Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 973731, 22 pages
doi:10.1155/2010/973731

Research Article

Solutions and Green's Functions for Boundary Value Problems of Second-Order Four-Point Functional Difference Equations

Yang Shujie and Shi Bao

Institute of Systems Science and Mathematics, Naval Aeronautical and Astronautical University, Yantai, Shandong 264001, China

Correspondence should be addressed to Yang Shujie, yangshujie@163.com

Received 23 April 2010; Accepted 11 July 2010

Academic Editor: Irena Rachůnková

We consider the Green's functions and the existence of positive solutions for a second-order functional difference equation with four-point boundary conditions.

1. Introduction

In recent years, boundary value problems (BVPs) of differential and difference equations have been studied widely and there are many excellent results (see Gai et al. [1], Guo and Tian [2], Henderson and Peterson [3], and Yang et al. [4]). By using the critical point theory, Deng and Shi [5] studied the existence and multiplicity of the boundary value problems to a class of second-order functional difference equations

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}) \tag{1.1}$$

with boundary value conditions

$$\Delta u_0 = A, \qquad u_{k+1} = B, \tag{1.2}$$

where the operator L is the Jacobi operator

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n. \tag{1.3}$$

Ntouyas et al. [6] and Wong [7] investigated the existence of solutions of a BVP for functional differential equations

$$\begin{aligned} x''(t) &= f\left(t, x_t, x'(t)\right), \quad t \in [0, T], \\ \alpha_0 x_0 - \alpha_1 x'(0) &= \phi \in C_r, \\ \beta_0 x(T) + \beta_1 x'(T) &= A \in \mathbb{R}^n, \end{aligned} \tag{1.4}$$

where $f : [0, T] \times C_r \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $\phi \in C_r = C([-r, 0], \mathbb{R}^n)$, $A \in \mathbb{R}^n$, and $x_t(\theta) = x(t + \theta), \theta \in [-r, 0]$.

Weng and Guo [8] considered the following two-point BVP for a nonlinear functional difference equation with p-Laplacian operator

$$\begin{aligned} \Delta\Phi_p(\Delta x(t)) + r(t) f(x_t) &= 0, \quad t \in \{1, \ldots, T\}, \\ x_0 = \phi \in C^+, &\qquad \Delta x(T+1) = 0, \end{aligned} \tag{1.5}$$

where $\Phi_p(u) = |u|^{p-2}u$, $p > 1$, $\phi(0) = 0$, $T, \tau \in \mathbb{N}$, $C^+ = \{\phi \mid \phi(k) \geq 0, k \in [-\tau, 0]\}$, $f : C^+ \to \mathbb{R}^+$ is continuous, $\sum_{t=\tau+1}^{T} r(t) > 0$.

Yang et al. [9] considered two-point BVP of the following functional difference equation with p-Laplacian operator:

$$\begin{aligned} \Delta\Phi_p(\Delta x(t)) + r(t) f(x(t), x_t) &= 0, \quad t \in \{1, \ldots, T\}, \\ \alpha_0 x_0 - \alpha_1 \Delta x(0) &= h, \\ \beta_0 x(T+1) + \beta_1 \Delta x(T+1) &= A, \end{aligned} \tag{1.6}$$

where $h \in C_\tau^+ = \{\phi \in C_\tau \mid \phi(\theta) \geq 0, \theta \in \{-\tau, \ldots, 0\}\}$, $A \in \mathbb{R}^+$, and α_0, α_1, β_0, and β_1 are nonnegative real constants.

For $a, b \in \mathbb{N}$ and $a < b$, let

$$\begin{gathered} \mathbb{R}^+ = \{x \mid x \in \mathbb{R}, x \geq 0\}, \\ [a, b] = \{a, a+1, \ldots, b\}, \qquad [a, b) = \{a, a+1, \ldots, b-1\}, \qquad [a, +\infty) = \{a, a+1, \ldots, \}, \\ C_\tau = \{\phi \mid \phi : [-\tau, 0] \to R\}, \qquad C_\tau^+ = \{\phi \in C_\tau \mid \phi(\theta) \geq 0, \ \ \theta \in [-\tau, 0]\}. \end{gathered} \tag{1.7}$$

Then C_τ and C_τ^+ are both Banach spaces endowed with the max-norm

$$\|\phi\|_\tau = \max_{k \in [-\tau, 0]} |\phi(k)|. \tag{1.8}$$

For any real function x defined on the interval $[-\tau, T]$ and any $t \in [0, T]$ with $T \in \mathbb{N}$, we denote by x_t an element of C_τ defined by $x_t(k) = x(t + k)$, $k \in [-\tau, 0]$.

In this paper, we consider the following second-order four-point BVP of a nonlinear functional difference equation:

$$\begin{aligned} -\Delta^2 u(t-1) &= r(t) f(t, u_t), \quad t \in [1, T], \\ u_0 &= \alpha u(\eta) + h, \quad t \in [-\tau, 0], \\ u(T+1) &= \beta u(\xi) + \gamma, \end{aligned} \tag{1.9}$$

where $\xi, \eta \in (1, T)$ and $\xi < \eta$, $0 < \tau < T$, $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $f : \mathbb{R} \times C_\tau \to \mathbb{R}^+$ is a continuous function, $h \in C_\tau^+$ and $h(t) \geq h(0) \geq 0$ for $t \in [-\tau, 0]$, α, β, and γ are nonnegative real constants, and $r(t) \geq 0$ for $t \in [1, T]$.

At this point, it is necessary to make some remarks on the first boundary condition in (1.9). This condition is a generalization of the classical condition

$$u(0) = \alpha u(\eta) + C \tag{1.10}$$

from ordinary difference equations. Here this condition connects the history u_0 with the single $u(\eta)$. This is suggested by the well-posedness of BVP (1.9), since the function f depends on the term u_t (i.e., past values of u).

As usual, a sequence $\{u(-\tau), \ldots, u(T+1)\}$ is said to be a positive solution of BVP (1.9) if it satisfies BVP (1.9) and $u(k) \geq 0$ for $k \in [-\tau, T]$ with $u(k) > 0$ for $k \in [1, T]$.

