left(4\omega^{iv}(0) - \omega^{v}(0)\right)\frac{\widetilde{e}_i^5}{e_i^{10}} \\
&\quad + \frac{1}{12}\left(2A_3\left(6\omega''(0) - \omega^{iv}(0)\right) + A_2^2\left(6\omega''(0) - 18\omega'''(0) + 5\omega^{iv}(0)\right)\right)\frac{\widetilde{e}_i^4}{e_i^8} \\
&\quad + \left(-A_2A_3\left(\omega'(0) + 2\omega''(0) - 2\omega'''(0)\right) + A_2^3\left(3\omega''(0) - \frac{5}{3}\omega'''(0)\right) - \frac{1}{2}A_4\,\omega'''(0)\right)\widetilde{e}_i^3 \\
&\quad + \left(2A_2^2A_3\left(\omega'(0) - 3\omega''(0)\right) - A_2A_4\left(\omega'(0) - 3\omega''(0)\right) + \frac{1}{2}\left(3A_3^2 - 2A_5\right)\omega''(0)\right. \\
&\qquad \left. + A_2^4\left(-\omega'(0) + \frac{5}{2}\omega''(0)\right)\right)\frac{\widetilde{e}_i^2}{e_i^4} \\
&\quad + \left(-A_2^5 + 4A_2^3A_3 - 3A_2A_3^2 - 3A_2^2A_4 + 2A_3A_4 + 2A_2A_5 - A_6\right)\omega'(0)\frac{\widetilde{e}_i}{e_i^2}.
\end{aligned}
\tag{4.9}
$$

Using (4.6) and (4.8), we have

$$
\begin{aligned}
z_i &= w_i - \omega(\lambda_i)\frac{f(w_i)}{f'(x_i)} \\
&= r + [1-\omega(0)]\tilde{e}_i - \left[\omega'(0)\frac{\tilde{e}_i^2}{e_i} - 2\omega(0)A_2\tilde{e}_i e_i\right] \\
&\quad - \left[\frac{1}{2}\omega''(0)\frac{\tilde{e}_i^3}{e_i^2} + A_2(\omega(0) - 3\omega'(0))\tilde{e}_i^2 + \omega(0)\left(4A_2^2 - 3A_3\right)\tilde{e}_i e_i^2\right] \\
&\quad - \left[\frac{1}{2}\omega''(0)\frac{\tilde{e}_i^3}{e_i^2} + A_2(\omega(0) - 3\omega'(0))\tilde{e}_i^2 + \omega(0)\left(4A_2^2 - 3A_3\right)\tilde{e}_i e_i^2\right]e_i^5 \\
&\quad - M_6 e_i^6 - M_7 e_i^7 - M_8 e_i^8 + O\left(e_i^9\right),
\end{aligned}
\tag{4.10}
$$

where M_n $(n = 6, 7, 8)$ are the expression about A_n $(n = 2, 3, \ldots, 8)$.

If $(0) = 1$, $\omega'(0) = 2$ and substituting the value of $\tilde{e}_i$ from (4.4), we get

$$
\begin{aligned}
\hat{e}_i &= z_i - r \\
&= \left[\left(5 - \frac{1}{2}\omega''(0)\right)A_2^3 - A_2A_3\right]e_i^4 + \left(-36 + 5\omega''(0) - \frac{1}{6}\omega'''(0)\right)A_2^4 + (32 - 3\omega''(0))A_2^2A_3 \\
&\quad -2A_3^2 - 2A_2A_4\Big]e_i^5 + M_6e_i^6 + M_7e_i^7 + M_8e_i^8 + O\left(e_i^9\right).
\end{aligned}
\tag{4.11}
$$

Using Taylor's series expansion, we get

$$
f(z_i) = f'(r)\left[\hat{e}_i + A_2\hat{e}_i^2 + O\left(e_i^9\right)\right],
\tag{4.12}
$$

furthermore,

$$
\begin{aligned}
f[x_i, w_i] &= \frac{f(w_i) - f(x_i)}{w_i - x_i} \\
&= f'(r)\Big[1 + A_2e_i + \left(A_2^2 + A_3\right)e_i^2 + \left(-2A_2^3 + 3A_2A_3 + A_4\right)e_i^3 \\
&\qquad + \left(4A_2^4 - 8A_2^2A_3 + 2A_3^2 + 4A_2A_4 + A_5\right)e_i^4 + O\left(e_i^5\right)\Big].
\end{aligned}
$$

