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Research Article

Positive Solutions to Nonlinear First-Order Nonlocal BVPs with Parameter on Time Scales

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We discuss the existence of solutions for the first-order multipoint BVPs on time scale $\mathbb{T}$: $u^{\Delta}(t) + p(t)u(\sigma(t)) = \lambda f(t, u(\sigma(t)))$, $t \in [0, T]_{\mathbb{T}}$, $u(0) - \sum_{i=1}^{m} \alpha_i u(\xi_i) = 0$, where $\lambda > 0$ is a parameter, $T > 0$ is a fixed number, $0, T \in \mathbb{T}$, $f : [0, T]_{\mathbb{T}} \times [0, \infty) \to [0, \infty)$ is continuous, p is regressive and rd-continuous, $\alpha_i \geq 0$, $\xi_i \in \mathbb{T}$, $i = 1, 2, \ldots, m$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-1} < \xi_m = \sigma(T)$, and $1 - \sum_{i=1}^{m}(\alpha_i / e_p(\xi_i, 0)) > 0$. For suitable $\lambda > 0$, some existence, multiplicity, and nonexistence criteria of positive solutions are established by using well-known results from the fixed-point index.

1. Introduction

Let $\mathbb{T}$ be a time scale (a nonempty closed subset of the real line $\mathbb{R}$). We discuss the existence of positive solutions to the first-order multipoint BVPs on time scale $\mathbb{T}$:

$$\begin{gathered} u^{\Delta}(t) + p(t)u(\sigma(t)) = \lambda f(t, u(\sigma(t))), \quad t \in [0, T]_{\mathbb{T}}, \\ u(0) - \sum_{i=1}^{m} \alpha_i u(\xi_i) = 0, \end{gathered} \tag{1.1}$$

where $T > 0$ is a fixed number, $0, T \in \mathbb{T}$, $f : [0, T]_{\mathbb{T}} \times [0, \infty) \to [0, \infty)$ is continuous, p is regressive and rd-continuous, $\alpha_i \geq 0$, $\xi_i \in \mathbb{T}$, $i = 1, 2, \ldots, m$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-1} < \xi_m = \sigma(T)$, and $1 - \sum_{i=1}^{m}(\alpha_i / e_p(\xi_i, 0)) > 0$, $e_p(t, s)$ is defined in its standard form; see [1, page 59] for details.

The multipoint boundary value problems arise in a variety of different areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary value problem [2]; also many problems in the theory of elastic stability can be handled by a multipoint problem [3]. So, the existence of solutions to multipoint boundary value problems have been studied by many authors; see [4–13] and the reference therein. Especially, in recent years the existence of positive solutions to multipoint boundary value problems on time scales has caught considerable attention; see [10–14]. For other background on dynamic equations on time scales, one can see [1, 15–18].

Our ideas arise from [13, 16]. In [13], Tian and Ge discussed the existence of positive solutions to nonlinear first-order three-point boundary value problems on time scale $\mathbb{T}$:

$$\begin{gathered} u^{\Delta}(t) + p(t)u(\sigma(t)) = f(t, u(\sigma(t))), \quad t \in [0, T]_{\mathbb{T}}, \\ u(0) - \alpha u(\eta) = \beta u(\sigma(T)), \end{gathered} \tag{1.2}$$

where $f : [0, T]_{\mathbb{T}} \times [0, \infty) \to [0, \infty)$ is continuous, p is regressive and rd-continuous, $\alpha, \beta \geq 0$ and $\alpha / e_p(\xi, 0) + \beta / e_p(\sigma(T), 0) < 1$. The existence results are based on Krasnoselskii's fixed-point theorem in cones and Leggett-Williams's theorem.

As we can see, if we take $\lambda = 1$, $\xi_1 = \eta$, $\xi_m = \sigma(T)$, and $\alpha_i = 0$ for $i = 2, \ldots, m-1$, then (1.1) is reduced to (1.2). Because of the influence of the parameter λ, it will be more difficult to solve (1.1) than to solve (1.2).

In 2009, by using the fixed-point index theory, Sun and Li [16] discussed the existence of positive solutions to the first-order PBVPs on time scale $\mathbb{T}$:

$$\begin{gathered} u^{\Delta}(t) + p(t)u(\sigma(t)) = \lambda f(u(t)), \quad t \in [0, T]_{\mathbb{T}}, \\ u(0) = u(\sigma(T)). \end{gathered} \tag{1.3}$$

For suitable $\lambda > 0$, they gave some existence, multiplicity, and nonexistence criteria of positive solutions.

Motivated by the above results, by using the well-known fixed-point index theory [16, 19], we attempt to obtain some existence, multiplicity and nonexistence criteria of positive solutions to (1.1) for suitable $\lambda > 0$.

