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Research Article

Existence and Uniqueness Results for Perturbed Neumann Boundary Value Problems

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Using a fixed point theorem of general α-concave operators, we present in this paper criteria which guarantee the existence and uniqueness of positive solutions for two classes of nonlinear perturbed Neumann boundary value problems for second-order differential equations. The theorems for Neumann boundary value problems obtained are very general.

1. Introduction and Preliminaries

In this paper, we are interested in the existence and uniqueness of positive solutions for the following nonlinear perturbed Neumann boundary value problems (NBVPs):

$$(P_{\pm})\begin{cases} \pm u''(t) + m^2 u(t) = f(t, u(t)) + g(t), & 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where m is a positive constant, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \to [0,+\infty)$ are continuous.

It is well known that Neumann boundary value problem for the ordinary differential equations and elliptic equations is an important kind of boundary value problems. During the last two decades, Neumann boundary value problems have deserved the attention of many researchers [1–10]. By using-fixed point theorems in cone, in [1, 5, 7–9], the authors discussed the existence of positive solutions for ordinary differential equation Neumann boundary value problems.

Recently, the authors [4] discussed second-order superlinear repulsive singular Neumann boundary value problems by using a nonlinear alternative of Leray-Schauder and Krasnosel'skii fixed-point theorem on compression and expansion of cones, and obtained the existence of at least two positive solutions under reasonable conditions. In [6], the authors established the existence of sign-changing solutions and positive solutions for fourth-order Neumann boundary value problem by using the fixed-point index and the critical group. Besides the above methods mentioned, the method of upper and lower solutions is also used in the literature [2, 3, 10]. However, to the best of our knowledge, few papers can be found in the literature on the existence and uniqueness of positive solutions for the NBVPs $(P_\pm)$. Different from the above works mentioned, in this paper, we will use a fixed-point theorem of general α-concave operators to show the existence and uniqueness of positive solutions for the NBVPs $(P_\pm)$.

By a positive solution of $(P_\pm)$, we understand a function $u(t) \in C^2[0,1]$ which is positive on $0 < t < 1$ and satisfies the differential equation and the boundary conditions in $(P_\pm)$.

We now present a fixed point theorem of general α-concave operators which will be used in the latter proofs. Let E be a real Banach space and P be a cone in E, θ denotes the null element. Given $h > \theta(i.e., h \geq \theta$ and $h \neq \theta)$, we denote by P_h the set

$$P_h = \{x \in P \mid \exists \lambda(x), \mu(x) > 0 \text{ such that } \lambda(x)h \leq x \leq \mu(x)h\}. \tag{1.2}$$

See [11] for further information.

Theorem 1.1 (see [11]). *Assume that cone P is normal and operator A satisfies the following conditions:*

(B_1) $A : P_h \to P_h$ *is increasing in* P_h,

(B_2) *for for all* $x \in P_h$ *and* $t \in (0,1)$, *there exists* $\alpha(t) \in (0,1)$ *such that* $A(tx) \geq t^{\alpha(t)} Ax$,

(B_3) *there is a constant* $l \geq 0$ *such that* $x_0 \in [\theta, lh]$.

Then operator equation $x = Ax + x_0$ *has a unique solution in* P_h.

Remark 1.2. An operator A is said to be *general α-concave* if A satisfies condition (B_2).

2. Positive Solutions for the Problems $(P_\pm)$

In this section, we will apply Theorem 1.1 to study the general NBVPs $(P_\pm)$ and then we will obtain new results on the existence and uniqueness of positive solutions for the problems $(P_\pm)$. The following conditions will be assumed:

(H_1) $f(t,x)$ is increasing in x for fixed t,

(H_2) for any $\gamma \in (0,1)$ and $x \geq 0$, there exists $\varphi(\gamma) \in (\gamma, 1]$ such that $f(t, \gamma x) \geq \varphi(\gamma) f(t,x)$ for $t \in [0,1]$,

(H_3) for any $t \in [0,1]$, $f(t,a) > 0$, where $a = 1/2(ch\, m + 1)$.

