Hindawi Publishing Corporation
Boundary Value Problems
Volume 2011, Article ID 297026, 12 pages
doi:10.1155/2011/297026

Research Article

Positive Solution of Singular Boundary Value Problem for a Nonlinear Fractional Differential Equation

Changyou Wang,[1,2,3] Ruifang Wang,[2,4] Shu Wang,[3] and Chunde Yang[1]

[1] *College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*
[2] *Key Laboratory of Network Control & Intelligent Instrument, Chongqing University of Posts and Telecommunications, Ministry of Education, Chongqing 400065, China*
[3] *College of Applied Sciences, Beijing University of Technology, Beijing 100124, China*
[4] *Automation Institute, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*

Correspondence should be addressed to Changyou Wang, wangcy@cqupt.edu.cn

Received 16 August 2010; Revised 16 November 2010; Accepted 9 January 2011

Academic Editor: M. Salim

The method of upper and lower solutions and the Schauder fixed point theorem are used to investigate the existence and uniqueness of a positive solution to a singular boundary value problem for a class of nonlinear fractional differential equations with non-monotone term. Moreover, the existence of maximal and minimal solutions for the problem is also given.

1. Introduction

Fractional differential equation can be extensively applied to various disciplines such as physics, mechanics, chemistry, and engineering, see [1–3]. Hence, in recent years, fractional differential equations have been of great interest, and there have been many results on existence and uniqueness of the solution of boundary value problems for fractional differential equations, see [4–7]. Especially, in [8] the authors have studied the following type of fractional differential equations:

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad u(0) = u(1) = 0, \quad 0 < t < 1, \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and D_{0+}^{α} is the fractional derivative in the sense of Riemann-Liouville. Recently, Qiu and Bai [9] have proved the existence of a positive solution to boundary value problems of the nonlinear fractional differential equations

$$D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad u(0) = u'(1) = u''(0) = 0, \quad 0 < t < 1, \tag{1.2}$$

where $2 < \alpha \leq 3$, D_{0+}^{α} denotes Caputo derivative, and $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ with $\lim_{t\to 0+} f(t,\cdot) = +\infty$ (i.e., f is singular at $t = 0$). Their analysis relies on Krasnoselskii's fixed-point theorem and nonlinear alternative of Leray-Schauder type in a cone. More recently, Caballero Mena et al. [10] have proved the existence and uniqueness of a positive and non-decreasing solution to this problem by a fixed-point theorem in partially ordered sets. Other related results on the boundary value problem of the fractional differential equations can be found in the papers [11–23]. A study of a coupled differential system of fractional order is also very significant because this kind of system can often occur in applications [24–26].

However, in the previous works [9, 10], the nonlinear term has to satisfy the monotone or other control conditions. In fact, the nonlinear fractional differential equation with non-monotone term can respond better to impersonal law, so it is very important to weaken control conditions of the nonlinear term. In this paper, we mainly investigate the fractional differential (1.2) without any monotone requirement on nonlinear term by constructing upper and lower control function and exploiting the method of upper and lower solutions and Schauder fixed-point theorem. The existence and uniqueness of positive solution for (1.2) is obtained. Some properties concerning the maximal and minimal solutions are also given. This work is motivated by the above references and my previous work [27]. This paper is organized as follows. In Section 2, we recall briefly some notions of the fractional calculus and the theory of the operators for integration and differentiation of fractional order. Section 3 is devoted to the study of the existence and uniqueness of positive solution for (1.2) utilizing the method of upper and lower solutions and Schauder fixed-point theorem. The existence of maximal and minimal solutions for (1.2) is given in Section 4.

2. Preliminaries and Notations

For the convenience of the reader, we present here the necessary definitions and properties from fractional calculus theory, which are used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,+\infty) \to R$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to R$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \quad 0 < t < +\infty, \tag{2.2}$$

where $n-1 < \alpha \leq n$, $n \in N$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 2.3 (see [28]). *Let $n-1 < \alpha \leq n$, $n \in N$, $u(t) \in C^n[0,1]$, then*

$$\begin{gathered} I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - C_1 - C_2 t - \cdots - C_n t^{n-1}, \quad C_i \in R, \ i = 1,2,\ldots,n, \ 0 \leq t \leq 1, \\ D_{+}^{\alpha} I_{+}^{\alpha} u(t) = u(t), \quad 0 \leq t \leq 1. \end{gathered} \tag{2.3}$$

Lemma 2.4 (see [28]). *The relation*

$$I_{0+}^{\alpha} I_{0+}^{\beta} \varphi(t) = I_{0+}^{\alpha+\beta} \varphi(t) \tag{2.4}$$

is valid when $\operatorname{Re}\beta > 0$, $\operatorname{Re}(\alpha+\beta) > 0$, $\varphi(t) \in L^1(a,b)$.