The rest of this paper is arranged as follows. Some preliminary results including Green's function are given in Section 2. In Section 3, we obtain some useful lemmas for the proof of the main result. In Section 4, some existence and multiplicity results are established. At last, some nonexistence results are given in Section 5.

2. Preliminaries

Throughout the rest of this paper, we make the following assumptions:

$(H1)$ $f : [0, T]_{\mathbb{T}} \times [0, \infty) \to [0, \infty)$ is continuous and $f(\cdot, u) > 0$ for $u > 0$,

$(H2)$ $p : [0, T]_{\mathbb{T}} \to (0, \infty)$ is rd-continuous, which implies that $p \in \mathcal{R}^+$ (where $\mathcal{R}^+$ is defined in [16, 18, 20]).

Moreover, let

$$f_0 = \liminf_{u\to 0^+} \inf_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{u}, \qquad f_\infty = \liminf_{u\to +\infty} \inf_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{u},$$
$$f^0 = \limsup_{u\to 0^+} \sup_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{u}, \qquad f^\infty = \limsup_{u\to +\infty} \sup_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{u}. \tag{2.1}$$

Our main tool is the well-known results from the fixed-point index, which we state here for the convenience of the reader.

Theorem 2.1 (see [19]). *Let X be a Banach space and P be a cone in X. For $r > 0$, we define $P_r = \{x \in P \mid \|x\| < r\}$. Assume that $\Phi : \overline{P}_r \to P$ is completely continuous such $\Phi x \neq x$ for $x \in \partial P_r = \{x \in P \mid \|x\| = r\}$.*

(i) *If $\|\Phi x\| \geq \|x\|$ for $x \in \partial P_r$, then*

$$i(\Phi, P_r, P) = 0. \tag{2.2}$$

(ii) *If $\|\Phi x\| \leq \|x\|$ for $x \in \partial P_r$, then*

$$i(\Phi, P_r, P) = 1. \tag{2.3}$$

Let $E := \{x \mid x : [0,\sigma(T)]_{\mathbb{T}} \to \mathbb{R}$ is continuous$\}$ be equipped with the norm $\|x\| = \max_{t\in[0,\sigma(T)]_{\mathbb{T}}}|x(t)|$. It is easy to see that $(E, \|\cdot\|)$ is a Banach space.

For $h \in \mathbb{E}$, we consider the following linear BVP:

$$u^{\Delta}(t) + p(t)u(\sigma(t)) = \lambda h(\sigma(t)), \quad t \in [0,T]_{\mathbb{T}}, \tag{2.4}$$

$$u(0) - \sum_{i=1}^{m} \alpha_i u(\xi_i) = 0. \tag{2.5}$$

For $\xi_i, s \in [0,\sigma(T)]_{\mathbb{T}}$, define

$$H_{\xi_i}(s) = \begin{cases} \dfrac{\alpha_i}{e_p(\xi_i,0)}, & s \leq \xi_i, \\ 0, & s > \xi_i. \end{cases} \tag{2.6}$$

Lemma 2.2. *For $h \in \mathbb{E}$, the linear BVP* (2.4)-(2.5) *has a solution u if and only if u satisfies*

$$u(t) = \frac{\lambda}{e_p(t,0)} \left\{ \Gamma \sum_{i=1}^{m} \frac{\alpha_i}{e_p(\xi_i,0)} \int_0^{\xi_i} e_p(s,0)h(\sigma(s))\Delta s + \int_0^t e_p(s,0)h(\sigma(s))\Delta s \right\}, \tag{2.7}$$

where

$$\Gamma = \frac{1}{1 - \sum_{i=1}^{m} (\alpha_i / e_p(\xi_i,0))}. \tag{2.8}$$

Proof. By (2.4), we have

$$\left[u(t)e_p(t,0)\right]^{\Delta} = \lambda e_p(t,0)h(\sigma(t)). \tag{2.9}$$

So,

$$u(t)e_p(t,0) = u(0) + \lambda \int_0^t e_p(s,0)h(\sigma(s))\Delta s. \tag{2.10}$$

And so,

$$u(t) = \frac{1}{e_p(t,0)} \left[u(0) + \lambda \int_0^t e_p(s,0)h(\sigma(s))\Delta s \right]. \tag{2.11}$$

Combining this with (2.5), we get

$$u(t) = \frac{\lambda}{e_p(t,0)} \left\{ \Gamma \sum_{i=1}^{m} \frac{\alpha_i}{e_p(\xi_i,0)} \int_0^{\xi_i} e_p(s,0)h(\sigma(s))\Delta s + \int_0^t e_p(s,0)h(\sigma(s))\Delta s \right\}. \tag{2.12}$$