$$\begin{aligned}
f[x_i, z_i] &= \frac{f(z_i) - f(x_i)}{z_i - x_i} \\
&= f'(r)\Bigg[1 + A_2 e_i + A_3 e_i^2 + A_4 e_i^3 \\
&\quad + \left(\left(5 - \frac{1}{2}\omega''(0)\right)A_2^4 - A_2^2 A_3 + A_5\right)e_i^4 + O\left(e_i^5\right)\Bigg], \\
f[w_i, z_i] &= \frac{f(z_i) - f(w_i)}{z_i - w_i} \\
&= f'(r)\Bigg[1 + A_2^2 e_i^2 - 2\left(A_2^3 - A_2 A_3\right)e_i^3 \\
&\quad + \left(\left(9 - \frac{1}{2}\omega''(0)\right)A_2^4 - 7A_2^2 A_3 + 3A_2 A_4\right)e_i^4 + O\left(e_i^5\right)\Bigg].
\end{aligned} \tag{4.13}$$

Using the above results, we obtain

$$\begin{aligned}
\frac{f[x_i, w_i]}{f[w_i, z_i] f[x_i, z_i]} = \frac{1}{f'(r)}\Big[1 &+ \left(-A_2^3 + A_2 A_3\right)e_i^3 \\
&+ \left((-7 + \omega''(0))A_2^4 - 4A_2^2 A_3 + 2A_3^2 + A_2 A_4\right)e_i^4 + O\left(e_i^5\right)\Big].
\end{aligned} \tag{4.14}$$

Also

$$\mu_i = \frac{f(z_i)}{f(x_i)} = \frac{\hat{e}_i}{e_i} - A_2 \hat{e}_i + O\left(e_i^5\right). \tag{4.15}$$

Thus, using the Taylor expansion and $|W''(0)| < \infty$,we get

$$W(\mu_i) = W(0) + W'(0)\mu_i + O\left(\mu_i^2\right) = W(0) + W'(0)\left(\frac{\hat{e}_i}{e_i} - A_2 \hat{e}_i\right) + O\left(e_i^5\right). \tag{4.16}$$

Using these results in

$$x_{i+1} = z_i - W(\mu_i)\frac{f[x_i, w_i] f(z_i)}{f[w_i, z_i]\ f[x_i, z_i]}, \tag{4.17}$$

we obtain

$$\begin{aligned}
e_{i+1} &= \widehat{e}_i - \left[W(0) + W'(0)\frac{\widehat{e}_i}{e_i} - W'(0)A_2\widehat{e}_i + O\left(e_i^5\right)\right]\left[\widehat{e}_i + A_2\widehat{e}_i^2 + O\left(\widehat{e}_i^3\right)\right] \\
&\quad \times \left[1 + \left(-A_2^3 + A_2A_3\right)e_i^3 + \left(\left(-7 + \omega''(0)\right)A_2^4 - 4A_2^2A_3 + 2A_3^2 + A_2A_4\right)e_i^4 + O\left(e_i^5\right)\right] \\
&= \widehat{e}_i - \left[W(0) + W'(0)\frac{\widehat{e}_i}{e_i} - W'(0)A_2\widehat{e}_i\right]\left(\widehat{e}_i + A_2\widehat{e}_i^2\right) \\
&\quad \times \left[1 + \left(-A_2^3 + A_2A_3\right)e_i^3 + \left(\left(-7 + \omega''(0)\right)A_2^4 - 4A_2^2A_3 + 2A_3^2 + A_2A_4\right)e_i^4\right] + O\left(e_i^9\right) \\
&= [1 - W(0)]\widehat{e}_i - \left[W(0)\left(-A_2^3 + A_2A_3\right)e_i^3 + W'(0)\frac{\widehat{e}_i}{e_i}\right]\widehat{e}_i - [W(0) - W'(0)]A_2\widehat{e}_i^2 \\
&\quad - \left[\left(-7 + \omega''(0)\right)A_2^4 - 4A_2^2A_3 + 2A_3^2 + A_2A_4\right]W(0)e_i^4\widehat{e}_i + O\left(e_i^9\right) \\
&= [1 - W(0)]\widehat{e}_i - \left[W(0)\left(-A_2^3 + A_2A_3\right) + W'(0)\left(5 - \frac{1}{2}\omega''(0)A_2^3 - A_2A_3\right)\right]e_i^3\widehat{e}_i \\
&\quad + [W'(0) - W(0)]A_2\widehat{e}_i^2 + \left[W(0)\left(\left(7 - \omega''(0)\right)A_2^4 + 4A_2^2A_3 - 2A_3^2 - A_2A_4\right)\right. \\
&\quad \left. - W'(0)\left(\left(-36 + 5\omega''(0) - \frac{1}{6}\omega'''(0)\right)A_2^4 + \left(32 - 3\omega''(0)\right)A_2^2A_3 - 2A_3^2 - 2A_2A_4\right)\right] \\
&\quad \times e_i^4\widehat{e}_i + O\left(e_i^9\right).
\end{aligned} \tag{4.18}$$