Lemma 2.5 (see [9]). *Let $2 < \alpha \leq 3$, $0 < \sigma < \alpha - 2$; $F : (0,1] \to R$ is a continuous function and $\lim_{t\to 0^+} F(t) = \infty$. If $t^{\sigma} F(t)$ is continuous function on $[0,1]$, then the function*

$$H(t) = \int_0^1 G(t,s) F(s) ds \tag{2.5}$$

is continuous on $[0,1]$, where

$$G(t,s) = \begin{cases} \dfrac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\[2ex] \dfrac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.6}$$

Lemma 2.6. *Let $2 < \alpha \leq 3$, $0 < \sigma < \alpha - 2$; $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function and $\lim_{t\to 0^+} f(t,\cdot) = +\infty$. If $t^{\sigma} f(t,u(t))$ is continuous function on $[0,1] \times [0,+\infty)$, then the boundary value problems* (1.2) *are equivalent to the Volterra integral equations*

$$u(t) = \int_0^1 G(t,s) f(s,u(s)) ds. \tag{2.7}$$

Proof. From Lemma 2.5, the Volterra integral equation (2.7) is well defined. If $u(t)$ satisfies the boundary value problems (1.2), then applying I^{α} to both sides of (1.2) and using Lemma 2.3, one has

$$u(t) = -I_{0+}^{\alpha} f(t,u(t)) + C_1 + C_2 t + C_3 t^2, \tag{2.8}$$

where $C_i \in R$, $i = 1, 2, 3$. Since $t^{\sigma} f(t, u(t))$ is continuous in $[0, 1]$, there exists a constant $M > 0$, such that $|t^{\sigma} f(t, u(t))| \le M$, for $t \in [0, 1]$. Hence

$$\begin{aligned}
I_{0+}^{\alpha} f(t, u(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\sigma} s^{\sigma} f(s, u(s)) ds \\
&\le M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} ds \\
&= \frac{M}{\Gamma(\alpha)} t^{\alpha-\sigma} B(1-\sigma, \alpha) \\
&= \frac{\Gamma(1-\sigma) M}{\Gamma(1+\alpha-\sigma)} t^{\alpha-\sigma},
\end{aligned} \tag{2.9}$$

where B denotes the beta function. Thus, $I_{0+}^{\alpha} f(t, u(t)) \to 0$ as $t \to 0$. In the similar way, we can prove that $I_{0+}^{\alpha-2} f(t, u(t)) \to 0$ as $t \to 0$.

By Lemma 2.4 we have

$$\begin{aligned}
u'(t) &= -D_{0+}^{1} I_{0+}^{\alpha} f(t, u(t)) + C_2 + 2C_3 t \\
&= -D_{0+}^{1} I_{0+}^{1} I_{0+}^{\alpha-1} f(t, u(t)) + C_2 + 2C_3 t \\
&= -I_{0+}^{\alpha-1} f(t, u(t)) + C_2 + 2C_3 t, \\
u''(t) &= -D_{0+}^{1} I_{0+}^{\alpha-1} f(t, u(t)) + 2C_3 = -I_{0+}^{\alpha-2} f(t, u(t)) + 2C_3.
\end{aligned} \tag{2.10}$$

From the boundary conditions $u(0) = u'(1) = u''(0) = 0$, one has

$$C_1 = 0, \qquad C_2 = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds, \qquad C_3 = 0. \tag{2.11}$$

Therefore, it follows from (2.8) that

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} f(s, u(s)) ds \\
&= \int_0^t \left[\frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right] f(s, u(s)) ds + \int_t^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s)) ds \\
&= \int_0^1 G(t, s) f(s, u(s)) ds.
\end{aligned} \tag{2.12}$$

Namely, (2.7) follows.