2. The Green's Function of (1.9)

First we consider the nonexistence of positive solutions of (1.9). We have the following result.

Lemma 2.1. *Assume that*

$$\beta \xi > T + 1, \tag{2.1}$$

or

$$\alpha(T + 1 - \eta) > T + 1. \tag{2.2}$$

Then (1.9) *has no positive solution.*

Proof. From $\Delta^2 u(t-1) = -r(t) f(t, u_t) \leq 0$, we know that $u(t)$ is convex for $t \in [0, T+1]$.

Assume that $x(t)$ is a positive solution of (1.9) and (2.1) holds.

(1) Consider that $\gamma = 0$.

If $x(T+1) > 0$, then $x(\xi) > 0$. It follows that

$$\begin{aligned}\frac{x(T+1)-x(0)}{T+1} &= \frac{\beta x(\xi)-x(0)}{T+1} \\ &> \frac{x(\xi)}{\xi} - \frac{x(0)}{T+1} \\ &\geq \frac{x(\xi)-x(0)}{\xi},\end{aligned} \tag{2.3}$$

which is a contradiction to the convexity of $x(t)$.

If $x(T+1) = 0$, then $x(\xi) = 0$. If $x(0) > 0$, then we have

$$\begin{aligned}\frac{x(T+1)-x(0)}{T+1} &= -\frac{x(0)}{T+1}, \\ \frac{x(\xi)-x(0)}{\xi} &= -\frac{x(0)}{\xi}.\end{aligned} \tag{2.4}$$

Hence

$$\frac{x(T+1)-x(0)}{T+1} > \frac{x(\xi)-x(0)}{\xi}, \tag{2.5}$$

which is a contradiction to the convexity of $x(t)$. If $x(t) \equiv 0$ for $t \in [1,T]$, then $x(t)$ is a trivial solution. So there exists a $t_0 \in [1,\xi) \cup (\xi,T]$ such that $x(t_0) > 0$.

We assume that $t_0 \in [1,\xi)$. Then

$$\begin{aligned}\frac{x(T+1)-x(t_0)}{T+1-t_0} &= -\frac{x(t_0)}{T+1-t_0}, \\ \frac{x(\xi)-x(t_0)}{\xi-t_0} &= -\frac{x(t_0)}{\xi-t_0}.\end{aligned} \tag{2.6}$$

Hence

$$\frac{x(T+1)-x(t_0)}{T+1-t_0} > \frac{x(\xi)-x(t_0)}{\xi-t_0}, \tag{2.7}$$

which is a contradiction to the convexity of $x(t)$.

If $t_0 \in (\xi,T]$, similar to the above proof, we can also get a contradiction.

(2) Consider that $\gamma > 0$.

Now we have

$$\begin{aligned}\frac{x(T+1)-x(0)}{T+1} &= \frac{\beta x(\xi)-x(0)+\gamma}{T+1}\\ &\geq \frac{x(\xi)}{\xi}-\frac{x(0)}{T+1}+\frac{\gamma}{T+1}\\ &\geq \frac{x(\xi)-x(0)}{\xi}+\frac{\gamma}{T+1}\\ &> \frac{x(\xi)-x(0)}{\xi},\end{aligned} \tag{2.8}$$

which is a contradiction to the convexity of $x(t)$.

Assume that $x(t)$ is a positive solution of (1.9) and (2.2) holds.

(1) Consider that $h(0)=0$.

If $x(T+1)>0$, then we obtain

$$\begin{aligned}\frac{x(T+1)-x(0)}{T+1} &= \frac{x(\mathrm{T}+1)-\alpha x(\eta)}{T+1}\\ &< \frac{x(T+1)}{T+1-\eta}-\frac{\alpha x(\eta)}{T+1}\\ &\leq \frac{x(T+1)-x(\eta)}{T+1-\eta},\end{aligned} \tag{2.9}$$

which is a contradiction to the convexity of $x(t)$.

If $x(\eta)>0$, similar to the above proof, we can also get a contradiction.

If $x(T+1)=x(\eta)=0$, and so $x(0)=0$, then there exists a $t_0 \in [1,\eta)\cup(\eta,T]$ such that $x(t_0)>0$. Otherwise, $x(t)\equiv 0$ is a trivial solution. Assume that $t_0\in[1,\eta)$, then

$$\begin{aligned}\frac{x(T+1)-x(t_0)}{T+1-t_0} &= -\frac{x(t_0)}{T+1-t_0},\\ \frac{x(\eta)-x(t_0)}{\eta-t_0} &= -\frac{x(t_0)}{\eta-t_0},\end{aligned} \tag{2.10}$$

which implies that

$$\frac{x(T+1)-x(t_0)}{T+1-t_0} > \frac{x(\eta)-x(t_0)}{\eta-t_0}. \tag{2.11}$$

A contradiction to the convexity of $x(t)$ follows.

If $t_0\in(\eta,T]$, we can also get a contradiction.

(2) Consider that $h(0)>0$.

Now we obtain

$$\begin{aligned}\frac{x(T+1)-x(0)}{T+1} &= \frac{x(T+1)-\alpha x(\eta)-h(0)}{T+1}\\ &\le \frac{x(T+1)}{T+1-\eta}-\frac{x(\eta)}{T+1-\eta}-\frac{h(0)}{T+1}\\ &< \frac{x(T+1)-x(\eta)}{T+1-\eta},\end{aligned} \tag{2.12}$$

which is a contradiction to the convexity of $x(t)$. □

Next, we consider the existence of the Green's function of equation

$$\begin{aligned}-\Delta^2 u(t-1) &= f(t),\\ u(0) &= \alpha u(\eta),\\ u(T+1) &= \beta u(\xi).\end{aligned} \tag{2.13}$$

We always assume that

(H_1) $0 \le \alpha, \beta \le 1$ and $\alpha\beta < 1$.

Motivated by Zhao [10], we have the following conclusions.