In the following, we will work in the Banach space $C[0,1]$ and only the sup-norm is used. Set $P = \{x \in C[0,1] \mid x(t) \geq 0, t \in [0,1]\}$, the standard cone. It is easy to see that P is a normal cone of which the normality constant is 1. Let $G(t,s)$ be the Green's function for the boundary value problem

$$-u''(t)+m^2u(t)=0,\quad 0<t<1,$$
$$u'(0)=u'(1)=0. \tag{2.1}$$

Then,

$$G(t,s)=\frac{1}{\rho}\begin{cases}\psi(s)\psi(1-t), & 0\le s\le t\le 1,\\ \psi(t)\psi(1-s), & 0\le t\le s\le 1,\end{cases} \tag{2.2}$$

where $\rho=m\cdot sh\,m$, $\psi(t)=ch\,mt$. It is obvious that $\psi(t)$ is increasing on $[0,1]$, and

$$0<G(t,s)\le G(t,t),\quad 0\le t,\ s\le 1. \tag{2.3}$$

Lemma 2.1 (see [9]). *Let $G(t,s)$ be the Green's function for the NBVP* (2.1)*. then,*

$$G(t,s)\ge\frac{1}{ch^2m}ch\,mt\cdot ch(1-t)m\cdot G(t_0,s),\quad t,t_0,s\in[0,1]. \tag{2.4}$$

Theorem 2.2. *Assume $(H_1)-(H_3)$ hold. Then the NBVP (P_-) has a unique positive solution u^* in P_h, where $h(t)=\psi(t)\psi(1-t)=(1/2)(ch\,m+ch(m-2mt))$, $t\in[0,1]$.*

Remark 2.3. Let $b=(1/2)(e^m+e^{-m})$. Then it is easy to check that $a=\min\{h(t):t\in[0,1]\}=(1/2)(ch\,m+1), b=\max\{h(t):t\in[0,1]\}=ch\,m$.

Proof of Theorem 2.2. It is well known that u is a solution of the NBVP (P_-) if and only if

$$u(\mathrm{t})=\int_0^1 G(t,s)\left[f(s,u(s))+g(s)\right]ds, \tag{2.5}$$

where $G(t,s)$ is the Green's function for the NBVP (2.1). For any $u\in P$, we define

$$Au(t)=\int_0^1 G(t,s)f(s,u(s))ds,\qquad x_0(t)=\int_0^1 G(t,s)g(s)ds. \tag{2.6}$$

It is easy to check that $A:P\to P$. From (H_1), we know that $A:P\to P$ is an increasing operator. Next we show that A satisfies the conditions $(B_1),(B_2)$ in Theorem 1.1. From (H_2), for any $\gamma\in(0,1)$ and $u\in P$, there exists $\varphi(\gamma)\in(\gamma,1]$ such that

$$A(\gamma u)(t)=\int_0^1 G(t,s)f(s,\gamma u(s))ds\ge\int_0^1 G(t,s)\varphi(\gamma)f(s,u(s))ds=\varphi(\gamma)Au(t),\quad t\in[0,1]. \tag{2.7}$$

That is, $A(\gamma u) \geq \varphi(\gamma) Au$, for all $u \in P, \gamma \in (0,1)$. Set

$$\alpha(\gamma) = \frac{\ln \varphi(\gamma)}{\ln \gamma}, \tag{2.8}$$

then $\alpha(\gamma) \in (0,1)$ and

$$A(\gamma u) \geq \gamma^{\alpha(\gamma)} Au, \quad \text{for } \gamma \in (0,1),\ u \in P. \tag{2.9}$$

In the following, we show that $A : P_h \to P_h$. On one hand, it follows from $(H_1), (H_3)$, Lemma 2.1 and Remark 2.3, that

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s) f(s,h(s)) ds \\ &\geq \int_0^1 \frac{1}{ch^2 m} \psi(t)\varphi(1-t) G(t_0,s) f(s,a) ds \\ &= \frac{1}{ch^2 m} h(t) \int_0^1 G(t_0,s) f(s,a) ds, \quad t \in [0,1]. \end{aligned} \tag{2.10}$$

On the other hand, from (2.3), (H_1), and Remark 2.3, we obtain

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s) f(s,h(s)) ds \\ &\leq \int_0^1 G(t,t) f(s,b) ds \\ &= \frac{1}{\rho} h(t) \int_0^1 f(s,b) ds, \quad t \in [0,1]. \end{aligned} \tag{2.11}$$