This means that convergence order of the family (3.10) is seventh-order with $W(0) = 1$ and the error equation is

$$\begin{aligned}
e_{i+1} = &\left[\left(A_2^3 - A_2A_3\right)\left(\left(5 - \frac{1}{2}\omega''(0)A_2^3 - A_2A_3\right)\right.\right. \\
&\left.\left. - W'(0)\left(\left(5 - \frac{1}{2}\omega''(0)A_2^3 - A_2A_3\right)^2\right]e_i^7 + O\left(e_i^8\right),
\end{aligned} \tag{4.19}$$

and if W is any function with $W(0) = 1, W'(0) = 1$, and $\omega''(0) = 8$, then the convergence order of any method of the family (3.10) arrives to eight, and the error equation is

$$e_{i+1} = A_2^2\left(A_2^2 - A_3\right)\left[\left(-5 + \frac{1}{6}\omega'''(0)\right)A_2^3 - 4A_2A_3 + A_4\right]e_i^8 + O\left(e_i^9\right). \tag{4.20}$$

Thus if ω and $W$are any functions with $(0) = 1, \omega'(0) = 2, \omega''(0) = 8, W(0) = 1$, and $W'(0) = 1$, then the eighth-order convergence is established. This completes the proof of the theorem. □

Note that per iteration every method of the family (3.10) uses four pieces of information, namely, $f(x_i), f'(x_i), f(w_i), f(z_i)$ and has eighth-order convergence with the conditions $\omega(0) = 1,\ \omega'(0) = 2,\ \omega''(0) = 8,\ W(0) = 1$, and $W'(0) = 1$, which is in accordance with Kung-Traub conjecture for 4 evaluations.

5. Some Particular Forms

Here, we consider some forms of the functions $\omega(t)$ and $W(t)$ satisfying the conditions of the Theorem 4.1. Based on these forms some methods of the family (3.10) are also presented.

5.1. Forms of $\omega(t)$

Form 1. For the function ω given by the following:

$$\omega_1(t) = 1 + 2t + 4t^2 + \alpha t^3, \tag{5.1}$$

where $\alpha \in R$ is a constant, it is clear that the conditions of Theorem 4.1 are satisfied.

Form 2. For the function ω defined by the following:

$$\omega_2(t) = 1 + \frac{2t}{1 - 2t + \alpha t^2}, \tag{5.2}$$

where $\alpha \in R$, it can be easily seen that this function satisfies the conditions of Theorem 4.1.

Form 3. For the function ω defined by the following:

$$\omega_3(t) = \left(1 + 2\alpha t + \beta t^2\right)^{1/\alpha}, \quad \beta = 2\alpha(\alpha + 1), \tag{5.3}$$

where $\alpha \in R - \{0\}$, it can be seen that this function also satisfies the conditions of Theorem 4.1.

5.2. Forms of $W(t)$

Form 1. For the function W given by the following:

$$W_1(t) = 1 + t + \gamma t^2, \tag{5.4}$$

where $\gamma \in R$, it can be seen the function $W_1(t)$ satisfies the conditions of Theorem 4.1.

Form 2. For the function W defined by the following:

$$W_2(t) = 1 + \frac{t}{1 + \gamma t}, \tag{5.5}$$

where $\gamma \in R$, it is simple to see that $W_2(t)$ satisfies the conditions of Theorem 4.1

Form 3. For the function W defined by the following:

$$W_3(t) = (1 + \gamma t)^{1/\gamma}, \tag{5.6}$$

where $\gamma \in R - \{0\}$, again it can be seen that $W_3(t)$ satisfies the conditions of Theorem 4.1.