□

Lemma 2.3. *If the function u is defined in* (2.7), *then u may be expressed by*

$$u(t) = \lambda \int_0^T G(t,s)h(\sigma(s))\Delta s, \tag{2.13}$$

where

$$G(t,s)=\frac{1}{e_p(t,0)}\begin{cases}\Gamma\sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)}e_p(s,0)+e_p(s,0), & \xi_i\le s,\ \sigma(s)\le t,\ \sigma(t)\le\xi_{i+1},\\ & i=0,1,\dots,m-1,\\ \Gamma\left(\sum_{j=1}^{i}H_{\xi_j}(s)+\sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)}\right) & \\ \times e_p(s,0)+e_p(s,0), & \sigma(s)\le\xi_i\le t,\ \sigma(t)\le\xi_{i+1},\\ & i=1,2,\dots,m-1,\\ \Gamma\sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)}e_p(s,0), & \xi_i\le t,\ \sigma(t)\le s,\ \sigma(s)\le\xi_{i+1},\\ & i=0,1,\dots,m-1,\\ \Gamma\sum_{j=i+2}^{m}H_{\xi_j}(s)e_p(s,0), & \xi_i\le t,\ \sigma(t)\le\xi_{i+1}\le s,\\ & i=0,1,\dots,m-2.\end{cases}\tag{2.14}$$

Proof. When $\sigma(t)\le\xi_1$,

$$\begin{aligned}u(t)=\frac{\lambda}{e_p(t,0)}\Bigg\{&\Gamma\left[\sum_{i=1}^{m}\frac{\alpha_i}{e_p(\xi_i,0)}\int_0^t e_p(s,0)h(\sigma(s))\Delta s+\sum_{i=1}^{m}\frac{\alpha_i}{e_p(\xi_i,0)}\int_t^{\xi_1}e_p(s,0)h(\sigma(s))\Delta s\right]\\&+\Gamma\sum_{i=2}^{m}\frac{\alpha_i}{e_p(\xi_i,0)}\int_{\xi_1}^{\xi_i}e_p(s,0)h(\sigma(s))\Delta s+\int_0^t e_p(s,0)h(\sigma(s))\Delta s\Bigg\}.\end{aligned}\tag{2.15}$$

(1) For $\sigma(s)\le t$,

$$G(t,s)=\frac{1}{e_p(t,0)}\left[\Gamma\sum_{i=1}^{m}\frac{\alpha_i}{e_p(\xi_i,0)}e_p(s,0)+e_p(s,0)\right].\tag{2.16}$$

(2) For $\sigma(t)\le s,\ \sigma(s)\le\xi_1$,

$$G(t,s)=\frac{1}{e_p(t,0)}\left[\Gamma\sum_{i=1}^{m}\frac{\alpha_i}{e_p(\xi_i,0)}e_p(s,0)\right].\tag{2.17}$$

(3) For $\sigma(t)\le\xi_1\le s$,

$$G(t,s)=\frac{1}{e_p(t,0)}\left[\Gamma\sum_{i=2}^{m}H_{\xi_i}(s)e_p(s,0)\right].\tag{2.18}$$

When $\xi_i \le t,\ \sigma(t) \le \xi_{i+1},\ i = 1,2,\ldots,m-2$,

$$u(t) = \frac{\lambda}{e_p(t,0)}\left\{\Gamma\left[\sum_{j=1}^{i}\frac{\alpha_j}{e_p(\xi_j,0)}\int_0^{\xi_j} e_p(s,0)h(\sigma(s))\Delta s + \sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)}\int_0^{\xi_i} e_p(s,0)h(\sigma(s))\Delta s\right]\right.$$
$$\left. +\Gamma\left[\sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)}\int_{\xi_i}^{\xi_j} e_p(s,0)h(\sigma(s))\Delta s\right] + \int_0^t e_p(s,0)h(\sigma(s))\Delta s\right\}. \tag{2.19}$$

(1) For $\sigma(s) \le \xi_i$,

$$G(t,s) = \frac{e_p(s,0)}{e_p(t,0)}\left[\Gamma\left(\sum_{j=1}^{i}H_{\xi_j}(s) + \sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)}\right) + 1\right]. \tag{2.20}$$

(2) For $\xi_i \le s,\ \sigma(s) \le t$,

$$G(t,s) = \frac{e_p(s,0)}{e_p(t,0)}\left[\Gamma\sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)} + 1\right]. \tag{2.21}$$

(3) For $\sigma(t) \le s,\ \sigma(s) \le \xi_{i+1}$,

$$G(t,s) = \frac{e_p(s,0)}{e_p(t,0)}\Gamma\sum_{j=i+1}^{m}\frac{\alpha_j}{e_p(\xi_j,0)}. \tag{2.22}$$

(4) For $\xi_{i+1} \le s$,

$$G(t,s) = \frac{e_p(s,0)}{e_p(t,0)}\Gamma\sum_{j=i+2}^{m}H_{\xi_j}(s). \tag{2.23}$$

