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Research Article

Global Optimal Regularity for the Parabolic Polyharmonic Equations

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We show the global regularity estimates for the following parabolic polyharmonic equation $u_t + (-\Delta)^m u = f$ in $\mathbb{R}^n \times (0,\infty), m \in \mathbb{Z}^+$ under proper conditions. Moreover, it will be verified that these conditions are necessary for the simplest heat equation $u_t - \Delta u = f$ in $\mathbb{R}^n \times (0,\infty)$.

1. Introduction

Regularity theory in PDE plays an important role in the development of second-order elliptic and parabolic equations. Classical regularity estimates for elliptic and parabolic equations consist of Schauder estimates, L^p estimates, De Giorgi-Nash estimates, Krylov-Safonov estimates, and so on. L^p and Schauder estimates, which play important roles in the theory of partial differential equations, are two fundamental estimates for elliptic and parabolic equations and the basis for the existence, uniqueness, and regularity of solutions.

The objective of this paper is to investigate the generalization of L^p estimates, that is, regularity estimates in Orlicz spaces, for the following parabolic polyharmonic problems:

$$u_t(x,t) + (-\Delta)^m u(x,t) = f(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

$$u(x,0) = 0 \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $x = (x^1,\dots,x^n)$, $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ and m is a positive integer. Since the 1960s, the need to use wider spaces of functions than Sobolev spaces arose out of various practical problems. Orlicz spaces have been studied as the generalization of Sobolev spaces since they were introduced by Orlicz [1] (see [2–6]). The theory of Orlicz spaces plays a crucial role in many fields of mathematics (see [7]).

We denote the distance in $\mathbb{R}^{n+1}$ as

$$\delta(z_1, z_2) = \max\left\{|x_1 - x_2|, |t_1 - t_2|^{1/2m}\right\} \quad \text{for } z_1 = (x_1, t_1),\ z_2 = (x_2, t_2) \tag{1.3}$$

and the cylinders in $\mathbb{R}^{n+1}$ as

$$Q_R = B_R \times \left(-R^{2m}, R^{2m}\right], \qquad Q_R(z) = Q_R + z, \quad z = (x,t) \in \mathbb{R}^{n+1}, \tag{1.4}$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ is an open ball in $\mathbb{R}^n$. Moreover, we denote

$$D_x^{\nu} u = \frac{\partial^{|\nu|} u}{\partial_{x_1}^{\nu_1} \cdots \partial_{x_n}^{\nu_n}}, \tag{1.5}$$

where $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ is a multiple index, $\nu_i \geq 0$ $(i = 1, 2, \dots, n)$, and $|\nu| = \sum_{i=1}^{n} |\nu_i|$. For convenience, we often omit the subscript x in $D_x^{\nu} u$ and write $D^k u = \{D^{\nu} u : |\nu| = k\}$.

Indeed if $m = 1$, then (1.1) is simplified to be the simplest heat equation. L^p estimates and Schauder estimates for linear second-order equations are well known (see [8, 9]). When $m \neq 1$, the corresponding regularity results for the higher-order parabolic equations are less. Solonnikov [10] studied L^p and Schauder estimates for the general linear higher-order parabolic equations with the help of fundamental solutions and Green functions. Moreover, in [11] we proved global Schauder estimates for the initial-value parabolic polyharmonic problem using the uniform approach as the second-order case. Recently we [6] generalized the local L^p estimates to the Orlicz space

$$\int_{Q_{1/6}} \phi\left(\left|D^{2m} u\right|^2\right) dz + \int_{Q_{1/6}} \phi\left(|u_t|^2\right) dz \leq C\left\{\int_{Q_{1/2}} \phi\left(|f|^2\right) dz + \int_{Q_{1/2}} \phi\left(|u|^2\right) dz\right\} \tag{1.6}$$

for

$$u_t(z) + (-\Delta)^m u(z) = f(z) \quad \text{in } \Omega \times (0, T], \tag{1.7}$$

where $\phi \in \Delta_2 \cap \nabla_2$ (see Definition 1.2) and Ω is an open bounded domain in $\mathbb{R}^n$. When $\phi(x) = |x|^{p/2}$ with $p > 2$, (1.6) is reduced to the local L^p estimates. In fact, we can replace 2 of $\phi(|\cdot|^2)$ in (1.6) by the power of p_1 for any $p_1 > 1$.