Let

$$r_1 = \min_{t\in[0,1]} f(t,a), \qquad r_2 = \max_{t\in[0,1]} f(t,b). \tag{2.12}$$

Then $0 < r_1 \leq r_2$. Note that

$$\int_0^1 G(t_0,s) ds = \frac{1}{\rho} \int_0^{t_0} \psi(s)\varphi(1-t_0) ds + \frac{1}{\rho} \int_{t_0}^1 \psi(t_0)\varphi(1-s) ds = \frac{1}{m^2}. \tag{2.13}$$

Consequently,

$$\frac{r_1}{ch^2 m} \cdot \frac{1}{m^2} h(t) \leq Ah(t) \leq r_2 \cdot \frac{1}{m\, sh\, m} h(t), \quad t \in [0,1]. \tag{2.14}$$

Hence $Ah \in P_h$. For any $u \in P_h$, we can choose a small number $t_0 \in (0,1)$ such that

$$t_0 h \le u \le \frac{1}{t_0} h. \tag{2.15}$$

By (2.9), we get

$$A\left(\frac{1}{\gamma} u\right) \le \frac{1}{\gamma^{\alpha(\gamma)}} Au, \quad \forall \gamma \in (0,1),\ u \in P. \tag{2.16}$$

Thus, from (2.9), (2.16), we have

$$Au \ge A(t_0 h) \ge t_0{}^{\alpha(t_0)} Ah, \qquad Au \le A\left(\frac{1}{t_0} h\right) \le \frac{1}{t_0{}^{\alpha(t_0)}} Ah. \tag{2.17}$$

Thus, $Au \in P_h$. Therefore, $A : P_h \to P_h$. This together with (2.9) implies that A is general α-concave. That is, A satisfies the conditions (B_1), (B_2) in Theorem 1.1.

Next we show that the condition (B_3) is satisfied. If $g(t) \equiv 0$, then $x_0(t) \equiv 0$; if $g(t) \not\equiv 0$, let $l = \rho \max_{t \in [0,1]} g(t)$, then $l > 0$. It is easy to prove that

$$0 \le x_0(t) \le \frac{l}{\rho} \int_0^1 G(t,t) ds = lh(t). \tag{2.18}$$

Hence, $0 \le x_0 \le lh$. Finally, using Theorem 1.1, $u = Au + x_0$ has a unique solution u^* in P_h. That is, u^* is a unique positive solution of the NBVP (P_-) in P_h. □

In the following, using the same technique, we study the general NBVP (P_+) with $m \in (0, \pi/2)$. Let $G(t,s)$ be the Green's function for the boundary value problem

$$\begin{gathered} u''(t) + m^2 u(t) = 0, \quad 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{gathered} \tag{2.19}$$

Then,

$$G(t,s) = \frac{1}{m \sin m} \begin{cases} \cos ms \cos m(1-t), & 0 \le s \le t \le 1, \\ \cos mt \cos m(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.20}$$

It is obvious that cos mt is decreasing on $[0,1]$, and

$$G(t,s) \ge G(t,t), \quad 0 \le t,\ s \le 1. \tag{2.21}$$

Lemma 2.4. *Let $G(t,s)$ be the Green's function for the* NBVP (2.19). *Then,*

$$G(t,s) \le \frac{1}{\cos^2 m} \cos mt \cos m(1-t) \cdot G(t_0, s), \quad t, t_0, s \in [0,1]. \tag{2.22}$$

Proof. When $t, t_0 \leq s$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{\cos m(1-s)\cos mt}{\cos m(1-s)\cos mt_0} = \frac{\cos m(1-t)\cos mt}{\cos m(1-t)\cos mt_0}\\ &\leq \frac{1}{\cos^2 m}\cos m(1-t)\cos mt = C\ \cos m(1-t)\cos mt.\end{aligned} \tag{2.23}$$