5.3. Forms of Methods

To form a concrete method we can take any combination of the above defined $\omega(t)$ and $W(t)$. For simplicity, we consider only three such combinations. For example, by taking $\omega_2(t)$ with $W_i(t),\ i = 1, 2, 3$ the following methods can be formed

Method 1. Taking $\omega_2(t)$ and $W_1(t)$, we get a new two-parameter family of eighth-order methods

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f^2(x_i) + \alpha f^2(w_i)}{f^2(x_i) - 2f(x_i)f(w_i) + \alpha f^2(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \left[1 + \frac{f(z_i)}{f(x_i)} + \gamma \frac{f^2(z_i)}{f^2(x_i)}\right] \frac{f[x_i, w_i]\ f(z_i)}{f[w_i, z_i] f[x_i, z_i]}. \end{aligned} \tag{5.7}$$

Method 2. Considering $\omega_2(t)$ and $W_2(t)$, we get another new two-parameter family of eighth-order methods

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f^2(x_i) + \alpha f^2(w_i)}{f^2(x_i) - 2f(x_i)f(w_i) + \alpha f^2(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \left[\frac{f(x_i) + (1 + \gamma) f(z_i)}{f(x_i) + \gamma f(z_i)}\right] \frac{f[x_i, w_i]\ f(z_i)}{f[w_i, z_i]\ f[x_i, z_i]}. \end{aligned} \tag{5.8}$$

Table 1: Test functions.

$f(x)$	r
$f_1(x) = x^5 + x^4 + 4x^2 - 15$	1.3474280989683050
$f_2(x) = \sin(x) - x/3$	2.2788626600758283
$f_3(x) = 10xe^{-x^2} - 1$	1.6796306104284499
$f_4(x) = \cos(x) - x$	0.7390851332151606
$f_5(x) = e^{-x^2+x+2} - 1$	−1.0000000000000000
$f_6(x) = e^{-x} + \cos(x)$	1.7461395304080124
$f_7(x) = \ln(x^2 + x + 2) - x + 1$	4.1525907367571583
$f_8(x) = \arcsin(x^2 - 1) - x/2 + 1$	0.5948109683983692
$f_9(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$	−1.2076478271309189

Method 3. Considering now $\omega_2(t)$ and $W_3(t)$, we get another new two-parameter family of eighth-order methods

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f^2(x_i) + \alpha f^2(w_i)}{f^2(x_i) - 2f(x_i)f(w_i) + \alpha f^2(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \left[1 + \gamma \frac{f(z_i)}{f(x_i)}\right]^{1/\gamma} \frac{f[x_i, w_i]\ f(z_i)}{f[w_i, z_i]\ f[x_i, z_i]}. \end{aligned} \tag{5.9}$$

The proposed families require three evaluations of the function f and one evaluation of first derivative f'per iteration, and achieve eighth-order convergence. Thus the efficiency index (E) defined by (2.3) of the present methods (3.10) is $E = \sqrt[4]{8} \approx 1.682$ which is better than $E = \sqrt{2} \approx 1.414$ of Newton's method, $E = \sqrt[3]{4} \approx 1.587$ of King's [3] and Ostrowski's [4] methods, $E = \sqrt[4]{6} \approx 1.565$ of sixth-order methods [10–12] and $E = \sqrt[4]{7} \approx 1.627$ of seventh-order methods [15, 18].

6. Numerical Examples

We employ the present methods (4.1), and (4.4) denoted by M81, M82 and M83, respectively to solve some nonlinear equations and compare with Newton's method (NM) defined by (1.1), the eighth-order method developed by Cordero et al. [23] denoted by C8 and defined as follows:

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= x_i - \frac{f(x_i) - f(w_i)}{f(x_i) - 2f(w_i)} \frac{f(x_i)}{f'(x_i)}, \\ u_i &= z_i - \left(\frac{f(x_i) - f(w_i)}{f(x_i) - 2f(w_i)} + \frac{1}{2}\frac{f(z_i)}{f(w_i) - 2f(z_i)}\right)\frac{f(z_i)}{f'(x_i)}, \\ x_{i+1} &= u_i - \frac{3(\beta_2 + \beta_3)(u_i - z_i)}{\beta_1(u_i - z_i) + \beta_2(w_i - x_i) + \beta_3(z_i - x_i)} \frac{f(z_i)}{f'(x_i)}, \end{aligned} \tag{6.1}$$

Table 2: Comparison of methods using same total number of function evaluations for all methods (TFE = 12).