Our purpose in this paper is to extend local regularity estimate (1.3) in [6] to global regularity estimates, assuming that $\phi \in \Delta_2 \cap \nabla_2$. Moreover, we will also show that the $\Delta_2 \cap \nabla_2$ condition is necessary for the simplest heat equation $u_t - \Delta u = f$ in $\mathbb{R}^n \times (0, \infty)$. In particular, we are interested in the estimate like

$$\int_{\mathbb{R}^n \times (0,\infty)} \phi\left(\left|D^{2m} u\right|\right) dz + \int_{\mathbb{R}^n \times (0,\infty)} \phi(|u_t|) dz \leq C \int_{\mathbb{R}^n \times (0,\infty)} \phi\left(|f|\right) dz, \tag{1.8}$$

where C is a constant independent from u and f. Indeed, if $\phi(x) = |x|^p$ with $p > 1$, (1.8) is reduced to classical L^p estimates. We remark that although we use similar functional framework and iteration-covering procedure as in [6, 12], more complicated analysis should be carefully carried out with a proper dilation of the unbounded domain.

Here for the reader's convenience, we will give some definitions on the general Orlicz spaces.

Definition 1.1. A convex function $\phi : \mathbb{R} \to \mathbb{R}^+$ is said to be a Young function if

$$\phi(-s) = \phi(s), \qquad \phi(0) = 0, \qquad \lim_{s \to \infty} \phi(s) = +\infty. \tag{1.9}$$

Definition 1.2. A Young function ϕ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that for every $s > 0$,

$$\phi(2s) \leq K\phi(s). \tag{1.10}$$

Moreover, a Young function ϕ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a number $a > 1$ such that for every $s > 0$,

$$\phi(s) \leq \frac{\phi(as)}{2a}. \tag{1.11}$$

Example 1.3. (i) $\phi_1(s) = (1 + |s|)\log(1 + |s|) - |s| \in \Delta_2$, but $\phi_1(s) \notin \nabla_2$.

(ii) $\phi_2(s) = e^{|s|} - |s| - 1 \in \nabla_2$, but $\phi_2(s) \notin \Delta_2$.

(iii) $\phi_3(s) = |s|^\alpha(1 + |\log|s||) \in \Delta_2 \cap \nabla_2$, $\alpha > 1$.

Remark 1.4. If a function ϕ satisfies (1.10) and (1.11), then

$$\phi(\theta_1 s) \leq K\theta_1^{\alpha_1}\phi(s), \qquad \phi(\theta_2 s) \leq 2a\theta_2^{\alpha_2}\phi(s), \tag{1.12}$$

for every $s > 0$ and $0 < \theta_2 \leq 1 \leq \theta_1 < \infty$, where $\alpha_1 = \log_2 K$ and $\alpha_2 = \log_a 2 + 1$.

Remark 1.5. Under condition (1.12), it is easy to check that ϕ satisfies

$$\phi(0) = 0, \qquad \lim_{s \to \infty} \phi(s) = +\infty, \qquad \lim_{s \to 0+} \frac{\phi(s)}{s} = \lim_{s \to +\infty} \frac{s}{\phi(s)} = 0. \tag{1.13}$$

Definition 1.6. Assume that ϕ is a Young function. Then the Orlicz class $K^\phi(\mathbb{R}^n)$ is the set of all measurable functions $g : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^n} \phi(|g|)dx < \infty. \tag{1.14}$$

The Orlicz space $L^\phi(\mathbb{R}^n)$ is the linear hull of $K^\phi(\mathbb{R}^n)$.