When $t \ge \xi_{m-1}$,

$$u(t) = \frac{\lambda}{e_p(t,0)}\left\{\Gamma\left[\sum_{j=1}^{m-1}\frac{\alpha_j}{e_p(\xi_j,0)}\int_0^{\xi_j} e_p(s,0)h(\sigma(s))\Delta s + \frac{\alpha_m}{e_p(\xi_m,0)}\int_0^{\xi_{m-1}} e_p(s,0)h(\sigma(s))\Delta s\right]\right.$$
$$+ \Gamma\frac{\alpha_m}{e_p(\xi_m,0)}\int_{\xi_{m-1}}^{t} e_p(s,0)h(\sigma(s))\Delta s$$
$$\left. +\Gamma\frac{\alpha_m}{e_p(\xi_m,0)}\int_t^{\xi_m} e_p(s,0)h(\sigma(s))\Delta s + \int_0^t e_p(s,0)h(\sigma(s))\Delta s\right\}. \tag{2.24}$$

(1) For $\sigma(s) \le \xi_{m-1}$,

$$G(t,s) = \frac{e_p(s,0)}{e_p(t,0)}\left[\Gamma\left(\sum_{i=1}^{m-1} H_{\xi_i}(s) + \frac{\alpha_m}{e_p(\xi_m,0)}\right) + 1\right]. \tag{2.25}$$

(2) For $\xi_{m-1} \le s,\ \sigma(s) \le t$,

$$G(t,s) = \frac{e_p(s,0)}{e_p(t,0)}\left[\Gamma\frac{\alpha_m}{e_p(\xi_m,0)} + 1\right]. \tag{2.26}$$

(3) For $\sigma(t) \le s$,

$$G(t,s) = \frac{e_p(s,0)}{e_p(t,0)}\Gamma\frac{\alpha_m}{e_p(\xi_m,0)}. \tag{2.27}$$

□

Lemma 2.4. *Green's function $G(t,s)$ has the following properties.*

(i) $G(t,s) \ge 0,\ (t,s) \in [0,\sigma(T)]_{\mathbb{T}} \times [0,T]_{\mathbb{T}}$,

(ii) $m \le G(t,s) \le M$, *where* $m = \Gamma\sum_{i=1}^{m}\alpha_i/(e_p(\sigma(T),0))^2$; $M = \Gamma\sum_{i=1}^{m}\alpha_i + e_p(\sigma(T),0)$;

(iii) $G(t,s) \ge (m/M)\sup_{(t,s)\in[0,\sigma(T)]_{\mathbb{T}}\times[0,T]_{\mathbb{T}}} G(t,s)$, $(t,s) \in [0,\sigma(T)]_{\mathbb{T}} \times [0,T]_{\mathbb{T}}$.

Proof. This proof is similar to [13, Lemma 2.3], so we omit it.

Now, we define a cone P in $\mathbb{E}$ as follows:

$$P = \{x \in \mathbb{E} \mid x(t) \ge 0,\ x(t) \ge \delta\|x\| \text{ on } [0,\sigma(T)]_{\mathbb{T}}\}, \tag{2.28}$$

where $\delta = m/M$. For $r > 0$, let $P_r = \{u \in P \mid \|u\| < r\}$ and $\partial P_r = \{u \in P \mid \|u\| = r\}$.

For $\lambda > 0$, define an operator $\Phi_\lambda : P \to \mathbb{E}$:

$$(\Phi_\lambda u)(t) = \lambda\int_0^T G(t,s)f(s,u(\sigma(s)))\Delta s, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \tag{2.29}$$

Similar to the proof of [13, Lemma 2.4], we can see that $\Phi_\lambda : P \to P$ is completely continuous. By the above discussions, its not difficult to see that u being a solution of BVP (1.1) equals the solution that u is a fixed point of the operator Φ_λ. □

3. Some Lemmas

Lemma 3.1. *Let $\eta > 0$. If $u \in P$ and $f(t,u(\sigma(t))) \ge \eta u(\sigma(t))$, $t \in [0,T]_{\mathbb{T}}$, then*

$$\|\Phi_\lambda x\| \ge \lambda\eta\delta m T\|u\|. \tag{3.1}$$

Proof. Since $u \in P$ and $f(t, u(\sigma(t))) \geq \eta u(\sigma(t))$, $t \in [0, T]_{\mathbb{T}}$, we have

$$\Phi_\lambda u(t) = \lambda \int_0^T G(t,s) f(s, u(\sigma(s)))\Delta s \geq \lambda\eta \int_0^T G(t,s) u(\sigma(s))\Delta s \geq \lambda\eta\delta m T\|u\|. \tag{3.2}$$