Theorem 2.2. *The Green's function for second-order four-point linear BVP* (2.13) *is given by*

$$\begin{aligned}G_1(t,s) = G(t,s) &+ \frac{\alpha(T+1-t)}{\alpha\eta+(1-\alpha)(T+1)} \times \frac{\alpha\eta+\beta(1-\alpha)t+(1-\alpha)(T+1-\beta\xi)}{(1-\beta)\alpha\eta+(1-\alpha)(T+1-\beta\xi)}G(\eta,s)\\ &+ \frac{\beta(1-\alpha)t+\alpha\beta\eta}{(1-\beta)\alpha\eta+(1-\alpha)(T+1-\beta\xi)}G(\xi,s),\end{aligned} \tag{2.14}$$

where

$$G(t,s) = \begin{cases} \dfrac{s(T+1-t)}{T+1}, & 0 \le s \le t-1,\\ \dfrac{t(T+1-s)}{T+1}, & t \le s \le T+1.\end{cases} \tag{2.15}$$

Proof. Consider the second-order two-point BVP

$$\begin{aligned}-\Delta^2 u(t-1) &= f(t), \quad t \in [1,T],\\ u(0) &= 0,\\ u(T+1) &= 0.\end{aligned} \tag{2.16}$$

It is easy to find that the solution of BVP (2.16) is given by

$$u(t) = \sum_{s=1}^{T} G(t,s) f(s), \tag{2.17}$$

$$u(0) = 0, \qquad u(T+1) = 0, \qquad u(\eta) = \sum_{s=1}^{T} G(\eta,s) f(s). \tag{2.18}$$

The three-point BVP

$$\begin{aligned} -\Delta^2 u(t-1) &= f(t), \quad t \in [1,T], \\ u(0) &= \alpha u(\eta), \quad t \in [-\tau,0], \\ u(T+1) &= 0 \end{aligned} \tag{2.19}$$

can be obtained from replacing $u(0) = 0$ by $u(0) = \alpha u(\eta)$ in (2.16). Thus we suppose that the solution of (2.19) can be expressed by

$$v(t) = u(t) + (c + dt) u(\eta), \tag{2.20}$$

where c and d are constants that will be determined.

From (2.18) and (2.20), we have

$$\begin{aligned} v(0) &= u(0) + c u(\eta), \\ v(\eta) &= u(\eta) + (c + d\eta) u(\eta) = (1 + c + d\eta) u(\eta), \\ v(T+1) &= u(T+1) + (c + d(T+1)) u(\eta) = (c + d(T+1)) u(\eta). \end{aligned} \tag{2.21}$$

Putting the above equations into (2.19) yields

$$\begin{aligned} (1-\alpha) c - \alpha \eta d &= \alpha, \\ c + (T+1) d &= 0. \end{aligned} \tag{2.22}$$

By (H_1), we obtain c and d by solving the above equation:

$$\begin{aligned} c &= \frac{\alpha (T+1)}{\alpha \eta + (1-\alpha)(T+1)}, \\ d &= \frac{-\alpha}{\alpha \eta + (1-\alpha)(T+1)}. \end{aligned} \tag{2.23}$$

By (2.19) and (2.20), we have

$$v(0) = \alpha v(\eta),$$
$$v(T+1) = 0, \tag{2.24}$$
$$v(\xi) = u(\xi) + (c + d\xi)u(\eta).$$

The four-point BVP (2.13) can be obtained from replacing $u(T+1) = 0$ by $u(T+1) = \beta u(\xi)$ in (2.19). Thus we suppose that the solution of (2.13) can be expressed by

$$w(t) = v(t) + (a + bt)v(\xi), \tag{2.25}$$

where a and b are constants that will be determined.

From (2.24) and (2.25), we get

$$\begin{aligned} w(0) &= v(0) + av(\xi) = \alpha v(\eta) + av(\xi), \\ w(\eta) &= v(\eta) + (a + b\eta)v(\xi), \\ w(T+1) &= v(T+1) + (a + b(T+1))v(\xi) = (a + b(T+1))v(\xi), \\ w(\xi) &= v(\xi) + (a + b\xi)v(\xi). \end{aligned} \tag{2.26}$$

Putting the above equations into (2.13) yields

$$\begin{aligned} (1-\alpha)a - \alpha\eta b &= 0, \\ (1-\beta)a + (T + 1 - \beta\xi)b &= \beta. \end{aligned} \tag{2.27}$$

By (H_1), we can easily obtain

$$\begin{aligned} a &= \frac{\alpha\beta\eta}{(1-\beta)\alpha\eta + (1-\alpha)(T+1-\beta\xi)}, \\ b &= \frac{\beta(1-\alpha)}{(1-\beta)\alpha\eta + (1-\alpha)(T+1-\beta\xi)}. \end{aligned} \tag{2.28}$$

Then by (2.17), (2.20), (2.23), (2.25), and (2.28), the solution of BVP (2.13) can be expressed by

$$w(t) = \sum_{s=1}^{T} G_1(t,s)f(s), \tag{2.29}$$

where $G_1(t,s)$ is defined in (2.14). That is, $G_1(t,s)$ is the Green's function of BVP (2.13). □

Remark 2.3. By (H_1), we can see that $G_1(t,s) > 0$ for $(t,s) \in [0, T+1]^2$. Let

$$m = \min_{(t,s)\in[1,T]^2} G_1(t,s), \qquad M = \max_{(t,s)\in[1,T]^2} G_1(t,s). \tag{2.30}$$

Then $M \geq m > 0$.

Lemma 2.4. *Assume that (H_1) holds. Then the second-order four-point BVP* (2.13) *has a unique solution which is given in* (2.29).

Proof. We need only to show the uniqueness.