Lemma 1.7 (see [2]). *Assume that $\phi \in \Delta_2 \cap \nabla_2$ and $g \in L^{\phi}(\Omega)$. Then*

(1) $K^{\phi}(\Omega) = L^{\phi}(\Omega)$,

(2) $C_0^{\infty}(\Omega)$ *is dense in* $L^{\phi}(\Omega)$,

(3)

$$\int_{\Omega} \phi(|g|)dx = \int_0^{\infty} |\{x \in \Omega : |g| > \mu\}| d[\phi(\mu)]. \tag{1.15}$$

Now let us state the main results of this work.

Theorem 1.8. *Assume that ϕ is a Young function and u satisfies*

$$\begin{gathered} u_t(x,t) - \Delta u(x,t) = f(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = 0 \quad \text{in } \mathbb{R}^n. \end{gathered} \tag{1.16}$$

Then if the following inequality holds

$$\int_{\mathbb{R}^n \times (0,\infty)} \phi\left(\left|D^2 u\right|\right) dz + \int_{\mathbb{R}^n \times (0,\infty)} \phi(|u_t|) dz \le C \int_{\mathbb{R}^n \times (0,\infty)} \phi(|f|) dz, \tag{1.17}$$

One has

$$\phi \in \Delta_2 \cap \nabla_2. \tag{1.18}$$

Theorem 1.9. *Assume that $\phi \in \Delta_2 \cap \nabla_2$. If u is the solution of* (1.1)-(1.2) *with $f \in L^{\phi}(\mathbb{R}^n \times (0,\infty))$, then* (1.8) *holds.*

Remark 1.10. We would like to point out that the Δ_2 condition is necessary. In fact, if the local L^{ϕ} estimate (1.6) ($m = 1$) is true, then by choosing

$$u = \sqrt{2} s^{1/2} x_1 x_2, \quad f = 0 \quad \forall s > 0 \tag{1.19}$$

we have

$$\begin{aligned} \int_{Q_{1/6}} \phi(2s) dz &= \int_{Q_{1/6}} \phi\left(\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2\right) dz \\ &\le C\left\{\int_{Q_{1/2}} \phi\left(|f|^2\right) dz + \int_{Q_{1/2}} \phi\left(|u|^2\right) dz\right\} \\ &\le C \int_{Q_{1/2}} \phi(s) dz, \end{aligned} \tag{1.20}$$

which implies that

$$\phi(2s) \le C\phi(s), \quad \text{for any } s > 0. \tag{1.21}$$

2. Proof of Theorem 1.8

In this section we show that ϕ satisfies the global ∇_2 condition if u satisfies (1.16) and estimate (1.17) is true.

Proof. Now we consider the special case in (1.16) when

$$f(z) = \rho\eta(z) \tag{2.1}$$

for any constant $\rho > 0$, where $z = (x,t)$ and $\eta \in C_0^\infty(\mathbb{R}^{n+1})$ is a cutoff function satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \quad \text{in } B_1 \times (-1,1), \quad \eta \equiv 0 \quad \text{in } \frac{\mathbb{R}^{n+1}}{\{B_2 \times (-2,2)\}}. \tag{2.2}$$

Therefore the problem (1.16) has the solution

$$u(x,t) = \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4(t-s)} f(y,s)\,dy\,ds. \tag{2.3}$$

It follows from (1.17), (2.1), and (2.2) that

$$\int_{\mathbb{R}^n\times(0,\infty)} \phi(|u_t|)dz \le C\int_{\mathbb{R}^n\times(0,\infty)} \phi(|f|)dz \le C_1\phi(\rho). \tag{2.4}$$

We know from (2.3) that

$$u_t(x,t) = \frac{1}{(4\pi)^{n/2}} \int_0^t \frac{1}{(t-s)^{(n+2)/2}} \int_{B_2} \left(\frac{|x-y|^2}{4(t-s)} - \frac{n}{2}\right) e^{-|x-y|^2/4(t-s)} f(y,s)\,dy\,ds. \tag{2.5}$$

Define

$$D =: \left\{z = (x,t) \in \mathbb{R}^n \times (0,\infty) : |x| > 4,\ |x| \ge 4\sqrt{nt}\right\}. \tag{2.6}$$

