Hindawi Publishing Corporation
Boundary Value Problems
Volume 2011, Article ID 591219, 11 pages
doi:10.1155/2011/591219

Research Article

Multiple Positive Solutions for *m*-Point Boundary Value Problem on Time Scales

Jie Liu[1,2] and Hong-Rui Sun[2]

[1] *Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu, 730050, China*
[2] *School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, China*

Correspondence should be addressed to Hong-Rui Sun, hrsun@lzu.edu.cn

Received 29 May 2010; Accepted 6 August 2010

Academic Editor: Feliz Manuel Minhós

The purpose of this article is to establish the existence of multiple positive solutions of the dynamic equation on time scales $(\phi(u^{\Delta}(t)))^{\nabla} + h(t)f(t, u(t), u^{\Delta}(t)) = 0$, $t \in (0, T)_{\mathbb{T}}$, subject to the multi-point boundary condition $u^{\Delta}(0) = 0$, $u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and satisfies the relation $\phi(xy) = \phi(x)\phi(y)$ for $x, y \in \mathbb{R}$, which generalizes the usually p-Laplacian operator. An example applying the result is also presented. The main tool of this paper is a generalization of Leggett-Williams fixed point theorem, and the interesting points are that the nonlinearity f contains the first-order derivative explicitly and the operator ϕ is not necessarily odd.

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1], and is a new area of still fairly theoretical exploration in mathematics. On one hand, the time scales approach not only unifies calculus and difference equations, but also solves other problems that have a mix of stop-start and continuous behavior. On the other hand, the time scales calculus has tremendous potential for application in biological, phytoremediation of metals, wound healing, stock market and epidemic models [2–6].

Let $\mathbb{T}$ be a time scale (an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$). For each interval $\mathbf{I}$ of $\mathbb{R}$, we define $\mathbf{I}_{\mathbb{T}} = \mathbf{I} \cap \mathbb{T}$. For more details on time scales, one can refer to [1–3, 5]. In this paper we are concerned with the existence of at least triple positive solutions to the following m-point boundary value problems on time scales

$$\left(\phi\left(u^{\Delta}(t)\right)\right)^{\nabla} + h(t)f\left(t, u(t), u^{\Delta}(t)\right) = 0, \quad t \in (0, T)_{\mathbb{T}}, \tag{1.1}$$

$$u^{\Delta}(0) = 0, \qquad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and $\phi(xy) = \phi(x)\phi(y)$ for $x, y \in \mathbb{R}$.

Multipoint boundary value problem (BVP) arise in a variety of different areas of applied mathematics and physics, such as the vibrations of a guy wire of a uniform cross section and composed of N parts of different densities can be set up as a multipoint boundary value problem [7]. Small size bridges are often designed with two supported points, which leads to a standard two-point boundary value condition. And large size bridges are sometimes contrived with multipoint supports, which corresponds to a multipoint boundary value condition [8]. Especially, if we let $u(t)$ denotes the displacement of the bridge from the unloaded position, and we emphasize the position of the bridge at supporting points near $t = 0$, we can obtain the multipoint boundary condition (1.2). The study of multipoint BVPs for linear second-order ordinary differential equations was initiated by Ilin and Moiseev [9], since then many authors studied more general nonlinear multipoint boundary value problems. We refer readers to [8, 10–14] and the references therein.

Recently, when ϕ is p-Laplacian operator, that is $\phi(u) = |u|^{p-2}u(p > 1)$, and the nonlinear term does not depend on the first-order derivative, the existence problems of positive solutions of boundary value problems have attracted much attention, see [10, 12, 15–22] in the continuous case, see [15, 23–25] in the discrete case and [11, 13, 14, 26, 27] in the general time scale setting. From the process of proving main results in the above references, one can notice that the oddness of the p-Laplacian operator is key to the proof. However in this paper the operator ϕ is not necessary odd, so it improves and generalizes the p-Laplacian operator. One may note this from Example 3.3 in Section 3. In addition, Bai and Ge [16] generalized the Leggett-Williams fixed point theorems by using fixed point index theory. An application of the theorem is given to prove the existence of three positive solutions to the following second-order BVP:

$$u''(t) + f\left(t, u(t), u'(t)\right) = 0, \qquad t \in (0,1), \tag{1.3}$$

with Dirichlet boundary condition. They also extended the results to four-point BVP in [12].