Conversely, suppose that $u(t)$ satisfies (2.7), then we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s) f(s,u(s)) ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s)) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} f(s,u(s)) ds \\ &= -I_{0+}^{\alpha} f(t,u(t)) + \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds, \end{aligned} \tag{2.13}$$

From Lemmas 2.3 and 2.4 and Definition 2.2, one has

$$\begin{aligned} u'(t) &= -D_{0+}^{1} I_{0+}^{\alpha} f(t,u(t)) + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds \\ &= -I_{0+}^{\alpha-1} f(t,u(t)) + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds \\ &= -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s,u(s)) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds, \\ u''(t) &= D_{0+}^{1} \left[-I_{0+}^{\alpha-1} f(t,u(t)) + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds \right] \\ &= -I_{0+}^{\alpha-2} f(t,u(t)) = -\frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} f(s,u(s)) ds, \end{aligned} \tag{2.14}$$

as well as

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= D_{0+}^{\alpha} \left[-I_{0+}^{\alpha} f(t,u(t)) + \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds \right] \\ &= -D_{0+}^{\alpha} I_{0+}^{\alpha} f(t,u(t)) + I_{0+}^{3-\alpha} D_{0+}^{3} \left[\frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s,u(s)) ds \right] \\ &= -f(t,u(t)). \end{aligned} \tag{2.15}$$

Thus, from (2.12), (2.14), and (2.15), it is follows that

$$D_{0+}^{\alpha} u(t) + f(t,u(t)) = 0, \quad u(0) = u'(1) = u''(0) = 0, \quad 0 < t < 1. \tag{2.16}$$

Namely, (1.2) holds. The proof is therefore completed. □

Remark 2.7. For $G(t,s)$, since $2 < \alpha \le 3$, $0 \le s \le t \le 1$ we can obtain

$$(\alpha-1)t(1-s)^{\alpha-2} \ge t(1-s)^{\alpha-2} \ge t(t-s)^{\alpha-2} \ge (t-s)^{\alpha-1}. \tag{2.17}$$

Hence, it is follow from (2.6) that $G(t,s) > 0$, for $0 < t < 1$ and $G(0,s) = G(1,1) = 0$.

Let $X = C^3[0,1]$ is the Banach space endowed with the infinity norm, K is a nonempty closed subset of X defined as $K = \{u(t) \in X \mid 0 < u(t), 0 < t \le 1, u(0) = 0\}$. The positive solution which we consider in this paper is a function such that $u(t) \in K$.

According to Lemma 2.6, (1.2) is equivalent to the fractional integral equation (2.7). The integral equation (2.7) is also equivalent to fixed-point equation $Tu(t) = u(t)$, $u(t) \in C^3[0,1]$, where operator $T : K \to K$ is defined as

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))ds, \tag{2.18}$$

then we have the following lemma.

Lemma 2.8 (see [9]). *Let $2 < \alpha \le 3$, $0 < \sigma < \alpha - 2$, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function and $\lim_{t\to 0+} f(t,\cdot) = +\infty$. If $t^\sigma f(t,u(t))$ is continuous function on $[0,1]\times[0,+\infty)$, then the operator $T : K \to K$ is completely continuous.*

Let $2 < \alpha \le 3$, $0 < \sigma < \alpha - 2$, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $\lim_{t\to 0+} f(t,\cdot) = +\infty$, and $t^\sigma f(t,u(t))$ is continuous function on $[0,1]\times[0,+\infty)$. Take $a,b \in R^+$, and $a < b$. For any $u(t) \in X$, $a \le u(t) \le b$, we define the upper-control function $H(t,u) = \sup_{a\le\eta\le u} f(t,\eta)$, and lower-control function $h(t,u) = \inf_{u\le\eta\le b} f(t,\eta)$, it is obvious that $H(t,u), h(t,u)$ are monotonous non-decreasing on u and $h(t,u) \le f(t,u) \le H(t,u)$.

Definition 2.9. Let $\tilde{u}(t), \hat{u}(t) \in K$, $b \ge \tilde{u}(t) \ge \hat{u}(t) \ge a$, and satisfy, respectively

$$\begin{aligned}\tilde{u}(t) &\ge \int_0^1 G(t,s)H(s,\tilde{u}(s))ds,\\ \hat{u}(t) &\le \int_0^1 G(t,s)h(s,\hat{u}(s))ds,\end{aligned} \tag{2.19}$$

then the function $\tilde{u}(t), \hat{u}(t)$ are called a pair of order upper and lower solutions for (1.2).

3. Existence and Uniqueness of Positive Solution

Now, we give and prove the main results of this paper.