Then when $z \in D$, $t > s$ and $y \in B_2$, we have

$$\frac{|x-y|^2}{4(t-s)} \ge \frac{|x|^2}{16t} \ge n, \tag{2.7}$$

since

$$\left|x-y\right| \geq |x| - \left|y\right| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2}. \tag{2.8}$$

Therefore, since $|x-y| \leq |x| + |y| \leq 2|x|$ for $z \in D$ and $y \in B_2$, we conclude that

$$\begin{aligned} |u_t(x,t)| &\geq \frac{n\rho}{2\cdot(4\pi)^{n/2}} \int_0^t \frac{1}{(t-s)^{(n+2)/2}} \int_{B_1} e^{-|x-y|^2/4(t-s)} dy\, ds \\ &\geq \frac{n\rho}{2\cdot(4\pi)^{n/2}} \int_0^t \frac{1}{(t-s)} \int_{y\in B_1} e^{-|\xi|^2/4} d\xi\, ds \quad \xi = \frac{(x-y)}{(t-s)} \\ &\geq \frac{n\rho}{2\cdot(4\pi)^{n/2}} \int_0^t \frac{(t-s)^{(n-2)/2}}{(2|x|)^n} \int_{y\in B_1} |\xi|^n e^{-|\xi|^2/4} d\xi\, ds \\ &\geq C\rho|x|^{-n} \int_0^t (t-s)^{(n-2)/2} ds \geq C_2\rho|x|^{-n} t^{n/2}. \end{aligned} \tag{2.9}$$

Recalling estimate (2.4) we find that

$$\int_D \phi\left(C_2\rho|x|^{-n}t^{n/2}\right) dx\, dt \leq C_1\phi(\rho), \tag{2.10}$$

which implies that

$$\int_{1/n}^1 \left\{ \int_{4\sqrt{n}}^{+\infty} \phi(C_2\rho r^{-n}) r^{n-1} dr \right\} dt \leq C_1\phi(\rho). \tag{2.11}$$

By changing variable we conclude that, for any $\rho > 0$,

$$\int_0^{\alpha\rho} \frac{\phi(\sigma)}{\sigma^2} d\sigma \leq \frac{C_3\phi(\rho)}{\rho}, \tag{2.12}$$

where $\alpha = C_2 4^{-n} n^{-n/2}$. Let $\rho_2 \geq \rho_1$ and $0 < \varepsilon \leq \alpha/2$. Then we conclude from (2.12) that

$$\begin{aligned} \frac{\phi(\rho_2)}{\rho_2} &\geq \frac{1}{C_3} \int_0^{\alpha\rho_2} \frac{\phi(\sigma)}{\sigma^2} d\sigma \geq \frac{1}{C_3} \int_{\varepsilon\rho_1}^{\alpha\rho_1} \frac{\phi(\sigma)}{\sigma^2} d\sigma \\ &\geq \frac{\phi(\varepsilon\rho_1)}{C_3} \left(\frac{1}{\varepsilon\rho_1} - \frac{1}{\alpha\rho_1} \right) \geq \frac{\phi(\varepsilon\rho_1)}{2C_3\varepsilon\rho_1}. \end{aligned} \tag{2.13}$$

Now we use (2.12) and (2.13) to obtain that

$$\frac{\phi(\rho)}{\rho} \geq \frac{1}{C_3} \int_{\varepsilon\rho}^{\alpha\rho} \frac{\phi(\sigma)}{\sigma} \frac{1}{\sigma} d\sigma \geq \frac{\phi(\varepsilon^2\rho)}{2C_3^2\varepsilon^2\rho} \ln\frac{\alpha}{\varepsilon}, \tag{2.14}$$

where we choose that $\rho_1 = \varepsilon\rho$, $\rho_2 = \sigma$ in (2.13). Set $a = 1/\varepsilon^2$. Then we have

$$\phi(\rho) \geq \frac{\ln(\alpha\sqrt{a})}{2C_3^2} a\phi\left(\frac{\rho}{a}\right) \geq 2a\phi\left(\frac{\rho}{a}\right), \tag{2.15}$$

when a is chosen large enough. This implies that ϕ satisfies the ∇_2 condition. Thus this completes our proof. □

3. Proof of the Main Result

In this section, we will finish the proof of the main result, Theorem 1.9. Just as [6], we will use the following two lemmas. The first lemma is the following integral inequality.