□

Lemma 3.2. *Let $\varepsilon > 0$. If $u \in P$ and $f(t, u(\sigma(t))) \leq \varepsilon u(\sigma(t))$, $t \in [0, T]_{\mathbb{T}}$, then*

$$\|\Phi_\lambda u\| \leq \lambda\varepsilon M T\|u\|. \tag{3.3}$$

Proof. Since $u \in P$ and $f(t, u(\sigma(t))) \leq \varepsilon u(\sigma(t))$, $t \in [0, T]_{\mathbb{T}}$, we have

$$\Phi_\lambda u(t) = \lambda \int_0^T G(t,s) f(s, u(\sigma(s)))\Delta s \leq \lambda\varepsilon \int_0^T G(t,s) u(\sigma(s))\Delta s \leq \lambda\varepsilon M T\|u\|. \tag{3.4}$$

□

Lemma 3.3. *Let $r > 0$. If $u \in \partial P_r$, then*

$$\lambda m(r) m T \leq \|\Phi_\lambda x\| \leq \lambda M(r) M T, \tag{3.5}$$

where $m(r) = \min_{(t,u)\in[0,T]_{\mathbb{T}}\times[\delta r, r]} f(t,u)$; $M(r) = \max_{(t,u)\in[0,T]_{\mathbb{T}}\times[\delta r, r]} f(t,u)$.

Proof. Since $u \in \partial P_r$, we have $\delta r \leq u(\sigma(t)) \leq r$, $t \in [0, T]_{\mathbb{T}}$. So,

$$\begin{aligned} \Phi_\lambda u(t) &= \lambda \int_0^T G(t,s) f(s, u(\sigma(s)))\Delta s \geq \lambda m(r) \int_0^T G(t,s)\Delta s \geq \lambda m(r) m T, \\ \Phi_\lambda u(t) &= \lambda \int_0^{\sigma(T)} G(t,s) f(s, u(\sigma(s)))\Delta s \leq \lambda M(r) \int_0^{\sigma(T)} G(t,s)\Delta s \leq \lambda M(r) M T. \end{aligned} \tag{3.6}$$

□

4. Some Existence and Multiplicity Results

Theorem 4.1. *Assume that (H1) and (H2) hold and that $f^0 \in (0, \infty)$, $f^\infty \in (0, \infty)$. Then the BVP* (1.1) *has at least two positive solutions for*

$$\frac{1}{m(1) m T} < \lambda < \frac{1}{2MT \max\{f^0, f^\infty\}}. \tag{4.1}$$

Proof. Let $r_1 = 1$. Then it follows from (4.1) and Lemma 3.3 that

$$\|\Phi_\lambda u\| \geq \lambda m(1) m T > 1 = \|u\|, \quad \text{for } u \in \partial P_{r_1}. \tag{4.2}$$

In view of Theorem 2.1, we have

$$i(\Phi_\lambda, P_{r_1}, P) = 0. \tag{4.3}$$

Now, combined with the definition of f^0, we may choose $0 < r_2 < r_1$ such that $f(t,u) \leq (f^0 + \varepsilon)u$ for $u \in [0, r_2]$ and $t \in [0,T]_{\mathbb{T}}$ uniformly, where $\varepsilon > 0$ satisfies

$$\lambda \varepsilon MT < \frac{1}{2}. \tag{4.4}$$

So,

$$f(t, u(\sigma(t))) \leq \left(f^0 + \varepsilon\right) u(\sigma(t)), \quad \text{for } u \in \partial P_{r_2},\ t \in [0,T]_{\mathbb{T}}. \tag{4.5}$$

In view of (4.1), (4.4), (4.5), and Lemma 3.2, we have

$$\|\Phi_\lambda u\| \leq \lambda \left(f^0 + \varepsilon\right) MT\|u\| < \|u\|, \quad \text{for } u \in \partial P_{r_2}. \tag{4.6}$$

It follows from Theorem 2.1 that

$$i(\Phi_\lambda, P_{r_2}, P) = 1. \tag{4.7}$$

By (4.3) and (4.7), we get

$$i\left(\Phi_\lambda, P_{r_1} \setminus \overline{P_{r_2}}, P\right) = -1. \tag{4.8}$$

This shows that Φ_λ has a fixed point in $P_{r_1} \setminus \overline{P_{r_2}}$, which is a positive solution of the BVP (1.1).