When $\phi(u) = |u|^{p-2}u$ and the nonlinearity f is not involved with the first-order derivative $u^{\Delta}(t)$, in [27], Sun and Li discussed the existence and multiplicity of positive solutions for problems (1.1) and (1.2). The main tools used are fixed point theorems in cones.

Thanks to the above-mentioned research articles [16, 27], in this paper we consider the existence of multiple positive solutions for the more general dynamic equation on time scales (1.1) with m-point boundary condition (1.2). An example is also given to illustrate the main results. The obtained results are even new for the special cases of difference equations and differential equations, as well as in the general time scale setting. The main result extends and generalizes the corresponding results of Liu [18] and Webb [21] ($\mathbb{T} = \mathbb{R}$, $\phi(u) = u$, $T = 1$, $m = 3$, $f(t,u,v) = f(u)$), Sun and Li [27] ($\phi(u) = |u|^{p-2}u, f(t,u,v) = f(t,u)$). We also emphasize that in this paper the nonlinear term f is involved with the first-order delta derivative $u^{\Delta}(t)$, the operator ϕ is not necessary odd and have the more generalized form, and the tool is a generalized Leggett-Williams fixed point theorem [16].

The rest of the paper is organized as follows: in Section 2, we give some preliminaries which are needed later. Section 3 is due to develop existence criteria for at least three and arbitrary odd number positive solution of the boundary value problem (1.1) and (1.2). In the final part of this section, we present an example to illustrate the application of the obtained result.

Throughout this paper, the following hypotheses hold:

(H1) $0, T \in \mathbb{T}$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$, $\xi_i \in \mathbb{T}$, $a_i \geq 0$ for $i = 1, \ldots, m-2$, and $d = 1 - \sum_{i=1}^{m-2} a_i > 0$;

(H2) $\eta = \max\{t \in \mathbb{T} : 0 < t \leq T/2\}$ exists and $h \in C_{\mathrm{ld}}((0,T)_{\mathbb{T}}, [0,\infty))$ such that $0 < \int_0^T h(s)\nabla s < \infty$ and $f : (0,T)_{\mathbb{T}} \times [0,\infty) \times (-\infty,\infty) \to [0,\infty)$ is continuous.

2. Preliminaries

In this section, we first present some basic definition, then we define an appropriate Banach space, cone, and integral operator, and finally we list the fixed-point theorem which is needed later.

Definition 2.1. Suppose P is a cone in a Banach space B. The map α is said to be a nonnegative continuous concave (convex) functional on P provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(\lambda x + (1-\lambda)y) \geq (\leq) \lambda\alpha(x) + (1-\lambda)\alpha(y) \quad \forall x, y \in P,\ 0 \leq \lambda \leq 1. \tag{2.1}$$

Let the Banach space $B = C^1_{\mathrm{ld}}([0,T]_{\mathbb{T}})$ be endowed with the norm $\|u\| = \max\{\|u\|_0, \|u^\Delta\|_0\}$, where

$$\|u\|_0 = \max_{t\in[0,T]_{\mathbb{T}}} |u(t)|, \qquad \left\|u^\Delta\right\|_0 = \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^\Delta(t)\right|, \tag{2.2}$$

and choose the cone $P \subset B$ as

$$P = \left\{u \in B : u(t) \geq 0 \text{ for } t \in [0,T]_{\mathbb{T}},\ u^\Delta(0) = 0 \text{ and } u \text{ is concave in } [0,T]_{\mathbb{T}}\right\}. \tag{2.3}$$

Now we define the operator $A : P \to B$ by

$$\begin{aligned} Au(t) = &-\int_t^T \phi^{-1}\left(-\int_0^s h(\tau) f\left(\tau, u(\tau), u^\Delta(\tau)\right)\nabla\tau\right)\Delta s \\ &- \frac{1}{d}\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi^{-1}\left(-\int_0^s h(\tau) f\left(\tau, u(\tau), u^\Delta(\tau)\right)\nabla\tau\right)\Delta s. \end{aligned} \tag{2.4}$$