Lemma 3.1 (see [6]). *Let $\phi \in \Delta_2 \cap \nabla_2$, $g \in L^\phi(\mathbb{R}^{n+1})$, and $p \in (1, \alpha_2)$, where α_2 is defined in* (1.12). *Then for any $b_1, b_2 > 0$ one has*

$$\int_0^\infty \frac{1}{\mu^p}\left\{\int_{\{z\in\mathbb{R}^{n+1}:|g|>b_1\mu\}} |g|^p dz\right\} d[\phi(b_2\mu)] \leq C(b_1, b_2, \phi)\int_{\mathbb{R}^{n+1}} \phi(|g|)dz. \tag{3.1}$$

Moreover, we recall the following result.

Lemma 3.2 (see [10, Theorem 5.5]). *Assume that $g \in L^p(\mathbb{R}^n \times (0,\infty))$ for $p > 1$. There exists a unique solution $v \in W_p^{2m,1}(\mathbb{R}^n \times (0,\infty))$ of* (1.1)-(1.2) *with the estimate*

$$\left\|D^{2m}v\right\|_{L^p(\mathbb{R}^n\times(0,\infty))} + \|v_t\|_{L^p(\mathbb{R}^n\times(0,\infty))} \leq C\|g\|_{L^p(\mathbb{R}^n\times(0,\infty))}. \tag{3.2}$$

Moreover, we give one important lemma, which is motivated by the iteration-covering procedure in [12]. To start with, let u be a solution of (1.1)-(1.2). Let

$$p = \frac{1+\alpha_2}{2} > 1. \tag{3.3}$$

In fact, in the subsequent proof we can choose any constant p with $1 < p < \alpha_2$. Now we write

$$\lambda_0^p = \int_{\mathbb{R}^n\times(0,\infty)} \left|D^{2m}u\right|^p dz + \frac{1}{\epsilon}\int_{\mathbb{R}^n\times(0,\infty)} |f|^p dz, \tag{3.4}$$

while $\epsilon \in (0,1)$ is a small enough constant which will be determined later. Set

$$u_\lambda = \frac{u}{\lambda_0\lambda}, \qquad f_\lambda = \frac{f}{\lambda_0\lambda} \tag{3.5}$$

for any $\lambda > 0$. Then u_λ is still the solution of (1.1)-(1.2) with f_λ replacing f. Moreover, we write

$$J_\lambda[Q] = \fint_Q \left|D^{2m}u_\lambda\right|^p dz + \frac{1}{\epsilon}\fint_Q |f_\lambda|^p dz \tag{3.6}$$

for any domain Q in $\mathbb{R}^{n+1}$ and the level set

$$E_\lambda(1) = \left\{z \in \mathbb{R}^n \times (0,\infty) : \left|D^{2m}u_\lambda\right| > 1\right\}. \tag{3.7}$$

Next, we will decompose the level set $E_\lambda(1)$.

Lemma 3.3. *For any $\lambda > 0$, there exists a family of disjoint cylinders $\{Q_{\rho_i}(z_i)\}_{i\in\mathbb{N}}$ with $z_i = (x_i, t_i) \in E_\lambda(1)$ and $\rho_i = \rho(z_i, \lambda) > 0$ such that*

$$J_\lambda\left[Q_{\rho_i}(z_i)\right] = 1, \quad J_\lambda\left[Q_\rho(z_i)\right] < 1 \quad \text{for any } \rho > \rho_i, \tag{3.8}$$

$$E_\lambda(1) \subset \bigcup_{i\in\mathbb{N}} Q_{5\rho_i}(z_i) \cup \text{negligible set}, \tag{3.9}$$

where $Q_{5\rho_i}(z_i) =: B_{5\rho_i}(x_i) \times (t_i - (5\rho_i)^{2m}, t_i + (5\rho_i)^{2m}]$. Moreover, one has

$$\left|Q_{\rho_i}(z_i)\right| \le 2\left(\int_{\{z\in Q_{\rho_i}(z_i):|D^{2m}u_\lambda|^p>1/4\}} \left|D^{2m}u_\lambda\right|^p dz + \frac{1}{\epsilon}\int_{\{z\in Q_{\rho_i}(z_i):|f_\lambda|^p>\epsilon/4\}} |f_\lambda|^p dz\right). \tag{3.10}$$