Theorem 3.1. *Let $2 < \alpha \le 3$, $0 < \sigma < \alpha - 2$; $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function with $\lim_{t\to 0+} f(t,\cdot) = +\infty$, and $t^\sigma f(t,u(t))$ is a continuous function on $[0,1]\times[0,+\infty)$. Assume that $\tilde{u}(t), \hat{u}(t)$ are a pair of order upper and lower solutions of (1.2), then the boundary value problem (1.2) has at least one solution $u(t) \in C^3[0,1]$, moreover,*

$$\tilde{u}(t) \ge u(t) \ge \hat{u}(t), \quad t \in [0,1]. \tag{3.1}$$

Proof. Let

$$S = \{z(t) \mid z(t) \in K,\ \widetilde{u}(t) \le z(t) \le \widehat{u}(t),\ t \in [0,1]\}, \tag{3.2}$$

endowed with the norm $\|z\| = \max_{t\in[0,1]} z(t)$, then we have $\|z\| \le b$. Hence S is a convex, bounded, and closed subset of the Banach space X. According to Lemma 2.8, the operator $T : K \to K$ is completely continuous. Then we need only to prove $T : S \to S$.

For any $z(t) \in S$, we have $\widetilde{u}(t) \ge z(t) \ge \widehat{u}(t)$. In view of Remark 2.7, Definition 2.9, and the definition of control function, one has

$$\begin{aligned} Tz(t) &= \int_0^1 G(t,s) f(s, z(s)) ds \le \int_0^1 G(t,s) H(s, z(s)) ds \\ &\le \int_0^1 G(t,s) H(s, \widetilde{u}(s)) ds \le \widetilde{u}(t), \\ Tz(t) &= \int_0^1 G(t,s) f(s, z(s)) ds \ge \int_0^1 G(t,s) h(s, z(s)) ds \\ &\ge \int_0^1 G(t,s) h(s, \widehat{u}(s)) ds \ge \widehat{u}(t). \end{aligned} \tag{3.3}$$

Hence $\widetilde{u}(t) \ge Tz(t) \ge \widehat{u}(t)$, $1 \ge t \ge 0$, that is, $T : S \to S$. According to Schauder fixed-point theorem, the operator T has at least a fixed-point $u(t) \in S$, $0 \le t \le 1$. Therefore the boundary value problem (1.2) has at least one solution $u(t) \in C^3[0,1]$, and $\widetilde{u}(t) \ge u(t) \ge \widehat{u}(t)$, $t \in [0,1]$. □

Corollary 3.2. *Let* $2 < \alpha \le 3$, $0 < \sigma < \alpha - 2$; $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ *is a continuous function with* $\lim_{t\to 0+} f(t,\cdot) = +\infty$, *and* $t^\sigma f(t,u(t))$ *is a continuous function on* $[0,1]\times[0,+\infty)$. *Assume that there exist two distinct positive constant* ρ, μ $(\rho > \mu)$, *such that*

$$\mu \le t^\sigma f(t,l) \le \rho, \quad (t,l) \in [0,1] \times [0,+\infty), \tag{3.4}$$

then the boundary value problem (1.2) *has at least a positive solution* $u(t) \in C[0,1]$, *moreover*

$$\mu \int_0^1 G(t,s) s^{-\sigma} ds \le u(t) \le \rho \int_0^1 G(t,s) s^{-\sigma} ds. \tag{3.5}$$

Proof. By assumption (3.4) and the definition of control function, we have

$$\mu t^{-\sigma} \le h(t,l) \le H(t,l) \le \rho t^{-\sigma}, \quad (t,l) \in (0,1] \times [a,b]. \tag{3.6}$$

Now, we consider the equation

$$D_{0+}^{\alpha} u(t) + \rho t^{-\sigma} = 0, \quad u(0) = u'(1) = u''(0) = 0, \quad 0 < t < 1. \tag{3.7}$$

From Lemmas 2.5 and 2.6, (3.7) has a positive continuous solution on $[0,1]$

$$\begin{aligned} w(t) &= \rho \int_0^1 G(t,s)s^{-\sigma}ds, \quad t \in [0,1], \\ w(t) &= \rho \int_0^1 G(t,s)s^{-\sigma}ds \geq \int_0^1 G(t,s)H(s,w(s))ds. \end{aligned} \tag{3.8}$$