From the definition of A and the assumptions of (H1), (H2), we can easily obtain that for each $u \in P$, $Au(t) \geq 0$ for $t \in [0,T]_{\mathbb{T}}$ and $(Au)^\Delta(0) = 0$. From the fact that

$$\left(\phi\left(Au^\Delta(t)\right)\right)^\nabla = -h(t) f\left(t, u(t), u^\Delta(t)\right) \leq 0, \quad t \in (0,T)_{\mathbb{T}}, \tag{2.5}$$

we know that Au is concave in $[0,T]_{\mathbb{T}}$. Thus $A : P \to P$ and $Au(0)$ is the maximum value of $Au(t)$. In addition, by direct calculation, we get that each fixed point of the operator A in P is a positive solution of (1.1) and (1.2). Similar as the proof of Lemma 2.3 in [27], it is easy to see that $A : P \to P$ is completely continuous.

Suppose α and β are two nonnegative continuous convex functionals satisfying

$$\|x\| \leq L \max\{\alpha(x), \beta(x)\}, \quad x \in P, \tag{2.6}$$

where L is a positive constant, and

$$\Omega = \{x \in P \mid \alpha(x) < r,\ \beta(x) < l\} \neq \emptyset, \quad r > 0,\ l > 0. \tag{2.7}$$

Let $r > a > 0$, $l > 0$ be given, α, β nonnegative continuous convex functionals on P satisfying the relation (2.6) and (2.7), and γ a nonnegative continuous concave functional on P. We define the following convex sets:

$$\begin{aligned} P(\alpha, r; \beta, l) &= \{u \in P : \alpha(u) < r,\ \beta(u) < l\}, \\ \overline{P}(\alpha, r; \beta, l) &= \{u \in P : \alpha(u) \leq r,\ \beta(u) \leq l\}, \\ P(\alpha, r; \beta, l; \gamma, a) &= \{u \in P : \alpha(u) < r,\ \beta(u) < l,\ \gamma(u) > a\}, \\ \overline{P}(\alpha, r; \beta, l; \gamma, a) &= \{u \in P : \alpha(u) \leq r,\ \beta(u) \leq l,\ \gamma(u) \geq a\}. \end{aligned} \tag{2.8}$$

In order to prove our main results, the following fixed point theorem is important in our argument.

Lemma 2.2 (see [16]). *Let B be Banach space, $P \subset B$ a cone, and $r_2 \geq d > b > r_1 > 0$, $l_2 \geq l_1 > 0$. Assume that α and β are nonnegative continuous convex functionals satisfying* (2.6) *and* (2.7), *γ is a nonnegative continuous concave functional on P such that $\gamma(u) \leq \alpha(u)$ for all $u \in \overline{P}(\alpha, r_2; \beta, l_2)$, and $A : \overline{P}(\alpha, r_2; \beta, l_2) \to \overline{P}(\alpha, r_2; \beta, l_2)$ is a completely continuous operator. Suppose*

(C1) $\{u \in \overline{P}(\alpha, d; \beta, l_2; \gamma, b) : \gamma(u) > b\} \neq \emptyset$, *$\gamma(Au) > b$ for $u \in \overline{P}(\alpha, d; \beta, l_2; \gamma, b)$;*

(C2) *$\alpha(Au) < r_1$, $\beta(Au) < l_1$ for $u \in \overline{P}(\alpha, r_1; \beta, l_1)$;*

(C3) *$\gamma(Au) > b$ for $u \in \overline{P}(\alpha, r_2; \beta, l_2; \gamma, b)$ with $\alpha(Au) > d$.*

Then A has at least three fixed points $u_1, u_2, u_3 \in \overline{P}(\alpha, r_2; \beta, l_2)$ with

$$\begin{gathered} u_1 \in P(\alpha, r_1; \beta, l_1), \qquad u_2 \in \left\{\overline{P}(\alpha, r_2; \beta, l_2; \gamma, b) \mid \gamma(u) > b\right\}, \\ u_3 \in \overline{P}(\alpha, r_2; \beta, l_2) \setminus \left(\overline{P}(\alpha, r_2; \beta, l_2; \gamma, b) \cup \overline{P}(\alpha, r_1; \beta, l_1)\right). \end{gathered} \tag{2.9}$$

3. Main Results

In this section, we impose some growth conditions on f which allow us to apply Lemma 2.2 to the operator A defined in Section 2 to establish the existence of three positive solutions of

(1.1) and (1.2). We note that, from the nonnegativity of h and f, the solution of (1.1) and (1.2) is nonnegative and concave on $[0,T]_{\mathbb{T}}$.