Proof. (1) Fix any $\lambda > 0$. We first claim that

$$\sup_{w\in\mathbb{R}^n\times(0,\infty)} \sup_{\rho\ge\rho_0} J_\lambda\left[Q_\rho(w)\right] \le 1, \tag{3.11}$$

where $\rho_0 = \rho_0(\lambda) > 0$ satisfies $\lambda^p |Q_{\rho_0}| = 1$. To prove this, fix any $w \in \mathbb{R}^n \times (0,\infty)$ and $\rho \ge \rho_0$. Then it follows from (3.4) that

$$\begin{aligned} J_\lambda\left[Q_\rho(w)\right] &\le \frac{1}{\lambda_0^p \lambda^p \left|Q_\rho(w)\right|}\left\{\int_{\mathbb{R}^n\times(0,\infty)} \left|D^{2m}u\right|^p dz + \frac{1}{\epsilon}\int_{\mathbb{R}^n\times(0,\infty)} |f|^p dz\right\} \\ &\le \frac{1}{\lambda^p \left|Q_{\rho_0}\right|} = 1. \end{aligned} \tag{3.12}$$

(2) For a.e. $w \in E_\lambda(1)$, from Lebesgue's differentiation theorem we have

$$\lim_{\rho\to 0} J_\lambda\left[Q_\rho(w)\right] > 1, \tag{3.13}$$

which implies that there exists some $\rho > 0$ satisfying

$$J_\lambda\left[Q_\rho(w)\right] > 1. \tag{3.14}$$

Therefore from (3.11) we can select a radius $\rho_w \in (0, \rho_0]$ such that

$$J_\lambda\left[Q_{\rho_w}(w)\right] = 1, \quad J_\lambda\left[Q_\rho(w)\right] < 1 \quad \text{for any } \rho > \rho_w. \tag{3.15}$$

Therefore, applying Vitali's covering lemma, we can find a family of disjoint cylinders $\{Q_{\rho_i}(z_i)\}$ such that (3.8) and (3.9) hold.

(3) Equation (3.8) implies that

$$⨍_{Q_{\rho_i}(z_i)} \left|D^{2m}u_\lambda\right|^p dz + \frac{1}{\epsilon} ⨍_{Q_{\rho_i}(z_i)} \left|f_\lambda\right|^p dz = 1. \tag{3.16}$$

Therefore, by splitting the two integrals above as follows we have

$$\begin{aligned} \left|Q_{\rho_i}(z_i)\right| \le{}& \int_{\{z\in Q_{\rho_i}(z_i):|D^{2m}u_\lambda|^p>1/4\}} \left|D^{2m}u_\lambda\right|^p dz + \frac{\left|cQ_{\rho_i}(z_i)\right|}{4} \\ &+ \frac{1}{\epsilon}\int_{\{z\in Q_{\rho_i}(z_i):|f_\lambda|^p>\epsilon/4\}} \left|f_\lambda\right|^p dz + \frac{\left|Q_{\rho_i}(z_i)\right|}{4}. \end{aligned} \tag{3.17}$$

Thus we can obtain the desired result (3.10). □

Now we are ready to prove the main result, Theorem 1.9.