Obviously, $w(t)$ in (2.29) is a solution of BVP (2.13). Assume that $v(t)$ is another solution of BVP (2.13). Let

$$z(t) = v(t) - w(t), \quad t \in [-\tau, T+1]. \tag{2.31}$$

Then by (2.13), we have

$$-\Delta^2 z(t-1) = -\Delta^2 v(t-1) + \Delta^2 w(t-1) \equiv 0, \quad t \in [1, T], \tag{2.32}$$

$$\begin{gathered} z(0) = v(0) - w(0) = \alpha z(\eta), \\ z(T+1) = v(T+1) - w(T+1) = \beta z(\xi). \end{gathered} \tag{2.33}$$

From (2.32) we have, for $t \in [1, T]$,

$$z(t) = c_1 t + c_2, \tag{2.34}$$

which implies that

$$z(0) = c_2, \qquad z(\eta) = c_1\eta + c_2, \qquad z(\xi) = c_1\xi + c_2, \qquad z(T+1) = c_1(T+1) + c_2. \tag{2.35}$$

Combining (2.33) with (2.35), we obtain

$$\begin{gathered} \alpha\eta c_1 - (1-\alpha)c_2 = 0, \\ (T+1-\beta\xi)c_1 + (1-\beta)c_2 = 0. \end{gathered} \tag{2.36}$$

Condition (H_1) implies that (2.36) has a unique solution $c_1 = c_2 = 0$. Therefore $v(t) \equiv w(t)$ for $t \in [-\tau, T+1]$. This completes the proof of the uniqueness of the solution. □

3. Existence of Positive Solutions

In this section, we discuss the BVP (1.9).

Assume that $h(0) = 0$, $\gamma = 0$.

We rewrite BVP (1.9) as

$$\begin{gathered} -\Delta^2 u(t-1) = r(t) f(t, u_t), \quad t \in [1, T], \\ u_0 = \alpha u(\eta) + h, \quad t \in [-\tau, 0], \\ u(T+1) = \beta u(\xi) \end{gathered} \tag{3.1}$$

with $h(0) = 0$.

Suppose that $u(t)$ is a solution of the BVP (3.1). Then it can be expressed as

$$u(t) = \begin{cases} \displaystyle\sum_{s=1}^{T} G_1(t,s) r(s) f(s, u_s), & t \in [1, T], \\ \alpha u(\eta) + h(t), & t \in [-\tau, 0], \\ \beta u(\xi), & t = T+1. \end{cases} \tag{3.2}$$

Lemma 3.1 (see Guo et al. [11]). *Assume that E is a Banach space and $K \subset E$ is a cone in E. Let $K_p = \{u \in K \mid \|u\| = p\}$. Furthermore, assume that $\Phi : K \to K$ is a completely continuous operator and $\Phi u \neq u$ for $u \in \partial K_p = \{u \in K \mid \|u\| = p\}$. Thus, one has the following conclusions:*

(1) if $\|u\| \leq \|\Phi u\|$ for $u \in \partial K_p$, then $i(\Phi, K_p, K) = 0$;

(2) if $\|u\| \geq \|\Phi u\|$ for $u \in \partial K_p$, then $i(\Phi, K_p, K) = 1$.

Assume that $f \equiv 0$. Then (3.1) may be rewritten as

$$\begin{gathered} -\Delta^2 u(t-1) = 0, \quad t \in [1, T], \\ u_0 = \alpha u(\eta) + h, \\ u(T+1) = \beta u(\xi). \end{gathered} \tag{3.3}$$

Let $\overline{u}(t)$ be a solution of (3.3). Then by (3.2) and $\xi, \eta \in (1, T)$, it can be expressed as

$$\overline{u}(t) = \begin{cases} 0, & t \in [1, T], \\ h(t), & t \in [-\tau, 0], \\ 0, & t = T+1. \end{cases} \tag{3.4}$$

Let $u(t)$ be a solution of BVP (3.1) and $y(t) = u(t) - \overline{u}(t)$. Then for $t \in [1, T]$ we have $y(t) \equiv u(t)$ and

$$y(t) = \begin{cases} \sum_{s=1}^{T} G_1(t,s) r(s) f\left(s, y_s + \overline{u}_s\right), & t \in [1, T], \\ \alpha y(\eta), & t \in [-\tau, 0], \\ \beta y(\xi), & t = T+1. \end{cases} \tag{3.5}$$

Let

$$\|u\| = \max_{t \in [-\tau, T+1]} |u(t)|, \qquad E = \{u \mid u : [-\tau, T+1] \to R\},$$
$$K = \left\{ u \in E \mid \min_{t \in [1,T]} u(t) \geq \frac{m}{M} \|u\|,\ u(t) = \alpha u(\eta),\ t \in [-\tau, 0],\ u(T+1) = \beta u(\xi) \right\}. \tag{3.6}$$

Then E is a Banach space endowed with norm $\|\cdot\|$ and K is a cone in E.

For $y \in K$, we have by (H_1) and the definition of K,

$$\|y\| = \max_{t \in [-\tau, T+1]} |y(t)| = \max_{t \in [1,T]} |y(t)|. \tag{3.7}$$

For every $y \in \partial K_p$, $s \in [1, T]$, and $k \in [-\tau, 0]$, by the definition of K and (3.5), if $s + k \leq 0$, we have

$$y_s = y(s+k) = \alpha y(\eta). \tag{3.8}$$

If $T \geq s + k \geq 1$, we have, by (3.4),

$$\overline{u}_s = \overline{u}(s+k) = 0, \qquad y_s = y(s+k) \geq \min_{t \in [1,T]} y(t) \geq \frac{m}{M} \|y\|, \tag{3.9}$$

hence by the definition of $\|\cdot\|_\tau$, we obtain for $s \in [\tau + 1, T]$

$$\|y_s\|_\tau \geq \frac{m}{M} \|y\|. \tag{3.10}$$

Lemma 3.2. *For every $y \in K$, there is $t_0 \in [\tau + 1, T]$, such that*

$$\|y_{t_0}\|_\tau = \|y\|. \tag{3.11}$$

Proof. For $s \in [\tau+1, T]$, $k \in [-\tau, 0]$, and $s+k \in [1, T]$, by the definitions of $\|\cdot\|_\tau$ and $\|\cdot\|$, we have

$$\|y_s\|_\tau = \max_{k\in[-\tau,0]} \left|y(s+k)\right|, \qquad \|y\| = \max_{t\in[1,T]} \left|y(t)\right|. \tag{3.12}$$

Obviously, there is a $t_0 \in [\tau+1, T]$, such that (3.11) holds.