First in view of Lemma 2.4 in [27], we know that for $u \in P$, there is $u(t) \geq ((T-t)/T)\|u\|$ for $t \in [0,T]_{\mathbb{T}}$. So we get

$$u(t) \geq \frac{T-\eta}{T}\|u\| \geq \frac{1}{2}\|u\| \quad \text{for } t \in [0,\eta]_{\mathbb{T}}. \tag{3.1}$$

Let the nonnegative continuous convex functionals α, β and the nonnegative continuous concave functional γ be defined on the cone P by

$$\alpha(u) = \max_{t\in[0,T]_{\mathbb{T}}} |u(t)|, \quad \beta(u) = \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right|, \quad \gamma(u) = \min_{t\in[0,\eta]_{\mathbb{T}}} u(t), \quad u \in P. \tag{3.2}$$

Then, it is easy to see that $\|u\| = \max\{\alpha(u), \beta(u)\}$ and (2.6), (2.7) hold.

Now, for convenience we introduce the following notations. Let

$$\begin{gathered}
S = \phi^{-1}\left(\int_0^T h(\tau)\nabla\tau\right), \qquad M = (T-\eta)\phi^{-1}\left(\int_0^\eta h(\tau)\nabla\tau\right), \\
N = \int_0^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s + \frac{1}{d}\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s.
\end{gathered} \tag{3.3}$$

Theorem 3.1. *Assume $f(t,0,0) \not\equiv 0$ for $t \in [0,T]_{\mathbb{T}}$. If there are positive numbers $r_2 \geq 2b > b > r_1 > 0$, $l_2 \geq l_1 > 0$ with $b/M \leq \min\{r_2/N, l_2/S\}$, such that the following conditions are satisfied*

(i) *$f(t,u,v) \leq \min\{-\phi(-r_2/N), -\phi(-l_2/S)\}$ for $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0,r_2] \times [-l_2,0]$;*

(ii) *$f(t,u,v) > -\phi(-b/M)$ for $(t,u,v) \in [0,\eta]_{\mathbb{T}} \times [b,2b] \times [-l_2,0]$;*

(iii) *$f(t,u,v) < \min\{-\phi(-r_1/N), -\phi(-l_1/S)\}$ for $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0,r_1] \times [-l_1,0]$.*

then the problem (1.1), (1.2) *has at least three positive solutions u_1, u_2, u_3 satisfying*

$$\begin{gathered}
\max_{t\in[0,T]_{\mathbb{T}}} u_1(t) < r_1, \qquad \sup_{t\in[0,T]_{\mathbb{T}}} \left|u_1^{\Delta}(t)\right| < l_1, \\
b < \min_{t\in[0,\eta]_{\mathbb{T}}} u_2(t) \leq \max_{t\in[0,T]_{\mathbb{T}}} u_2(t) \leq r_2, \qquad \sup_{t\in[0,T]_{\mathbb{T}}} \left|u_2^{\Delta}(t)\right| \leq l_2, \\
\max_{t\in[0,T]_{\mathbb{T}}} u_3(t) < 2b \quad \textit{with} \quad \min_{t\in[0,\eta]_{\mathbb{T}}} u_3(t) < b, \qquad \sup_{t\in[0,T]_{\mathbb{T}}} \left|u_3^{\Delta}(t)\right| \leq l_2.
\end{gathered} \tag{3.4}$$

Proof. By the definition of operator A and its properties, it suffices to show that the conditions of Lemma 2.2 hold with respect to the operator A.