Proof. In the following by the elementary approximation argument as [3, 12] it is sufficient to consider the proof of (1.8) under the additional assumption that $D^{2m}u \in L^\phi(\mathbb{R}^n \times (0,\infty))$. In view of Lemma 3.3, given any $\lambda > 0$, we can construct a family of cylinders $\{Q_{\rho_i}(z_i)\}_{i\in\mathbb{N}}$, where $z_i = (x_i, t_i) \in E_\lambda(1)$. Fix $i \ge 1$. It follows from (3.6) and (3.8) in Lemma 3.3 that

$$⨍_{Q_{10\rho_i}(z_i)} \left|D^{2m}u_\lambda\right|^p dz \le 1, \qquad ⨍_{Q_{10\rho_i}(z_i)} \left|f_\lambda\right|^p dz \le \epsilon. \tag{3.18}$$

We first extend f_λ from $Q_{10\rho_i}(z_i)$ to $\mathbb{R}^{n+1}$ by the zero extension and denote by $\overline{f}_\lambda$. From Lemma 3.2, there exists a unique solution $v \in W_p^{2m,1}(\mathbb{R}^n \times (0,\infty))$ of

$$\begin{aligned} v_t + (-\Delta)^{2m} v &= \overline{f}_\lambda \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ v &= 0 \quad \text{in } \mathbb{R}^n \times \{t = 0\} \end{aligned} \tag{3.19}$$

with the estimate

$$\left\|D^{2m}v\right\|_{L^p(\mathbb{R}^n\times(0,\infty))} \le C\left\|\overline{f}_\lambda\right\|_{L^p(\mathbb{R}^n\times(0,\infty))}. \tag{3.20}$$

Therefore we see that

$$\begin{aligned} \left\| D^{2m} v \right\|_{L^p(Q_{10\rho_i}(z_i))} &\leq \left\| D^{2m} v \right\|_{L^p(\mathbb{R}^n \times (0,\infty))} \\ &\leq C \left\| \overline{f}_\lambda \right\|_{L^p(\mathbb{R}^n \times (0,\infty))} \\ &= C \| f_\lambda \|_{L^p(Q_{10\rho_i}(z_i))}. \end{aligned} \tag{3.21}$$

Set $w = u_\lambda - v$. Then we know that

$$w_t + (-\Delta)^{2m} w = 0 \quad \text{in } Q_{10\rho_i}(z_i). \tag{3.22}$$

Moreover, by (3.18) and (3.21) we have

$$\begin{aligned} ⨍_{Q_{10\rho_i}(z_i)} \left| D^{2m} w \right|^p dz &\leq 2^p \left(⨍_{Q_{10\rho_i}(z_i)} \left| D^{2m} v \right|^p dz + ⨍_{Q_{10\rho_i}(z_i)} \left| D^{2m} v \right|^p dz \right) \\ &\leq 2^p + C ⨍_{Q_{10\rho_i}(z_i)} |f_\lambda|^p dz \leq C. \end{aligned} \tag{3.23}$$

Thus from the elementary interior $W_\infty^{2m,1}$ regularity, we know that there exists a constant $N_1 > 1$ such that

$$\sup_{Q_{5\rho_i}(z_i)} \left| D^{2m} w \right| \leq N_1. \tag{3.24}$$

Set $\mu = \lambda\lambda_0$. Therefore, we deduce from (3.5) and (3.24) that

$$\begin{aligned} &\left| \left\{ z \in Q_{5\rho_i}(z_i) : \left| D^{2m} u \right| > 2N_1 \mu \right\} \right| \\ &\quad = \left| \left\{ z \in Q_{5\rho_i}(z_i) : \left| D^{2m} u_\lambda \right| > 2N_1 \right\} \right| \\ &\quad \leq \left| \left\{ z \in Q_{5\rho_i}(z_i) : \left| D^{2m} w \right| > N_1 \right\} \right| + \left| \left\{ z \in Q_{5\rho_i}(z_i) : \left| D^{2m} v \right| > N_1 \right\} \right| \\ &\quad = \left| \left\{ z \in Q_{5\rho_i}(z_i) : \left| D^{2m} v \right| > N_1 \right\} \right| \leq \frac{1}{N_1^p} \int_{Q_{5\rho_i}(z_i)} \left| D^{2m} v \right|^p dz. \end{aligned} \tag{3.25}$$