Namely, $w(t)$ is a upper solution of (1.2). In the similar way, we obtain $v(t) = \mu \int_0^1 G(t,s)s^{-\sigma}ds$ is the lower solution of (1.2). An application of Theorem 3.1 now yields that the boundary value problem (1.2) has at least a positive solution $u(t) \in C^3[0,1]$, moreover

$$\mu \int_0^1 G(t,s)s^{-\sigma}ds \leq u(t) \leq \rho \int_0^1 G(t,s)s^{-\sigma}ds. \tag{3.9}$$

□

Theorem 3.3. *If the conditions in Theorem 3.1 hold. Moreover for any $u_1(t), u_2(t) \in X$, $0 < t < 1$, there exists $l > 0$, such that*

$$\left| f(t,u_1) - f(t,u_2) \right| \leq l|u_1 - u_2|, \tag{3.10}$$

then when $l \max_{0\leq t\leq 1} \int_0^1 G(t,s)ds < 1$, the boundary value problem (1.2) *has a unique positive solution $u(t) \in S$.*

Proof. According to Theorem 3.1, if the conditions in Theorem 3.1 hold, then the boundary value problems (1.2) have at least a positive solution in S. Hence we need only to prove that the operator T defined in (2.18) is the contraction mapping in X. In fact, for any $u_1(t), u_2(t) \in X$, by assumption (3.10), we have

$$\begin{aligned} |Tu_1(t) - Tu_2(t)| &= \left| \int_0^1 G(t,s)f(s,u_1(s))ds - \int_0^1 G(t,s)f(s,u_2(s))ds \right| \\ &= \left| \int_0^1 G(t,s)\left[f(s,u_1(s)) - f(s,u_2(s))\right]ds \right| \\ &\leq \int_0^1 G(t,s)\left|f(s,u_1(s)) - f(s,u_2(s))\right|ds \\ &\leq l \int_0^1 G(t,s)ds|u_1 - u_2|. \end{aligned} \tag{3.11}$$

Note that, from Lemma 2.5, $\int_0^1 G(t,s)ds$ is a continuous function on $[0,1]$. Thus, when $l \max_{0\leq t\leq 1} \int_0^1 G(t,s)ds < 1$, the operator T is the contraction mapping. Then by Banach contraction fixed-point theorem, the boundary value problem (1.2) has a unique positive solution $u(t) \in S$. □

4. Maximal and Minimal Solutions Theorem

In this section, we consider the existence of maximal and minimal solutions for (1.2).

Definition 4.1. Let $m(t)$ be a solution of (1.2) in $[0,1]$, then $m(t)$ is said to be a maximal solution of (1.2), if for every solution $u(t)$ of (1.2) existing on $[0,1]$, the inequality $u(t) \le m(t)$, $t \in [0,1]$, holds. A minimal solution may be defined similarly by reversing the last inequality.

Theorem 4.2. *Let $2 < \alpha \le 3$, $0 < \sigma < \alpha - 2$, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function with $\lim_{t\to 0+} f(t,\cdot) = +\infty$, and $t^\sigma f(t,u(t))$ is a continuous function on $[0,1]\times[0,+\infty)$. Assume that $f(t,u)$ is monotone non-decreasing with respect to the second variable, and there exist two positive constants λ, μ $(\mu > \lambda)$ such that*

$$\lambda \le t^\sigma f(t,u) \le \mu, \quad \text{for } (t,u) \in (0,1] \times [0,+\infty). \tag{4.1}$$

Then there exist maximal solution $\varphi(t)$ and minimal solution $\eta(t)$ of (1.2) on $[0,1]$, moreover

$$\lambda \int_0^1 G(t,s)s^{-\sigma} ds \le \eta(t) \le \varphi(t) \le \mu \int_0^1 G(t,s)s^{-\sigma} ds, \quad 0 \le t \le 1. \tag{4.2}$$

Proof. It is easy to know from Corollary 3.2 that $\mu \int_0^1 G(t,s)\, s^{-\sigma} ds$ and $\lambda \int_0^1 G(t,s)s^{-\sigma} ds$ are the upper and lower solutions of (1.2), respectively. Then by using $\overline{u}^{(0)} = \mu \int_0^1 G(t,s)s^{-\sigma} ds$, $\underline{u}^{(0)} = \lambda \int_0^1 G(t,s)s^{-\sigma} ds$ as a pair of coupled initial iterations we construct two sequences $\{\overline{u}^{(m)}\},\{\underline{u}^{(m)}\}$ from the following linear iteration process:

$$\begin{aligned} \overline{u}^{(m)}(t) &= \int_0^1 G(t,s) f\left(s, \overline{u}^{(m-1)}(s)\right) ds, \\ \underline{u}^{(m)}(t) &= \int_0^1 G(t,s) f\left(s, \underline{u}^{(m-1)}(s)\right) ds. \end{aligned} \tag{4.3}$$