We first show that if the condition (i) is satisfied, then

$$A : \overline{P}(\alpha, r_2; \beta, l_2) \longrightarrow \overline{P}(\alpha, r_2; \beta, l_2). \tag{3.5}$$

In fact, if $u \in \overline{P}(\alpha, r_2; \beta, l_2)$, then

$$\alpha(u) = \max_{t\in[0,T]_{\mathbb{T}}} |u(t)| \leq r_2, \qquad \beta(u) = \sup_{t\in[0,T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right| \leq l_2, \tag{3.6}$$

so assumption (i) implies

$$f\left(t, u(t), u^{\Delta}(t)\right) \leq \min\left\{-\phi\left(-\frac{r_2}{N}\right), -\phi\left(-\frac{l_2}{S}\right)\right\}, \quad t \in [0,T]_{\mathbb{T}}. \tag{3.7}$$

On the other hand, for $u \in P$, there is $Au \in P$; then $Au(t)$ is concave in $[0,T]_{\mathbb{T}}$, and $Au^{\Delta}(t) \leq 0$ for $t \in (0,T)_{\mathbb{T}}$, so

$$\begin{aligned}
\alpha(Au) &= \max_{t\in[0,T]_{\mathbb{T}}} |(Au)(t)| = Au(0) \\
&= -\int_0^T \phi^{-1}\left(-\int_0^s h(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\Delta s \\
&\quad - \frac{1}{d}\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi^{-1}\left(-\int_0^s h(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\Delta s \\
&\leq \frac{r_2}{N}\left(\int_0^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s + \frac{1}{d}\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s\right) \\
&= r_2, \\
\beta(Au) &= \sup_{t\in[0,T]_{\mathbb{T}}} \left|(Au)^{\Delta}(t)\right| = -Au^{\Delta}(T) \\
&= -\phi^{-1}\left(-\int_0^T h(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right) \\
&\leq \frac{l_2}{S}\phi^{-1}\left(\int_0^T h(\tau)\nabla\tau\right) = l_2.
\end{aligned} \tag{3.8}$$

Therefore, (3.5) holds.

In the same way, if $u \in \overline{P}(\alpha, r_1; \beta, l_1)$, then condition (iii) implies

$$f\left(t, u(t), u^{\Delta}(t)\right) < \min\left\{-\phi\left(-\frac{r_1}{N}\right), \phi\left(-\frac{l_1}{S}\right)\right\} \quad \text{for } t \in [0,T]_{\mathbb{T}}. \tag{3.9}$$

As in the argument above, we can get that $A : \overline{P}(\alpha, r_1; \beta, l_1) \to P(\alpha, r_1; \beta, l_1)$. Thus, condition $(C2)$ of Lemma 2.2 holds.

Next we show that condition $(C1)$ in Lemma 2.2 holds. We choose $\overline{u}(t) = 2b$ for $t \in [0,T]_{\mathbb{T}}$. It is easy to see that

$$\overline{u} \in \overline{P}(\alpha, 2b; \beta, l_2; \gamma, b), \qquad \gamma(\overline{u}) = 2b > b, \tag{3.10}$$

and consequently

$$\left\{u \in \overline{P}(\alpha, 2b; \beta, l_2; \gamma, b) : \gamma(u) > b\right\} \neq \emptyset. \tag{3.11}$$

Therefore, for $u \in \overline{P}(\alpha, 2b; \beta, l_2; \gamma, b)$, there are

$$b \leq u(t) \leq 2b, \qquad \left|u^{\Delta}(t)\right| \leq l_2 \quad \text{for } t \in [0, \eta]_{\mathbb{T}}. \tag{3.12}$$

Hence in view of hypothesis (ii), we have

$$f\left(t, u(t), u^{\Delta}(t)\right) > -\phi\left(-\frac{b}{M}\right) \quad \text{for } t \in [0, \eta]_{\mathbb{T}}. \tag{3.13}$$

So by the definition of the functional γ, we see that

$$\begin{aligned}
\gamma(Au) &= \min_{[0,\eta]_{\mathbb{T}}} Au(t) = Au(\eta) \\
&= -\int_{\eta}^{T} \phi^{-1}\left(-\int_0^s h(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\Delta s \\
&\quad - \frac{1}{d}\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{T} \phi^{-1}\left(-\int_0^s h(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\Delta s \\
&\geq -\int_{\eta}^{T} \phi^{-1}\left(-\int_0^s h(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\Delta s \\
&\geq -\int_{\eta}^{T} \phi^{-1}\left(-\int_0^{\eta} h(\tau) f\left(\tau, u(\tau), u^{\Delta}(\tau)\right)\nabla\tau\right)\Delta s \\
&> \frac{b}{M}\int_{\eta}^{T} \phi^{-1}\left(\int_0^{\eta} h(\tau)\nabla\tau\right)\Delta s = \frac{b}{M} M = b.
\end{aligned} \tag{3.14}$$

Therefore, we get $\gamma(Au) > b$ for $u \in \overline{P}(\alpha, 2b; \beta, l_2; \gamma, b)$, and condition (C1) in Lemma 2.2 is fulfilled.