Now, by the definition of f^∞, there exits an $\widehat{H} > 0$ such that $f(t,u) \leq (f^\infty + \varepsilon)u$ for $u \in [\widehat{H}, \infty)$ and $t \in [0,T]_{\mathbb{T}}$, where $\varepsilon > 0$ is chosen so that

$$\lambda \varepsilon MT < \frac{1}{2}. \tag{4.9}$$

Let $r_3 = \max\{2r_1, \widehat{H}/\delta\}$. Then for $u \in \partial P_{r_3}$, $u(t) \geq \delta\|u\| = \delta r_3 \geq \widehat{H},\ t \in [0, \sigma(T)]_{\mathbb{T}}$. So,

$$f(t, u(\sigma(t))) \leq (f^\infty + \varepsilon) u(\sigma(t)), \quad \text{for } u \in \partial P_{r_3},\ t \in [0,T]_{\mathbb{T}}. \tag{4.10}$$

In view of (4.1), (4.9), and Lemma 3.2, we have

$$\|\Phi_\lambda u\| \leq \lambda (f^\infty + \varepsilon) MT\|u\| < \|u\|, \quad \text{for } x \in \partial P_{r_3}. \tag{4.11}$$

It follows from Theorem 2.1 that

$$i(\Phi_\lambda, P_{r_3}, P) = 1. \tag{4.12}$$

By (4.3) and (4.12), we get

$$i\left(\Phi_\lambda, P_{r_3} \setminus \overline{P}_{r_1}, P\right) = 1. \tag{4.13}$$

This shows that Φ_λ has a fixed point in $P_{r_3} \setminus \overline{P}_{r_1}$, which is another positive solution of the BVP (1.1). □

Similar to the proof of Theorem 4.1, we have the following results.

Theorem 4.2. *Suppose that (H1) and (H2) hold and*

$$\lambda > \frac{1}{m(1)mT}. \tag{4.14}$$

Then,

(i) *equation* (1.1) *has at least one positive solution if* $f^0 = 0$,

(ii) *equation* (1.1) *has at least one positive solution if* $f^\infty = 0$,

(iii) *equation* (1.1) *has at least two positive solutions if* $f^0 = f^\infty = 0$.

Theorem 4.3. *Assume that (H1) and (H2) hold. If* $f_0 \in (0,\infty)$, $f_\infty \in (0,\infty)$, *then the BVP* (1.1) *has at least two positive solutions for*

$$\frac{2}{\delta mT \min\{f_0, f_\infty\}} < \lambda < \frac{1}{M(1)MT}. \tag{4.15}$$

Proof. Let $r_1 = 1$. Then it follows from (4.15) and Lemma 3.3 that

$$\|\Phi_\lambda u\| \le \lambda M(1)MT < 1 = \|u\|, \quad \text{for } u \in \partial P_{r_1}. \tag{4.16}$$

In view of Theorem 2.1, we have

$$i(\Phi_\lambda, P_{r_1}, P) = 1. \tag{4.17}$$

Since $f_0 > 0$, we may choose $0 < r_2 < r_1$ such that $f(t,u) \ge (f_0 - \eta_1)u$ for $u \in [0, r_2]$ and $t \in [0,T]_{\mathbb{T}}$, where $0 < \eta_1 < f_0$ satisfies $\lambda\eta_1\delta m\sigma(T) < 1$. So,

$$f(t, u(\sigma(t))) \ge \left(f_0 - \eta_1\right)u(\sigma(t)), \quad \text{for } u \in \partial P_2,\ t \in [0,T]_{\mathbb{T}}. \tag{4.18}$$

In view of (4.15), (4.18), and Lemma 3.1, we have

$$\|\Phi_\lambda u\| \ge \lambda\left(f_0 - \eta_1\right)\delta mT\|u\| > \|u\|, \quad \text{for } u \in \partial P_2. \tag{4.19}$$

It follows from Theorem 2.1 that

$$i(\Phi_\lambda, P_{r_2}, P) = 0. \tag{4.20}$$

By (4.17) and (4.20), we get

$$i\left(\Phi_\lambda, P_{r_1} \setminus \overline{P_{r_2}}, P\right) = 1. \tag{4.21}$$

This shows that Φ_λ has a fixed point in $P_{r_1} \setminus \overline{P_{r_2}}$, which is a positive solution of the BVP (1.1).

Now, by the definition of f_∞, there exists an $\widehat{H} > 0$ such that $f(t,u) \geq (f_\infty - \eta_2)u$ for $u \in [\widehat{H}, \infty)$ and $t \in [0,T]_{\mathbb{T}}$, where $0 < \eta_2 < f_\infty$ satisfies

$$\lambda \eta_2 \delta m T < 1. \tag{4.22}$$

Let $r_3 = \max\{2r_1, \widehat{H}/\delta\}$. Then for $u \in \partial P_{r_3}$, $u(t) \geq \delta\|u\| = \delta r_3 \geq \widehat{H}$, $t \in [0, \sigma(T)]_{\mathbb{T}}$. So,

$$f(t, u(\sigma(t))) \geq \left(f_\infty - \eta_2\right) u(\sigma(t)), \quad \text{for } u \in \partial P_{r_3},\ t \in [0,T]_{\mathbb{T}}. \tag{4.23}$$

Combined with (4.22) and Lemma 3.1, we have

$$\|\Phi_\lambda u\| \geq \lambda\left(f_\infty - \eta_2\right)\delta m T\|u\| > \|u\| \quad \text{for } u \in \partial P_{r_3}. \tag{4.24}$$

It follows from Theorem 2.1 that

$$i(\Phi_\lambda, P_{r_3}, P) = 0, \tag{4.25}$$

By (4.17) and (4.25), we get

$$i\left(\Phi_\lambda, P_{r_3} \setminus \overline{P_{r_1}}, P\right) = -1, \tag{4.26}$$

This shows that Φ_λ has a fixed point in $P_{r_3} \setminus \overline{P_{r_1}}$, which is another positive solution of the BVP (1.1). □

Similar to the proof of Theorem 4.3, we have the following results.