Define an operator $\Phi : K \to E$ by

$$(\Phi y)(t) = \begin{cases} \sum_{s=1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s), & t \in [1, T], \\ \alpha (\Phi y)(\eta), & t \in [-\tau, 0], \\ \beta (\Phi y)(\xi), & t = T+1. \end{cases} \tag{3.13}$$

Then we may transform our existence problem of positive solutions of BVP (3.1) into a fixed point problem of operator (3.13). □

Lemma 3.3. *Consider that* $\Phi(K) \subset K$.

Proof. If $t \in [-\tau, 0]$ and $t = T+1$, $(\Phi y)(t) = \alpha\Phi(\eta)$ and $(\Phi y)(T+1) = \beta\Phi(\xi)$, respectively. Thus, (H_1) yields

$$\|\Phi y\| = \max_{t\in[-\tau,T+1]} \left|(\Phi y)(t)\right| = \max_{t\in[1,T]} \left|(\Phi y)(t)\right| = \|\Phi y\|_{[1,T]}. \tag{3.14}$$

It follows from the definition of K that

$$\begin{aligned} \min_{t\in[1,T]} (\Phi y)(t) &= \min_{t\in[1,T]} \sum_{s=1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s) \\ &\geq m \sum_{s=1}^{T} r(s) f(s, y_s + \overline{u}_s) \\ &\geq \frac{m}{M} \sum_{s=1}^{T} \left\{ \max_{1\leq s,t\leq T} G_1(t,s) \right\} r(s) f(s, y_s + \overline{u}_s) \\ &\geq \frac{m}{M} \max_{t\in[1,T]} \sum_{s=1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s) \\ &= \frac{m}{M} \|\Phi y\|, \end{aligned} \tag{3.15}$$

which implies that $\Phi(K) \subset K$. □

Lemma 3.4. *Suppose that (H_1) holds. Then $\Phi : K \to K$ is completely continuous.*

We assume that

(H_2)$\sum_{t=1}^{T} r(t) > 0$,

(H_3)$\overline{h} = \|h\|_\tau = \max_{t\in[-\tau,0]} h(t) > 0$.

We have the following main results.

Theorem 3.5. *Assume that (H_1)–(H_3) hold. Then BVP* (3.1) *has at least one positive solution if the following conditions are satisfied:*

(H_4) *there exists a* $p_1 > \overline{h}$ *such that, for* $s \in [1,T]$, *if* $\|\phi\|_\tau \le p_1 + \overline{h}$, *then* $f(s,\phi) \le R_1 p_1$;

(H_5) *there exists a* $p_2 > p_1$ *such that, for* $s \in [1,T]$, *if* $\|\phi\|_\tau \ge (m/M)p_2$, *then* $f(s,\phi) \ge R_2 p_2$ *or*

(H_6)$1 > \alpha > 0$;

(H_7) *there exists a* $0 < r_1 < p_1$ *such that, for* $s \in [1,T]$, *if* $\|\phi\|_\tau \le r_1$, *then* $f(s,\phi) \ge R_2 r_1$;

(H_8) *there exists an* $r_2 \ge \max\{p_2 + \overline{h}, (M\overline{h}/m\alpha)\}$, *such that, for* $s \in [1,T]$, *if* $\|\phi\|_\tau \ge (m\alpha/M)r_2 - \overline{h}$, *then* $f(s,\phi) \le R_1 r_2$,

where

$$R_1 \le \frac{1}{M\sum_{s=1}^{T} r(s)}, \qquad R_2 \ge \frac{1}{m\sum_{s=\tau+1}^{T} r(s)}. \tag{3.16}$$

Proof. Assume that (H_4) and (H_5) hold. For every $y \in \partial K_{p_1}$, we have $\|y_s + \overline{u}_s\|_\tau \le p_1 + \overline{h}$, thus

$$\begin{aligned} \|\Phi y\| &= \|\Phi y\|_{[1,T]} \\ &\le M\sum_{s=1}^{T} r(s) f(s, y_s + \overline{u}_s) \\ &\le M R_1 p_1 \sum_{s=1}^{T} r(s) \\ &\le p_1 \\ &= \|y\|, \end{aligned} \tag{3.17}$$

which implies by Lemma 3.1 that

$$i(\Phi, K_{p_1}, K) = 1. \tag{3.18}$$

For every $y \in \partial K_{p_2}$, by (3.8)–(3.10) and Lemma 3.2, we have, for $s \in [\tau+1, T]$, $\|y_s\|_\tau \geq (m/M)\|y\| = (m/M)p_2$. Then by (3.13) and ($H_5$), we have

$$\begin{aligned}\|\Phi y\| = \|\Phi y\|_{[1,T]} &\geq m \sum_{s=\tau+1}^{T} r(s) f\left(s, y_s + \overline{u}_s\right) \\ &= m \sum_{s=\tau+1}^{T} r(s) f\left(s, y_s\right) \\ &\geq mR_2 p_2 \sum_{s=\tau+1}^{T} r(s) \geq p_2 = \|y\|,\end{aligned} \tag{3.19}$$

which implies by Lemma 3.1 that

$$i\left(\Phi, K_{p_2}, K\right) = 0. \tag{3.20}$$

So by (3.18) and (3.20), there exists one positive fixed point y_1 of operator Φ with $y_1 \in \overline{K}_{p_2} \setminus K_{p_1}$.

Assume that (H_6)–(H_8) hold, for every $y \in \partial K_{r_1}$ and $s \in [\tau+1, T]$, $\|y_s + \overline{u}_s\|_\tau = \|y_s\|_\tau \leq \|y\| = r_1$, by ($H_7$), we have

$$\|\Phi y\| \geq \|y\|. \tag{3.21}$$

Thus we have from Lemma 3.1 that

$$i(\Phi, K_{r_1}, K) = 0. \tag{3.22}$$

For every $y \in \partial K_{r_2}$, by (3.8)–(3.10), we have $\|y_s + \overline{u}_s\|_\tau \geq \|y_s\|_\tau - \overline{h} \geq (m\alpha/M)r_2 - \overline{h} > 0$,

$$\|\Phi y\| \leq \|y\|. \tag{3.23}$$

Thus we have from Lemma 3.1 that

$$i(\Phi, K_{r_2}, K) = 1. \tag{3.24}$$

So by (3.22) and (3.24), there exists one positive fixed point y_2 of operator Φ with $y_2 \in \overline{K}_{r_2} \setminus K_{r_1}$.