We finally prove that (C3) in Lemma 2.2 holds. In fact, for $u \in \overline{P}(\alpha, r_2; \beta, l_2; \gamma, b)$ with $\alpha(Au) > 2b$, we have

$$\gamma(Au) = \min_{[0,\eta]_{\mathbb{T}}} Au(t) = Au(\eta) \geq \frac{T-\eta}{T} \max_{t \in [0,T]_{\mathbb{T}}} |Au(t)| \geq \frac{1}{2}\alpha(Au) > b. \tag{3.15}$$

Thus from Lemma 2.2 and the assumption that $f(t,0,0) \not\equiv 0$ on $[0,T]_{\mathbb{T}}$, the BVP (1.1) and (1.2) has at least three positive solutions u_1, u_2, and u_3 in $\overline{P}(\alpha, r_2; \beta, l_2)$ with

$$u_1 \in P(\alpha, r_1; \beta, l_1), \qquad u_2 \in \{P(\alpha, r_2; \beta, l_2; \gamma, b) : \gamma(u) > b\},$$
$$u_3 \in \overline{P}(\alpha, r_2; \beta, l_2) \setminus \left(\overline{P}(\alpha, r_2; \beta, l_2; \gamma, b) \cup \overline{P}(\alpha, r_1; \beta, l_1)\right). \tag{3.16}$$

The fact that the functionals α and β on P satisfy an additional relation $(1/2)\alpha(u) \leq \gamma(u)$ for $u \in P$ implies that

$$\max_{t \in [0,T]_{\mathbb{T}}} u_3(t) < 2b. \tag{3.17}$$

The proof is complete. □

From Theorem 3.1, we see that, when assumptions as (i), (ii), and (iii) are imposed appropriately on f, we can establish the existence of an arbitrary odd number of positive solutions of (1.1) and (1.2).

Theorem 3.2. *Suppose that there exist constants*

$$0 < r_1 < b_1 < 2b_1 \leq r_2 < b_2 < 2b_2 \leq \cdots \leq r_n, \quad 0 < l_1 \leq l_2 \leq \cdots \leq l_{n-1}, \quad n \in \mathbb{N} \tag{3.18}$$

with

$$\frac{b_i}{M} \leq \min\left\{\frac{r_{i+1}}{N}, \frac{l_{i+1}}{S}\right\} \quad \text{for } 1 \leq i \leq n-1 \tag{3.19}$$

such that the following conditions hold:

(i) $f(t,u,v) < \min\{-\phi(-r_i/N), -\phi(-l_i/S)\}$ *for* $(t,u,v) \in [0,T]_{\mathbb{T}} \times [0,r_i] \times [-l_i, l_i], 1 \leq i \leq n$;

(ii) $f(t,u,v) > -\phi(-b_i/M)$ *for* $(t,u,v) \in [0,\eta]_{\mathbb{T}} \times [b_i, 2b_i] \times [-l_{i+1}, l_{i+1}], 1 \leq i \leq n$.

Then, BVP (1.1) *and* (1.2) *has at least* $2n-1$ *positive solutions.*

Proof. When $n = 1$, it is immediate from condition (i) that $A : \overline{P}(\alpha, r_1; \beta, l_1) \to P(\alpha, r_1; \beta, l_1)$, which means that A has at least one fixed point $u_1 \in P(\alpha, r_1; \beta, l_1)$ by the Schauder fixed point theorem. When $n = 2$, it is clear that the hypothesis of Theorem 3.1 holds. Then we can obtain at least three positive solutions u_1, u_2, and u_3. Following this way, we finish the proof by induction. The proof is complete. □

In the final part of this section, we give an example to illustrateour results.