Theorem 4.4. *Suppose that (H1) and (H2) hold and that*

$$\lambda < \frac{1}{M(1)MT}. \tag{4.27}$$

Then,

(i) *equation* (1.1) *has at least one positive solution if* $f_0 = \infty$,

(ii) *equation* (1.1) *has at least one positive solution if* $f_\infty = \infty$,

(iii) *equation* (1.1) *has at least two positive solutions if* $f_0 = f_\infty = \infty$.

Theorem 4.5. *Suppose that (H1) and (H2) hold. If* $f^0 \in (0,\infty)$, $f_0 \in (0,\infty)$, $f^\infty \in (0,\infty)$, $f_\infty \in (0,\infty)$, *then the BVP* (1.1) *has at least one positive solution for*

$$\frac{1}{\delta m T \min\{f_\infty, f_0\}} < \lambda < \frac{1}{MT \max\{f^0, f^\infty\}}. \tag{4.28}$$

Proof. We only deal with the case that $f_\infty \le f_0$, $f^0 \ge f^\infty$. The other three cases can be discussed similarly.

Let λ satisfy (4.28) and let $\varepsilon > 0$ be chosen such that

$$f_\infty - \varepsilon > 0, \quad \frac{1}{\delta m T(f_\infty - \varepsilon)} \le \lambda \le \frac{1}{MT(f^0+\varepsilon)}. \tag{4.29}$$

From the definition of f^0, we know that there exists a constant $r_1 > 0$ such that $f(t,u) \le (f^0+\varepsilon)u$ for $u \in [0,r_1]$ and $t \in [0,T]_{\mathbb{T}}$. So,

$$f(t,u(\sigma(t))) \le \left(f^0+\varepsilon\right)u(\sigma(t)), \quad \text{for } u \in \partial P_{r_1},\ t \in [0,T]_{\mathbb{T}}. \tag{4.30}$$

This combines with (4.29) and Lemma 3.2, we have

$$\|\Phi_\lambda u\| \le \lambda\left(f^0+\varepsilon\right)MT\|u\| \le \|u\|, \quad \text{for } u \in \partial P_{r_1}. \tag{4.31}$$

It follows from Theorem 2.1 that

$$i(\Phi_\lambda, P_{r_1}, P) = 1. \tag{4.32}$$

On the other hand, from the definition of f_∞, there exists an $\widehat{H} > 0$ such that $f(t,u) \ge (f_\infty - \varepsilon)u$ for $u \in [\widehat{H},\infty)$ and $t \in [0,T]_{\mathbb{T}}$. Let $r_2 = \max\{2r_1, \widehat{H}/\delta\}$. Then for $u \in \partial P_{r_2}$, $u(\sigma(t)) \ge \delta\|u\| = \delta r_2 \ge \widehat{H},\ t \in [0,T]_{\mathbb{T}}$. So,

$$f(t,u(t)) \ge (f_\infty - \varepsilon)u(t), \quad \text{for } u \in \partial P_{r_2},\ t \in [0,T]_{\mathbb{T}}. \tag{4.33}$$

Combined with (4.29) and Lemma 3.1, we have

$$\|\Phi_\lambda u\| \ge \lambda(f_\infty - \varepsilon)\delta m T\|u\| \ge \|u\|. \tag{4.34}$$

It follows from Theorem 2.1 that

$$i(\Phi_\lambda, P_{r_2}, P) = 0. \quad (4.35)$$

By (4.32) and (4.35), we get

$$i\left(\Phi_\lambda, P_{r_2} \setminus \overline{P_{r_1}}, P\right) = -1, \quad (4.36)$$

which implies that the BVP (1.1) has at least one positive solution in $P_{r_2} \setminus \overline{P_{r_1}}$. □

Remark 4.6. By making some minor modifications to the proof of Theorem 4.5, we can obtain the existence of at least one positive solution, if one of the following conditions is satisfied:

(i) $f^0 = 0$, $f_\infty \in (0,\infty)$and $\lambda > 1/\delta mT f_\infty$.

(ii) $f^0 = 0$, $f_\infty = \infty$and $\lambda \in (0,\infty)$.