Consequently, $u_1 = y_1 + \overline{u}$ or $u_2 = y_2 + \overline{u}$ is a positive solution of BVP (3.1). □

Theorem 3.6. *Assume that (H_1)–(H_3) hold. Then BVP* (3.1) *has at least one positive solution if (H_4) and (H_7) or (H_5) and (H_8) hold.*

Theorem 3.7. *Assume that (H_1)–(H_3) hold. Then BVP* (3.1) *has at least two positive solutions if (H_4), (H_5), and (H_7) or (H_4), (H_5), and (H_8) hold.*

Theorem 3.8. *Assume that* (H_1)–(H_3) *hold. Then BVP* (3.1) *has at least three positive solutions if* (H_4)–(H_8) *hold.*

Assume that $h(0) > 0$, $\gamma > 0$, and

(H_9) $(1-\beta)h(0) - (1-\alpha)\gamma > 0$.

Define $H(t) : [-\tau, T+1] \to \mathbb{R}$ as follows:

$$H(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ 0, & t \in [1, T], \\ H(T+1), & t = T+1, \end{cases} \tag{3.25}$$

which satisfies

(H_{10}) $(1-\alpha)H(T+1) - (1-\beta)h(0) > 0$.

Obviously, $H(t)$ exists.

Assume that $u(t)$ is a solution of (1.9). Let

$$w(t) = u(t) + pH(t) + B, \tag{3.26}$$

where

$$p = \frac{(1-\beta)h(0) - (1-\alpha)\gamma}{(1-\alpha)H(T+1) - (1-\beta)h(0)}, \qquad B = \frac{h(0)\{\gamma - H(T+1)\}}{(1-\alpha)H(T+1) - (1-\beta)h(0)}. \tag{3.27}$$

By (1.9), (3.26), (3.27), (H_7), (H_8), and the definition of $H(t)$, we have

$$\begin{aligned} w(0) &= u(0) + ph(0) + B \\ &= \alpha w(\eta) + ph(0) + (1-\alpha)B + h(0) \\ &= \alpha w(\eta), \end{aligned} \tag{3.28}$$

$$\begin{aligned} w(T+1) &= u(T+1) + ph(T+1) + B \\ &= \beta w(\xi) + pH(T+1) + (1-\beta)B + \gamma \\ &= \beta w(\xi), \end{aligned} \tag{3.29}$$

and, for $t \in [1, T]$,

$$\begin{aligned} -\Delta^2 w(t-1) &= -\Delta^2 u(t-1) - p\Delta^2 H(t-1) \\ &= r(t) f(t, u_t) - p\Delta^2 H(t-1) \\ &= r(t) f(t, w_t - pH_t - B) - p\{H(t+1) - H(t-1)\}. \end{aligned} \tag{3.30}$$

Let

$$F(t, w_t) = r(t) f(t, w_t - pH_t - B) - p\{H(t+1) - H(t-1)\}. \tag{3.31}$$

Then by (3.27), (H_9), (H_{10}), and the definition of $H(t)$, we have $F(t, w_t) > 0$ for $t \in [1, T]$. Thus, the BVP (1.9) can be changed into the following BVP:

$$\begin{gathered} -\Delta^2 w(t-1) = F(t, w_t), \quad t \in [1, T], \\ w_0 = \alpha w(\eta) + g, \quad t \in [-\tau, 0], \\ w(T+1) = \beta w(\xi), \end{gathered} \tag{3.32}$$

with $g = -B\alpha + h + pH_0 + B \in C_\tau^+$ and $g(0) = 0$.

Similar to the above proof, we can show that (1.9) has at least one positive solution. Consequently, (1.9) has at least one positive solution.

Example 3.9. Consider the following BVP:

$$\begin{gathered} -\Delta^2 u(t-1) = \frac{t}{120} f(t, u_t), \quad t \in [1, 5], \\ u_0 = u(2) + \frac{t^2}{4}, \quad t \in [-2, 0], \\ u(T+1) = \frac{1}{2} u(4). \end{gathered} \tag{3.33}$$

That is,

$$T = 5, \qquad \tau = 2, \qquad \alpha = \frac{1}{2}, \qquad \beta = 1, \qquad \xi = 2, \qquad \eta = 4, \qquad h(t) = \frac{t^2}{4}, \qquad r(t) = \frac{t}{120}. \tag{3.34}$$

Then we obtain

$$\overline{h} = 1, \qquad \frac{21}{24} \le G_1(t, s) \le \frac{163}{40}, \qquad \sum_{s=1}^{5} r(t) = \frac{1}{8}, \qquad \sum_{s=3}^{5} r(t) = \frac{1}{10}. \tag{3.35}$$

Let

$$\begin{gathered} f(t, \phi) = \begin{cases} \dfrac{2R_2(p_2 - r_1)}{\pi} \arctan\left(s - \dfrac{m}{M} p_2\right) + R_2 p_2, & s \le \dfrac{m}{M} p_2, \\ \dfrac{2(R_1 r_2 - R_2 p_2)}{\pi} \arctan\left(s - \dfrac{m}{M} p_2\right) + R_2 p_2, & s > \dfrac{m}{M} p_2, \end{cases} \\ R_1 = \frac{3}{2}, \quad R_2 = 12, \quad r_1 = 1, \quad r_2 = 400, \quad p_1 = 4, \quad p_2 = 40, \end{gathered} \tag{3.36}$$

where $s = \|\phi\|_\tau$.

By calculation, we can see that (H_4)–(H_8) hold, then by Theorem 3.8, the BVP (3.33) has at least three positive solutions.