	NM	C8	L8		T8	B81	B82	B83	M81	M82	M83
		$\beta_1 = \beta_3 = 0$, $\beta_2 = 1$	$\alpha_1 = \alpha_2 = 1$	P8	$\alpha = 0$	$\alpha = 1$	$\alpha = \gamma = 1$	$\alpha = 1$	$\alpha = \gamma = 1$	$\alpha = \gamma = 1$	$\alpha = \gamma = 1$
f_1, $x_0 = 1.6$											
$\lvert x_i - r\rvert$	$9.75e-41$	$4.35e-338$	$4.51e-300$	$0.00e+00$	$3.67e-305$	$6.55e-304$	$8.24e-304$	$9.60e-304$	$0.00e+00$	$1.40e-349$	$1.70e-350$
$\lvert f(x_i)\rvert$	$3.61e-39$	$1.61e-336$	$1.67e-298$	$0.00e+00$	$1.36e-303$	$2.43e-302$	$3.05e-302$	$3.55e-302$	$6.96e-350$	$5.19e-348$	$6.28e-349$
ρ	2.0000000	7.9999999	7.9999999	8.0000000	7.9999999	7.9999989	7.9999989	7.9999989	7.9999997	7.9999997	7.9999997
f_2, $x_0 = 2.0$											
$\lvert x_i - r\rvert$	$4.27e-57$	$0.00e+00$	$4.76e-334$	$0.00e+00$	$2.88e-333$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
$\lvert f(x_i)\rvert$	$4.20e-57$	$0.00e+00$	$4.68e-334$	$0.00e+00$	$2.83e-333$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
ρ	2.0000000	8.0000000	7.9999993	8.0000000	7.9999994	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000
f_3, $x_0 = 1.8$											
$\lvert x_i - r\rvert$	$4.41e-58$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
$\lvert f(x_i)\rvert$	$1.22e-57$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
ρ	2.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	7.9999997	7.9999997	7.9999997
f_4, $x_0 = 1.0$											
$\lvert x_i - r\rvert$	$1.80e-83$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
$\lvert f(x_i)\rvert$	$3.00e-83$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
ρ	2.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000
f_5, $x_0 = -0.5$											
$\lvert x_i - r\rvert$	$3.46e-27$	$7.71e-202$	$8.70e-163$	$5.99e-246$	$3.75e-162$	$3.12e-222$	$1.20e-221$	$2.98e-221$	$4.57e-195$	$7.97e-194$	$1.98e-194$
$\lvert f(x_i)\rvert$	$1.04e-26$	$2.31e-201$	$2.61e-162$	$1.80e-245$	$1.13e-161$	$9.36e-222$	$3.61e-221$	$8.93e-221$	$1.37e-194$	$2.39e-193$	$5.95e-194$
ρ	2.0000000	7.9999339	7.9996776	7.9999847	7.9996400	8.0000580	8.0000594	8.0000603	7.9995556	7.9995321	7.9995437
f_6, $x_0 = 2.0$											
$\lvert x_i - r\rvert$	$7.97e-85$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
$\lvert f(x_i)\rvert$	$9.24e-85$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
ρ	2.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000
f_7, $x_0 = 3.2$											
$\lvert x_i - r\rvert$	$4.66e-74$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
$\lvert f(x_i)\rvert$	$2.81e-74$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$	$0.00e+00$
ρ	2.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000	8.0000000
f_8, $x_0 = 1.0$											
$\lvert x_i - r\rvert$	$1.68e-54$	$4.42e-342$	$4.21e-309$	$0.00e+00$	$1.20e-309$	$0.00e+00$	$5.83e-346$	$1.68e-341$	$0.00e+00$	$0.00e+00$	$0.00e+00$
$\lvert f(x_i)\rvert$	$1.78e-54$	$4.68e-342$	$4.46e-309$	$0.00e+00$	$1.27e-309$	$0.00e+00$	$6.17e-346$	$1.78e-341$	$0.00e+00$	$0.00e+00$	$0.00e+00$
ρ	2.0000000	8.0000004	8.0000018	8.0000001	8.0000021	8.0000012	8.0000009	8.0000014	8.0000001	8.0000001	8.0000001
f_9, $x_0 = -1$											
$\lvert x_i - r\rvert$	$8.63e-33$	$0.00e+00$	$1.76e-339$	$0.00e+00$	$0.00e+00$	$4.78e-218$	$3.12e-214$	$8.95e-212$	$1.83e-216$	$3.85e-218$	$2.63e-217$
$\lvert f(x_i)\rvert$	$1.75e-31$	$0.00e+00$	$3.58e-338$	$0.00e+00$	$0.00e+00$	$9.71e-217$	$6.34e-213$	$1.82e-210$	$3.72e-215$	$7.81e-217$	$5.34e-216$
ρ	2.0000000	8.0000001	8.0000002	8.0000000	8.0000000	7.9999987	7.9999985	7.9999984	8.0001475	8.0001379	8.0001426

where $\beta_1, \beta_2, \beta_3 \in R$ and $\beta_2 + \beta_3 \neq 0$, eighth-order method developed by Liu and Wang [22] denoted by L8 and defined as follows:

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f(x_i)}{f(x_i) - 2f(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \left[\left(\frac{f(x_i) - f(w_i)}{f(x_i) - 2f(w_i)} \right)^2 + \frac{f(z_i)}{f(w_i) - \alpha_1 f(z_i)} + \frac{4f(z_i)}{f(x_i) + \alpha_2 f(z_i)} \right] \frac{f(z_i)}{f'(x_i)}, \end{aligned} \tag{6.2}$$

where $\alpha_1, \alpha_2 \in R$, eighth-order method developed by Petković et al. [20] denoted by P8 and defined as follows:

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f(x_i)}{f(x_i) - 2f(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \frac{(1 + a_4(z_i - x_i))^2 f(z_i)}{a_2 - a_1 a_4 + a_3(z_i - x_i)(2 + a_4(z_i - x_i))}, \end{aligned} \tag{6.3}$$

where $a_1 = f(x_i)$, $a_3 = (f'(x_i) f[w_i, z_i] - f[x_i, w_i] f[x_i, z_i]) / ((x_i f[w_i, z_i] + (w_i f(z_i) - z_i f(w_i)) / (w_i - z_i)) - f(x_i))$,

$$a_4 = \frac{a_3}{f[x_i, w_i]} + \frac{f'(x_i) - f[x_i, w_i]}{(w_i - x_i) f[x_i, w_i]}, \qquad a_2 = f'(x_i) + a_4 f(x_i), \tag{6.4}$$

eighth-order method developed by Thukral and Petković [21] denoted by T8 and defined as follows:

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f(x_i)}{f(x_i) - 2f(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \left[\frac{f^2(x_i)}{f^2(x_i) - 2f(x_i)f(w_i) - f^2(w_i)} + \frac{f(z_i)}{f(w_i) - \alpha f(z_i)} + \frac{4f(z_i)}{f(x_i)} \right] \frac{f(z_i)}{f'(x_i)}, \end{aligned} \tag{6.5}$$

where $\alpha \in R$, eighth-order methods presented in Section 3 of [19] by Bi et al. denoted by B81, B82 and B83.

The test functions and root r correct up to 16 decimal places are displayed in Table 1. The first eight functions we have selected are same as in [19]. The last function is selected from [18]. In Table 2, we exhibit the absolute values of the difference of root r and its approximation x_i, where r is computed with 350 significant digits and x_i is calculated by costing the same total number of function evaluations (TFE) for each method. The TFE is counted as sum of the number of evaluations of the function itself plus the number of evaluations of the derivatives. In the calculations, 12 TFE are used by each method. That means 6 iterations are used for NM and 3 iterations for the remaining methods. The absolute values of the function $|f(x_i)|$ and the computational order of convergence (ρ) are also displayed in Table 2. It can be observed that the computed results, displayed in Table 2, overwhelmingly support the theory of convergence and efficiency analyses discussed in the previous sections. From the results, it can be concluded that the proposed methods are competitive with existing methods and possess quick convergence for good initial approximations. Among the eighth-order methods, we are not able to select one as the best. For some initial guess one is better while for other initial guess the another one would be appropriate. Thus the present methods can be of practical interest.

7. Conclusions

In this work, we have obtained a new simple and elegant family of eighth-order multipoint methods for solving nonlinear equations. Thus, one requires three evaluations of the function f and one of its first-derivative f' per full step and therefore, the efficiency index of the present methods is 1.682 which is better than the efficiency index of Newton method, fourth-order methods, sixth-order methods, and seventh-order methods.

Many numerical applications use higher precision in their computations. In these types of applications, numerical methods of higher-order are important. The numerical results show that the methods associated with a multiprecision arithmetic floating point are very useful, because these methods yield a clear reduction in number of iterations. Finally, we conclude that the methods presented in this paper are preferable to other recognized efficient methods, namely, Newton's method, King's methods, sixth-order methods [10–12], seventh-order methods [15, 18], etc.

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