(iii) $f^\infty = 0$, $f_0 \in (0,\infty)$and $\lambda > 1/\delta mT f_0$.

(iv) $f^\infty = 0$, $f_0 = \infty$and $\lambda \in (0,\infty)$.

Remark 4.7. From Conditions (ii) and (iv) of Remark 4.6, we know that the conclusion in Theorem 4.5 holds for $\lambda = 1$ in these two cases. By $f^0 = 0$ and $f_\infty = \infty$, there exist two positive constants $0 < r < R < \infty$ such that, for $t \in [0,T]_{\mathbb{T}}$,

$$f(t,u) \le \frac{u}{MT} \quad \text{for } u \in [0,r], \qquad f(t,u) \ge \frac{u}{\delta mT} \quad \text{for } u \in [R,\infty). \quad (4.37)$$

This is the condition of Theorem 3.2 of [13]. By $f^\infty = 0$ and $f_0 = \infty$, there exist two positive constants $0 < r < R < \infty$ such that for $t \in [0,T]_{\mathbb{T}}$,

$$f(t,u) \le \frac{u}{MT} \quad \text{for } u \in [R,\infty); \qquad f(t,u) \ge \frac{u}{\delta mT} \quad \text{for } u \in [0,r]. \quad (4.38)$$

This is the condition of Theorem 3.3 of [13]. So, our conclusions extend and improve the results of [13].

5. Some Nonexistence Results

Theorem 5.1. *Assume that (H1) and (H2) hold. If $f^0 \in [0,\infty)$ and $f^\infty \in [0,\infty)$, then the BVP* (1.1) *has no positive solutions for sufficiently small $\lambda > 0$.*

Proof. In view of the definition of f^0, f^∞, there exist positive constants $\varepsilon_1, \varepsilon_2$, r_1 and r_2 satisfying $r_1 < r_2$ and

$$f(t,u) \le \left(f^0 + \varepsilon_1\right)u, \quad u \in [0,r_1], \qquad f(t,u) \le (f^\infty + \varepsilon_2)u, \quad u \in [r_2,\infty). \quad (5.1)$$

Let

$$c_1 = \max\left\{ f^0 + \varepsilon_1, f^\infty + \varepsilon_2, \max_{t\in[0,T]_{\mathbb{T}}, u\in[r_1,r_2]} \frac{f(t,u)}{u} \right\}. \tag{5.2}$$

Then $c_1 > 0$ and we have

$$f(t,u) \le c_1 u, \quad u \in [0,\infty),\ t \in [0,T]_{\mathbb{T}}. \tag{5.3}$$

We assert that the BVP (1.1) has no positive solutions for $0 < \lambda < 1/c_1 MT$.

Suppose on the contrary that the BVP (1.1) has a positive solution u for $0 < \lambda < 1/c_1 MT$. Then from (5.3) and Lemma 3.2, we get

$$\|u\| = \|\Phi_\lambda u\| \le \lambda c_1 MT \|u\| < \|u\|, \tag{5.4}$$

which is a contradiction. □

Theorem 5.2. *Assume that (H1) and (H2) hold. If $f_0 \in (0,\infty)$ and $f_\infty \in (0,\infty)$, then the BVP* (1.1) *has no positive solutions for sufficiently large $\lambda > 0$.*

Proof. By the definition of f_0, f_∞, there exist positive constants η_1, η_2, r_1, and r_2 satisfying $f_0 > \eta_1$, $f_\infty > \eta_2$, $r_1 < r_2$, and

$$f(t,u) \ge (f_0 - \eta_1)u, \quad u \in [0,r_1], \qquad f(t,u) \ge (f_\infty - \eta_2)u, \quad u \in [r_2,\infty). \tag{5.5}$$

Let

$$c_2 = \min\left\{ f_0 - \eta_1, f_\infty - \eta_2, \min_{t\in[0,T]_{\mathbb{T}}, u\in[r_1,r_2]} \frac{f(t,u)}{u} \right\}. \tag{5.6}$$

Then $c_2 > 0$ and we have

$$f(t,u) \ge c_2 u, \quad u \in [0,\infty),\ t \in [0,T]_{\mathbb{T}}. \tag{5.7}$$

We assert that the BVP (1.1) has no positive solutions for $\lambda > 1/c_2 \delta m T$.

Suppose on the contrary that the BVP (1.1) has a positive solution u for $\lambda > 1/c_2 \delta m T$. Then from (5.7) and Lemma 3.1 we get

$$\|u\| = \|\Phi_\lambda u\| \ge \lambda c_2 \delta m T \|u\| > \|u\|, \tag{5.8}$$

which is a contradiction. □

Corollary 5.3. *Assume that (H1) and (H2) hold. If $f_0 = \infty$ and $f_\infty = \infty$, then the BVP* (1.1) *has no positive solutions for sufficiently large $\lambda > 0$.*

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