4. Eigenvalue Intervals

In this section, we consider the following BVP with parameter λ:

$$\begin{gathered} -\Delta^2 u(t-1) = \lambda r(t) f(t, u_t), \quad t \in [1, T], \\ u_0 = \alpha u(\eta) + h, \quad t \in [-\tau, 0], \\ u(T+1) = \beta u(\xi) \end{gathered} \tag{4.1}$$

with $h(0) = 0$.

The BVP (4.1) is equivalent to the equation

$$u(t) = \begin{cases} \lambda \sum_{s=1}^{T} G_1(t,s) r(s) f(s, u_s), & t \in [1, T], \\ \alpha u(\eta) + h(t), & t \in [-\tau, 0], \\ \beta u(\xi), & t = T+1. \end{cases} \tag{4.2}$$

Let $\overline{u}(t)$ be the solution of (3.3), $y(t) = u(t) - \overline{u}(t)$. Then we have

$$y(t) = \begin{cases} \lambda \sum_{s=1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s), & t \in [1, T], \\ \alpha y(\eta), & t \in [-\tau, 0], \\ \beta y(\xi), & t = T+1. \end{cases} \tag{4.3}$$

Let E and K be defined as the above. Define $\Phi : K \to E$ by

$$\Phi y(t) = \begin{cases} \lambda \sum_{s=1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s), & t \in [1, T], \\ \alpha \Phi y(\eta), & t \in [-\tau, 0], \\ \beta \Phi y(\xi), & t = T+1. \end{cases} \tag{4.4}$$

Then solving the BVP (4.1) is equivalent to finding fixed points in K. Obviously Φ is completely continuous and keeps the K invariant for $\lambda \geq 0$.

Define

$$f_0 = \liminf_{\|\phi\|_\tau \to 0+} \min_{t\in[1,T]} \frac{f(t,\phi)}{\|\phi\|_\tau}, \qquad f_\infty = \liminf_{\|\phi\|_\tau \to \infty} \min_{t\in[1,T]} \frac{f(t,\phi)}{\|\phi\|_\tau}, \qquad f^\infty = \limsup_{\|\phi\|_\tau \to \infty} \max_{t\in[1,T]} \frac{f(t,\phi)}{\|\phi\|_\tau}, \tag{4.5}$$

respectively. We have the following results.

Theorem 4.1. *Assume that* (H_1), (H_2), (H_6),

(H_{11}) $\underline{r} = \min_{t\in[1,T]} r(t) > 0$,

(H_{12}) $\min\{1/m\underline{r}f_0, M/m^2 f_0 \sum_{s=\tau+1}^{T} r(s)\} < \lambda < 1/M\delta f^{\infty} \sum_{s=1}^{T} r(s)$

hold, where $\delta = \max\{1, (1+\mu)\alpha\}$, *then BVP* (4.1) *has at least one positive solution, where* μ *is a positive constant.*

Proof. Assume that condition (H_{12}) holds. If $\lambda > 1/m\underline{r}f_0$ and $f_0 < \infty$, there exists an $\epsilon > 0$ sufficiently small, such that

$$\lambda \geq \frac{1}{m\underline{r}(f_0 - \epsilon)}. \tag{4.6}$$

By the definition of f_0, there is an $r_1 > 0$, such that for $0 < \|\phi\|_{\tau} \leq r_1$,

$$\min_{t\in[1,T]} \frac{f(t,\phi)}{\|\phi\|_{\tau}} > f_0 - \epsilon. \tag{4.7}$$

It follows that, for $t \in [1,T]$ and $0 < \|\phi\|_{\tau} \leq r_1$,

$$f(t,\phi) > (f_0 - \epsilon)\|\phi\|_{\tau}. \tag{4.8}$$

For every $y \in \partial K_{r_1}$ and $s \in [\tau+1, T]$, by (3.9), we have

$$\|y_s + \overline{u}_s\|_{\tau} = \|y_s\|_{\tau} \leq \|y\| = r_1. \tag{4.9}$$

Therefore by (3.13) and Lemma 3.2, we have

$$\begin{aligned}
\|\Phi y\| &= \max_{t\in[1,T]} \lambda \sum_{s=1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s) \\
&\geq \lambda \max_{t\in[1,T]} \sum_{s=\tau+1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s) \\
&\geq m\lambda \underline{r}(f_0 - \epsilon)\|y_{t_0}\|_{\tau} \\
&= m\lambda \underline{r}(f_0 - \epsilon)\|y\| \\
&\geq \|y\|.
\end{aligned} \tag{4.10}$$

If $\lambda > M/m^2 f_0 \sum_{s=\tau+1}^{T} r(s)$, then for a sufficiently small $\epsilon > 0$, we have $\lambda \geq M/m^2(f_0 - \epsilon)\sum_{s=\tau+1}^{T} r(s)$. Similar to the above, for every $y \in \partial K_{r_1}$, we obtain by (3.10)

$$\begin{aligned}\|\Phi y\| &\geq m\lambda \sum_{s=\tau+1}^{T} r(s)(f_0 - \epsilon)\|y_s\|_\tau \\ &\geq m\lambda \sum_{s=\tau+1}^{T} r(s)(f_0 - \epsilon)\frac{m}{M}\|y\| \\ &\geq \frac{m^2\lambda(f_0 - \epsilon)}{M}\sum_{s=\tau+1}^{T} r(s)\|y\| \\ &\geq \|y\|.\end{aligned} \tag{4.11}$$