If $t \leq s \leq t_0$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{\cos m(1-s)\cos mt}{\cos m(1-t_0)\cos ms} = \frac{\cos m(1-t)\cos mt}{\cos m(1-t)\cos ms}\cdot\frac{\cos m(1-s)}{\cos m(1-t_0)}\\ &\leq \frac{1}{\cos^2 m}\cos m(1-t)\cos mt = C\ \cos m(1-t)\cos mt.\end{aligned} \tag{2.24}$$

If $t_0 \leq s \leq t$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{\cos m(1-t)\cos ms}{\cos m(1-s)\cos mt_0} = \frac{\cos m(1-t)\cos mt}{\cos m(1-s)\cos mt}\cdot\frac{\cos ms}{\cos mt_0}\\ &\leq \frac{1}{\cos^2 m}\cos m(1-t)\cos mt = C\cos m(1-t)\cos mt.\end{aligned} \tag{2.25}$$

For $s \leq t, t_0$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)} &= \frac{\cos m(1-t)\cos ms}{\cos m(1-t_0)\cos ms} = \frac{\cos m(1-t)\cos mt}{\cos m(1-t_0)\cos mt}\\ &\leq \frac{1}{\cos^2 m}\cos m(1-t)\cos mt = C\ \cos m(1-t)\cos mt.\end{aligned} \tag{2.26}$$

Therefore,

$$G(t,s) \leq \frac{1}{\cos^2 m}\cos m(1-t)\cos mt \cdot G(t_0,s), \quad t, t_0, s \in [0,1]. \tag{2.27}$$

This completes the proof. □

Theorem 2.5. *Assume* $(H_1), (H_2)$ *hold and* $f(t, \cos^2 m) > 0$ *for any* $t \in [0,1]$. *Then the* NBVP(P_+) *has a unique positive solution* u^* *in* P_h, *where* $h(t) = \cos m(1-t)\cos mt, t \in [0,1]$.

Remark 2.6. It is easy to check that $\cos^2 m \leq h(t) \leq 1$ for $t \in [0,1]$.

Proof of Theorem 2.5. It is well known that u is a solution of the NBVP (P_+) if and only if

$$u(t) = \int_0^1 G(t,s)\left[f(s,u(s)) + g(s)\right]ds, \tag{2.28}$$

where $G(t,s)$ is the Green's function for the NBVP (2.19). For any $u \in P$, we define

$$Au(t) = \int_0^1 G(t,s) f(s,u(s))ds, \qquad x_0(t) = \int_0^1 G(t,s) g(s) ds. \tag{2.29}$$

Similar to the proof of Theorem 2.2, we know that $A : P \to P$ is an increasing operator and satisfies the condition

$$A(\gamma u) \geq \varphi(\gamma) Au = \gamma^{\alpha(\gamma)} Au, \quad \forall u \in P, \ \gamma \in (0,1), \tag{2.30}$$

where $\alpha(\gamma) = \ln\varphi(\gamma) / \ln\gamma$.

It follows from condition (H_1), Lemma 2.4, and Remark 2.6 that

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s) f(s,h(s))ds \\ &\leq \int_0^1 \frac{1}{\cos^2 m} \cos mt \, \cos m(1-t) \cdot G(t_0,s) f(s,1) ds \\ &= \frac{1}{\cos^2 m} h(t) \int_0^1 G(t_0,s) f(s,1) ds, \quad t \in [0,1]. \end{aligned} \tag{2.31}$$

From (2.21), (H_1), and Remark 2.6, we obtain

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s) f(s,h(s))ds \\ &\geq \int_0^1 G(t,t) f\left(s, \cos^2 m\right) ds \\ &= \frac{1}{m \sin m} h(t) \int_0^1 f\left(s, \cos^2 m\right) ds, \quad t \in [0,1]. \end{aligned} \tag{2.32}$$

Let

$$r_1 = \min_{t\in[0,1]} f\left(t, \cos^2 m\right), \qquad r_2 = \max_{t\in[0,1]} f(t,1). \tag{2.33}$$

Then $0 < r_1 \leq r_2$. Consequently,

$$Ah(t) \leq r_2 \frac{1}{\cos^2 m} \int_0^1 G(t_0,s) ds \cdot h(t), \quad Ah(t) \geq r_1 \frac{1}{m \sin m} h(t), \quad t \in [0,1]. \tag{2.34}$$