It is easy to show from the monotone property of $f(t,u)$ and the condition (4.1) that the sequences $\{\overline{u}^{(m)}\},\{\underline{u}^{(m)}\}$ possess the following monotone property:

$$\underline{u}^{(0)} \le \underline{u}^{(m)} \le \underline{u}^{(m+1)} \le \overline{u}^{(m+1)} \le \overline{u}^{(m)} \le \overline{u}^{(0)} \quad (m = 1,2,\ldots). \tag{4.4}$$

The above property implies that

$$\lim_{m\to\infty} \overline{u}(t)^{(m)} = \varphi(t), \qquad \lim_{m\to\infty} \underline{u}(t)^{(m)} = \eta(t) \tag{4.5}$$

exist and satisfy the relation

$$\lambda \int_0^1 G(t,s)s^{-\sigma} ds \le \eta(t) \le \varphi(t) \le \mu \int_0^1 G(t,s)s^{-\sigma} ds, \quad 0 \le t \le 1. \tag{4.6}$$

Letting $m \to \infty$ in (4.3) shows that $\varphi(t)$ and $\eta(t)$ satisfy the equations

$$\varphi(t) = \int_0^1 G(t,s) f(s,\varphi(s)) ds,$$
$$\eta(t) = \int_0^1 G(t,s) f(s,\eta(s)) ds. \tag{4.7}$$

It is easy to verify that the limits $\varphi(t)$ and $\eta(t)$ are maximal and minimal solutions of (1.2) in

$$S^* = \left\{ \varphi(t) \mid \varphi(t) \in K,\ \lambda \int_0^1 G(t,s) s^{-\sigma} ds \le \varphi(t) \le \mu \int_0^1 G(t,s) s^{-\sigma} ds, \right.$$
$$\left. t \in [0,1], \|\varphi(t)\| = \max_{0 \le t \le 1} \varphi(t) \right\} \tag{4.8}$$

respectively, furthermore, if $\varphi(t) = \eta(t)$ ($\equiv \zeta(t)$) then $\zeta(t)$ is the unique solution in S^*, and hence the proof is completed. □

Finally, we give an example to illuminate our results.

Example 4.3. We consider the fractional order differential equation

$$D_{0+}^{\alpha} u(t) = t^{-\sigma} \left\{ 1 + \frac{u(t)}{u(t) + \sin u(t) + 1} \right\}, \quad 0 < t < 1,$$
$$u(0) = u'(1) = u''(0) = 0, \tag{4.9}$$

where $2 < \alpha \le 3$, $0 < \sigma < \alpha - 2$. It is obvious from $f(t,u(t)) = t^{-\sigma}\{1 + u(t)/(u(t) + \sin u(t) + 1)\}$ that $1 \le t^{\sigma} f(t,u) \le 2$, $(t,u) \in (0,1] \times [0,+\infty)$. By Corollary 3.2, then (4.9) has a positive solution. Nevertheless it is easy to prove that the conclusions of [9, 10] cannot be applied to the above example.

Acknowledgments

The authors are grateful to the referee for the comments. This work is supported by Natural Science Foundation Project of CQ CSTC (Grants nos. 2008BB7415, 2010BB9401) of China, Ministry of Education Project (Grant no. 708047) of China, Science and Technology Project of Chongqing municipal education committee (Grant no. KJ100513) of China, the NSFC (Grant no. 51005264) of China.

References

[1] F. Mainardi, "The fundamental solutions for the fractional diffusion-wave equation," *Applied Mathematics Letters*, vol. 9, no. 6, pp. 23–28, 1996.

[2] E. Buckwar and Y. Luchko, "Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations," *Journal of Mathematical Analysis and Applications*, vol. 227, no. 1, pp. 81–97, 1998.

[3] Z. Y. Zhu, G. G. Li, and C. J. Cheng, “Quasi-static and dynamical analysis for viscoelastic Timoshenko beam with fractional derivative constitutive relation,” *Applied Mathematics and Mechanics*, vol. 23, no. 1, pp. 1–12, 2002.