If $f_0 = \infty$, choose $K > 0$ sufficiently large, such that

$$\frac{m^2\lambda K}{M}\sum_{s=\tau+1}^{T} r(s) \geq 1. \tag{4.12}$$

By the definition of f_0, there is an $r_1 > 0$, such that, for $t \in [1, T]$ and $0 < \|\phi\|_\tau \leq r_1$,

$$f(t, \phi) > K\|\phi\|_\tau. \tag{4.13}$$

For every $y \in \partial K_{r_1}$, by (3.8)–(3.10) and (3.13), we have

$$\|\Phi y\| \geq \|y\|, \tag{4.14}$$

which implies that

$$i(\Phi, K_{r_1}, K) = 0. \tag{4.15}$$

Finally, we consider the assumption $\lambda < 1/M\delta f^\infty \sum_{s=1}^{T} r(s)$. By the definition of f^∞, there is
$r > max\{r_1, \overline{h}/\mu\alpha\}$, such that, for $t \in [1, T]$ and $\|\phi\| \geq r$,

$$f(t, \phi) < (f^\infty + \epsilon_1)\|\phi\|. \tag{4.16}$$

We now show that there is $r_2 \geq r$, such that, for $y \in \partial K_{r_2}$, $\|\Phi y\| \leq \|y\|$. In fact, for $s \in [1, T]$ $r_2 \geq (Mr/m\alpha)$ and every $y \in \partial K_{r_2}$, $\delta\|y\| \geq \|y_s + \overline{u}_s\|_\tau \geq r$; hence in a similar way, we have

$$\|\Phi y\| \le \|y\|, \tag{4.17}$$

which implies that

$$i(\Phi, K_{r_2}, K) = 1. \tag{4.18}$$

□

Theorem 4.2. *Assume that* (H_1),(H_2), *and* (H_{11}) *hold. If* $f_\infty = \infty$ *or* $f_0 = \infty$, *then there is a* $\lambda_0 > 0$ *such that for* $0 < \lambda \le \lambda_0$, *BVP* (4.1) *has at least one positive solution.*

Proof. Let $r > \overline{h}$ be given. Define

$$L = \max\{f(t,\phi) \mid (t,\phi) \in [1,T] \times C_\tau^r\}. \tag{4.19}$$

Then $L > 0$, where $C_\tau^r = \{\phi \in C_\tau^+ \mid \|\phi\|_\tau \le r\}$.

For every $y \in \partial K_{r-\overline{h}}$, we know that $\|y\| = r - \overline{h}$. By the definition of operator Φ, we obtain

$$\|\Phi y\| = \|\Phi y\|_{[1,T]} \le \lambda L M \sum_{s=1}^{T} r(s). \tag{4.20}$$

It follows that we can take $\lambda_0 = (r - \overline{h}/ML \sum_{s=1}^{T} r(s)) > 0$ such that, for all $0 < \lambda \le \lambda_0$ and all $y \in \partial K_{r-\overline{h}}$,

$$\|\Phi y\| \le \|y\|. \tag{4.21}$$

Fix $0 < \lambda \le \lambda_0$. If $f_\infty = \infty$, for $C = (1/\lambda m \underline{r})$, we obtain a sufficiently large $R > r$ such that, for $\|\phi\|_\tau \ge R$,

$$\min_{t\in[1,T]} \frac{f(t,\phi)}{\|\Phi\|_\tau} > C. \tag{4.22}$$

It follows that, for $\|\phi\|_\tau \ge R$ and $t \in [1,T]$,

$$f(t,\phi) \ge C\|\phi\|_\tau. \tag{4.23}$$

For every $y \in \partial K_R$, by the definition of $\|\cdot\|$, $\|\cdot\|_\tau$ and the definition of Lemma 3.2, there exists a $t_0 \in [\tau+1, T]$ such that $\|y\| = \|y_{t_0}\|_\tau = R$ and $\overline{u}_{t_0} = 0$, thus $\|y_{t_0} + \overline{u}_{t_0}\|_\tau \geq R$. Hence

$$\begin{aligned} \|\Phi y\| &= \max_{t\in[1,T]} \lambda \sum_{s=1}^{T} G_1(t,s) r(s) f(s, y_s + \overline{u}_s) \\ &\geq \max_{t\in[1,T]} \lambda G_1(t,t_0) r(t_0) f(t_0, y_{t_0} + \overline{u}_{t_0}) \\ &\geq \lambda m \underline{r} C \|y_{t_0}\|_\tau \\ &\geq mCR\lambda\underline{r} \\ &= R \\ &= \|y\|. \end{aligned} \tag{4.24}$$

If $f_0 = \infty$, there is $s < r$, such that, for $0 < \|\phi\|_\tau \leq s$ and $t \in [1,T]$,

$$f(t,\phi) > T\|\phi\|_\tau, \tag{4.25}$$

where $T > (1/\lambda m \underline{r})$.

For every $y \in \partial K_s$, by(3.8)–(3.10) and Lemma 3.2,

$$\begin{aligned} \|\Phi y\| &\geq m\lambda \sum_{s=\tau+1}^{T} r(s) f(s, y_s) \\ &\geq Tm\lambda \sum_{s=\tau+1}^{T} r(s) \|y_s\|_\tau \\ &\geq Tm\lambda\underline{r}\|y_{t_0}\|_\tau \\ &= Tm\lambda\underline{r}\|y\| \\ &\geq \|y\|, \end{aligned} \tag{4.26}$$

which by combining with (4.21) completes the proof. □

Example 4.3. Consider the BVP(3.33) in Example 3.9 with

$$\begin{gathered} f(t,\phi) = \begin{cases} A \arctan s, & s \leq \dfrac{m}{M} p_2, \\ \dfrac{A \arctan s + C}{1000}, & s > \dfrac{m}{M} p_2, \end{cases} \\ C = \left(1000 - \frac{m}{M} p_2\right) A \arctan\left(\frac{m}{M} p_2\right), \end{gathered} \tag{4.27}$$

where $s = \|\phi\|_\tau$, A is some positive constant, $p_2 = 40, m = (21/24)$, and $M = (163/40)$.

By calculation, $f_0 = A$, $f^\infty = \pi A/2000$, and $\underline{r} = 1/120$; let $\delta = 1$. Then by Theorem(4.1), for $\lambda \in ((2608/49A), (640000/163\pi A))$, the above equation has at least one positive solution.

Acknowledgments

The authors would like to thank the editor and the reviewers for their valuable comments and suggestions which helped to significantly improve the paper. This work is supported by Distinguished Expert Science Foundation of Naval Aeronautical and Astronautical University.

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