Example 3.3. Let $\mathbb{T} = [0,1] \cup \{1 + (1/2)^{\mathbb{N}_0}\}$, where $\mathbb{N}_0$ denote nonnegative integer numbers set. If we choose $T = 2$, $\eta = 1$, $m = 4$, $a_1 = a_2 = 1/3$, $\xi_1 = 1/2$, $\xi_2 = 3/2$, and $h(t) = 1$ and consider the following BVP on time scale $\mathbb{T}$:

$$\begin{gathered} \left(\phi\left(u^{\Delta}(t)\right)\right)^{\nabla} + f\left(t, u(t), u^{\Delta}(t)\right) = 0, \quad t \in [0,2]_{\mathbb{T}}, \\ u^{\Delta}(0) = 0, \qquad u(2) = \frac{1}{3}u\left(\frac{1}{2}\right) + \frac{1}{3}u\left(\frac{3}{2}\right), \end{gathered} \tag{3.20}$$

where

$$\begin{gathered} \phi(u) = \begin{cases} u, & u \le 0; \\ u^2, & u > 0. \end{cases} \\ f(t,u,v) = \begin{cases} \dfrac{1}{120}t + \dfrac{2}{3}u^3 + \left(\dfrac{v}{100}\right)^3, & t \in [0,2]_{\mathbb{T}},\ u \in (-\infty, 3],\ v \in (-\infty, \infty); \\ \dfrac{1}{120}t + 18 + \left(\dfrac{v}{100}\right)^3, & t \in [0,2]_{\mathbb{T}},\ u \in (3, +\infty),\ v \in (-\infty, \infty), \end{cases} \end{gathered} \tag{3.21}$$

obviously the hypotheses (H1), (H2) hold and $f(t,0,0) \not\equiv 0$ on $[0,2]_{\mathbb{T}}$. By simple calculations, we have

$$\begin{gathered} S = \phi^{-1}\left(\int_0^2 1 \nabla s\right) = \sqrt{2}, \qquad M = \phi^{-1}\left(\int_0^1 1 \nabla s\right) = 1, \\ \widetilde{N} = \left(T + \frac{1}{d}\sum_{i=1}^{m-2} a_i (T - \xi_i)\right)\phi^{-1}\left(\int_0^T h(s)\nabla s\right) = 4\sqrt{2}. \end{gathered} \tag{3.22}$$

Observe that

$$\begin{aligned} N &= \int_0^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s + \frac{1}{d}\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau\right)\Delta s \\ &< \left(T + \frac{1}{d}\sum_{i=1}^{m-2} a_i (T - \xi_i)\right)\phi^{-1}\left(\int_0^T h(s)\nabla s\right) = \widetilde{N}. \end{aligned} \tag{3.23}$$

If we choose $r_2 = 180$, $b = 2$, $r_1 = 1/3$, and $l_2 = 78$, $l_1 = 10$, then $f(t,u,v)$ satisfies

$$
\begin{aligned}
f(t,u,v) \le \frac{45\sqrt{2}}{2} &= \min\left\{-\phi\left(-\frac{r_2}{\widetilde{N}}\right), -\phi\left(-\frac{l_2}{S}\right)\right\} \\
&< \min\left\{-\phi\left(-\frac{r_2}{N}\right), -\phi\left(-\frac{l_2}{S}\right)\right\}, \quad (t,u,v) \in [0,2]_{\mathbb{T}} \times [0,180] \times [-78,78]; \\
f(t,u,v) > 4 = \frac{b}{M} &\quad \text{for } (t,u,v) \in [0,1]_{\mathbb{T}} \times [2,4] \times [-78,78], \\
f(t,u,v) < \frac{\sqrt{2}}{24} &= \min\left\{-\phi\left(-\frac{r_1}{\widetilde{N}}\right), -\phi\left(-\frac{l_1}{S}\right)\right\} \\
&< \min\left\{-\phi\left(-\frac{r_1}{\widetilde{N}}\right), -\phi\left(-\frac{l_1}{S}\right)\right\}, \quad (t,u,v) \in [0,2]_{\mathbb{T}} \times \left[0, \frac{1}{3}\right] \times [-10,10].
\end{aligned}
\tag{3.24}
$$