Then according to (3.18) and (3.21), we discover

$$\begin{aligned} &\left| \left\{ z \in Q_{5\rho_i}(z_i) : \left| D^{2m} u \right| > 2N_1 \mu \right\} \right| \\ &\quad \leq C \int_{Q_{10\rho_i}(z_i)} |f_\lambda|^p dz \leq C\epsilon \left| Q_{10\rho_i}(z_i) \right| = C\epsilon \left| Q_{\rho_i}(z_i) \right|. \end{aligned} \tag{3.26}$$

Therefore, from (3.10) in Lemma 3.3 we find that

$$\begin{aligned}&\left|\left\{z\in Q_{5\rho_i}(z_i):\left|D^{2m}u\right|>2N_1\mu\right\}\right|\\&\quad\le\frac{C}{\mu^p}\left(\epsilon\int_{\{z\in Q_{\rho_i}(z_i):|D^{2m}u|^p>\mu^p/4\}}\left|D^{2m}u\right|^p dz+\int_{\{z\in Q_{\rho_i}(z_i):|f|>\epsilon\mu^p/4\}}|f|^p dz\right),\end{aligned}\tag{3.27}$$

where $C=C(n,m)$. Recalling the fact that the cylinders $\{Q_{\rho_i}(z_i)\}_{i\in\mathbb{N}}$ are disjoint,

$$\bigcup_{i\in\mathbb{N}}Q_{5\rho_i}(z_i)\cup\text{negligible set}\supset E_\lambda(1)=\left\{z\in\mathbb{R}^n\times(0,\infty):\left|D^{2m}u_\lambda(z)\right|>1\right\},\tag{3.28}$$

and then summing up on $i\in\mathbb{N}$ in the inequality above, we have

$$\begin{aligned}&\left|\left\{z\in\mathbb{R}^n\times(0,\infty):\left|D^{2m}u\right|>2N_1\mu\right\}\right|\\&\quad\le\sum_{i\in\mathbb{N}}\left|\left\{z\in Q_{5\rho_i}(z_i):\left|D^{2m}u\right|>2N_1\mu\right\}\right|\\&\quad\le\frac{C}{\mu^p}\left(\epsilon\int_{\{z\in\mathbb{R}^n\times(0,\infty):|D^{2m}u|^p>\mu^p/4\}}\left|D^{2m}u\right|^p\,dz+\int_{\{z\in\mathbb{R}^n\times(0,\infty):|f|>\epsilon\mu^p/4\}}|f|^p\,dz\right).\end{aligned}\tag{3.29}$$

Therefore, from Lemma 1.7(3) and the inequality above we have

$$\begin{aligned}&\int_{\mathbb{R}^n\times(0,\infty)}\phi\left(\left|D^{2m}u\right|\right)dz\\&\quad=\int_0^\infty\left|\left\{z\in\mathbb{R}^n\times(0,\infty):\left|D^{2m}u\right|>2N_1\mu\right\}\right|d[\phi(2N_1\mu)]\\&\quad\le C\epsilon\int_0^\infty\frac{1}{\mu^p}\left\{\int_{\{z\in\mathbb{R}^n\times(0,\infty):|D^{2m}u|^p>\mu^p/4\}}\left|D^{2m}u\right|^p dz\right\}d[\phi(2N_1\mu)]\\&\qquad+C\int_0^\infty\frac{1}{\mu^p}\left\{\int_{\{z\in\mathbb{R}^n\times(0,\infty):|f|>\epsilon\mu^p/4\}}|f|^p dz\right\}d[\phi(2N_1\mu)].\end{aligned}\tag{3.30}$$

Consequently, from Lemma 3.1 we conclude that

$$\int_{\mathbb{R}^n\times(0,\infty)}\phi\left(\left|D^{2m}u\right|\right)dz\le C_1\epsilon\int_{\mathbb{R}^n\times(0,\infty)}\phi\left(\left|D^{2m}u\right|\right)dz+C_2\int_{\mathbb{R}^n\times(0,\infty)}\phi(|f|)dz,\tag{3.31}$$

where $C_1=C_1(n,m,\phi)$ and $C_2=C_2(n,m,\epsilon,\phi)$. Finally selecting a suitable $\epsilon\in(0,1)$ such that $C_1\epsilon\le 1/2$, we finish the proof. □

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