[4] A. M. Nahušev, “The Sturm-Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms,” *Doklady Akademii Nauk SSSR*, vol. 234, no. 2, pp. 308–311, 1977.

[5] T. S. Aleroev, “The Sturm-Liouville problem for a second-order differential equation with fractional derivatives in the lower terms,” *Differentsial'nye Uravneniya*, vol. 18, no. 2, pp. 341–342, 1982.

[6] S. Zhang, “Existence of solution for a boundary value problem of fractional order,” *Acta Mathematica Scientia B*, vol. 26, no. 2, pp. 220–228, 2006.

[7] S. Zhang, “Positive solutions for boundary-value problems of nonlinear fractional differential equations,” *Electronic Journal of Differential Equations*, vol. 36, pp. 1–12, 2006.

[8] Z. B. Bai and H. S. Lü, “Positive solutions for boundary value problem of nonlinear fractional differential equation,” *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[9] T. Qiu and Z. Bai, “Existence of positive solutions for singular fractional differential equations,” *Electronic Journal of Differential Equations*, vol. 149, pp. 1–9, 2008.

[10] J. Caballero Mena, J. Harjani, and K. Sadarangani, “Existence and unqiueness of positive and nondecreasing solutions for a class of singular fractional boundary value problems,” *Boundary Value Problems*, vol. 2009, Article ID 421310, 10 pages, 2009.

[11] B. Ahmad and J. J. Nieto, “Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions,” *Boundary Value Problems*, vol. 2009, Article ID 708576, 11 pages, 2009.

[12] Y.-K. Chang and J. J. Nieto, “Some new existence results for fractional differential inclusions with boundary conditions,” *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 605–609, 2009.

[13] B. Ahmad and J. J. Nieto, “Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions,” *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1838–1843, 2009.

[14] X. Su, “Boundary value problem for a coupled system of nonlinear fractional differential equations,” *Applied Mathematics Letters*, vol. 22, no. 1, pp. 64–69, 2009.

[15] S. Q. Zhang, “Positive solutions to singular boundary value problem for nonlinear fractional differential equation,” *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1300–1309, 2010.

[16] B. Ahmad, “Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations,” *Applied Mathematics Letters*, vol. 23, no. 4, pp. 390–394, 2010.

[17] B. Ahmad and S. Sivasundaram, “Existence of solutions for impulsive integral boundary value problems of fractional order,” *Nonlinear Analysis. Hybrid Systems*, vol. 4, no. 1, pp. 134–141, 2010.

[18] V. Daftardar-Gejji and H. Jafari, “Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives,” *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 1026–1033, 2007.

[19] S. Momani and R. Qaralleh, “An efficient method for solving systems of fractional integro-differential equations,” *Computers & Mathematics with Applications*, vol. 52, no. 3-4, pp. 459–470, 2006.

[20] S. H. Hosseinnia, A. Ranjbar, and S. Momani, “Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part,” *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3138–3149, 2008.

[21] O. Abdulaziz, I. Hashim, and S. Momani, “Solving systems of fractional differential equations by homotopy-perturbation method,” *Physics Letters. A*, vol. 372, no. 4, pp. 451–459, 2008.

[22] O. Abdulaziz, I. Hashim, and S. Momani, “Application of homotopy-perturbation method to fractional IVPs,” *Journal of Computational and Applied Mathematics*, vol. 216, no. 2, pp. 574–584, 2008.

[23] I. Hashim, O. Abdulaziz, and S. Momani, “Homotopy analysis method for fractional IVPs,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 3, pp. 674–684, 2009.

[24] Y. Chen and H.-L. An, “Numerical solutions of coupled Burgers equations with time- and space-fractional derivatives,” *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 87–95, 2008.

[25] V. Gafiychuk, B. Datsko, and V. Meleshko, “Mathematical modeling of time fractional reaction-diffusion systems,” *Journal of Computational and Applied Mathematics*, vol. 220, no. 1-2, pp. 215–225, 2008.

[26] W. H. Deng and C. P. Li, “Chaos synchronization of the fractional Lü system,” *Physica A*, vol. 353, no. 1–4, pp. 61–72, 2005.

[27] C. Wang, “Existence and stability of periodic solutions for parabolic systems with time delays,” *Journal of Mathematical Analysis and Applications*, vol. 339, no. 2, pp. 1354–1361, 2008.

[28] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science, Yverdon, Switzerland, 1993.