Note that

$$\int_0^1 G(t_0,s)ds=\frac{1}{m\sin m}\int_0^{t_0}\cos\ m(1-t_0)\cos\ msds+\frac{1}{m\sin m}\int_{t_0}^1\cos m(1-s)\cos mt_0\,ds=\frac{1}{m^2}, \tag{2.35}$$

we have $r_2(1/\cos^2 m)\int_0^1 G(t_0,s)ds > 0$. Hence $Ah \in P_h$. The same reasoning as Theorem 2.2 shows that A is general α-concave and (B_3) is satisfied. Using Theorem 1.1, $u = Au + x_0$ has a unique solution u^* in P_h. That is, u^* is a unique positive solution of the NBVP (P_+) in P_h. □

Remark 2.7. For the case of $g(t) \equiv 0$, the problems $(P_\pm)$ reduce to the usual forms of Neumann boundary value problems for ordinary differential equations. We can establish the existence and uniqueness of positive solutions for these problems by using the same method used in this paper, which is new to the literature. So the method used in this paper is different from previous ones in literature and the results obtained in this paper are new.

3. Examples

To illustrate how our main results can be used in practice we present two examples.

Example 3.1. Consider the following NBVP:

$$\begin{gathered} -u''(t) + (\ln 2)^2 u(t) = u^\beta(t) + q(t) + t^2, \quad 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{gathered} \tag{3.1}$$

where $\beta \in (0,1)$ and $q : [0,1] \to [0,+\infty)$ is a continuous function. In this example, we let $m = \ln 2, f(t,x) := x^\beta + q(t), g(t) := t^2$. After a simple calculation, we get $a = 9/8, b = 5/4$ and

$$h(t) = \frac{5}{8} + \frac{1}{4}\left(2^{1-2t} + 2^{2t-1}\right), \quad t \in [0,1]. \tag{3.2}$$

Evidently, $f(t,x)$ is increasing for $x \geq 0$, and $g(t) \not\equiv 0$,

$$f(t,a) = \left(\frac{9}{8}\right)^\beta + q(t) > 0. \tag{3.3}$$

Moreover, set $\varphi(\gamma) = \gamma^\beta, \gamma \in (0,1)$. Then,

$$f(t,\gamma x) = \gamma^\beta x^\beta + q(t) \geq \gamma^\beta\left(x^\beta + q(t)\right) = \varphi(\gamma) f(t,x), \quad x \geq 0. \tag{3.4}$$

Hence, all the conditions of Theorem 2.2 are satisfied. An application of Theorem 2.2 implies that the NBVP (3.1) has a unique positive solution u^* in P_h.

Example 3.2. Consider the following NBVP:

$$u''(t) + \left(\frac{\pi}{3}\right)^2 u(t) = u^{1/3}(t) + q(t) + t^3, \quad 0 < t < 1,$$
$$u'(0) = u'(1) = 0, \tag{3.5}$$

where $q : [0,1] \to [0,+\infty)$ is a continuous function. In this example, we let $m = \pi/3$, $f(t,x) := x^{1/3} + q(t)$, $g(t) := t^3$. Then, $m \in (0, \pi/2)$ and

$$h(t) = \cos\frac{\pi}{3}t \, \cos\frac{\pi}{3}(1-t), \quad t \in [0,1]. \tag{3.6}$$

Evidently, $f(t,x)$ is increasing for $x \geq 0$, and $g(t) \not\equiv 0$,

$$f\left(t, \cos^2\frac{\pi}{3}\right) + q(t) = \left(\frac{1}{4}\right)^{1/3} + q(t) > 0. \tag{3.7}$$

Moreover, set $\varphi(\gamma) = \gamma^{1/3}, \gamma \in (0,1)$. Then,

$$f(t, \gamma x) = \gamma^{1/3} x^{1/3} + q(t) \geq \gamma^{1/3}\left(x^{1/3} + q(t)\right) = \varphi(\gamma) f(t,x), \quad x \geq 0. \tag{3.8}$$

Hence, all the conditions of Theorem 2.5 are satisfied. An application of Theorem 2.5 implies that the NBVP (3.5) has a unique positive solution u^* in P_h.

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