So all conditions of Theorem 3.1 hold. Thus by Theorem 3.1, the problem (3.20) has at least three positive solutions u_1, u_2, u_3 such that

$$
\begin{gathered}
\max_{t\in[0,T]_{\mathbb{T}}} u_1(t) < \frac{1}{3}; \qquad \sup_{t\in[0,T]_{\mathbb{T}}} \left|u_1^{\Delta}(t)\right| < 10, \\
2 < \min_{t\in[0,\eta]_{\mathbb{T}}} u_2(t) \le \max_{t\in[0,T]_{\mathbb{T}}} u_2(t) \le 180; \qquad \sup_{t\in[0,T]_{\mathbb{T}}} \left|u_2^{\Delta}(t)\right| \le 78, \\
\max_{t\in[0,T]_{\mathbb{T}}} u_3(t) < 4 \quad \text{with } \min_{t\in[0,\eta]_{\mathbb{T}}} u_3(t) < 2, \qquad \sup_{t\in[0,T]_{\mathbb{T}}} \left|u_3^{\Delta}(t)\right| \le 78.
\end{gathered}
\tag{3.25}
$$

Acknowledgments

This work was supported by the NNSF of China (10801065) and NSF of Gansu Province of China (0803RJZA096).

References

[1] S. Hilger, "Analysis on measure chains-a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: an Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[3] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[4] M. A. Jones, B. Song, and D. M. Thomas, "Controlling wound healing through debridement," *Mathematical and Computer Modelling*, vol. 40, no. 9-10, pp. 1057–1064, 2004.

[5] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic systems on measure chains*, vol. 370 of *Mathematics and Its Applications*, Kluwer Academic, Boston, Mass, USA, 1996.

[6] V. Spedding, "Taming nature's numbers," *New Scientist*, vol. 7, no. 19, pp. 28–32, 2003.

[7] J. Adem and M. Moshinsky, "Self-adjointness of a certain type of vectorial boundary value problems," *Boletín de la Sociedad Matemática Mexicana*, vol. 7, pp. 1–17, 1950.

[8] Y. Zou, Q. Hu, and R. Zhang, "On numerical studies of multi-point boundary value problem and its fold bifurcation," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 527–537, 2007.

[9] V. A. Ilin and E. I. Moiseev, "Nonlocal boundary value problem of the second kind for a Sturm-CLiouville operator," *Difference Equation*, vol. 23, pp. 979–987, 1987.

[10] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Boston, Mass, USA, 1999.

[11] D. R. Anderson, R. Avery, and J. Henderson, "Existence of solutions for a one dimensional p-Laplacian on time-scales," *Journal of Difference Equations and Applications*, vol. 10, no. 10, pp. 889–896, 2004.

[12] Z. Bai, W. Ge, and Y. Wang, "Multiplicity results for some second-order four-point boundary-value problems," *Nonlinear Analysis*, vol. 60, no. 3, pp. 491–500, 2005.

[13] W. T. Li and H. R. Sun, "Positive solutions for second-order m-point boundary value problems on time scales," *Acta Mathematica Sinica*, vol. 22, no. 6, pp. 1797–1804, 2006.

[14] Y.-H. Su, W.-T. Li, and H.-R. Sun, "Triple positive pseudo-symmetric solutions of three-point BVPs for p-Laplacian dynamic equations on time scales," *Nonlinear Analysis*, vol. 68, no. 6, pp. 1442–1452, 2008.

[15] R. I. Avery, C. J. Chyan, and J. Henderson, "Twin solutions of boundary value problems for ordinary differential equations and finite difference equations," *Computers & Mathematics with Applications*, vol. 42, no. 3-5, pp. 695–704, 2001.

[16] Z. Bai and W. Ge, "Existence of three positive solutions for some second-order boundary value problems," *Computers & Mathematics with Applications*, vol. 48, no. 5-6, pp. 699–707, 2004.

[17] C. Bereanu and J. Mawhin, "Existence and multiplicity results for some nonlinear problems with singular φ -Laplacian," *Journal of Differential Equations*, vol. 243, no. 2, pp. 536–557, 2007.

[18] B. Liu, "Positive solutions of a nonlinear three-point boundary value problem," *Applied Mathematics and Computation*, vol. 132, no. 1, pp. 11–28, 2002.

[19] H. Lü, D. O'Regan, and C. Zhong, "Multiple positive solutions for the one-dimensional singular p-Laplacian," *Applied Mathematics and Computation*, vol. 133, no. 2-3, pp. 407–422, 2002.

[20] J. Wang, "The existence of positive solutions for the one-dimensional p-Laplacian," *Proceedings of the American Mathematical Society*, vol. 125, no. 8, pp. 2275–2283, 1997.

[21] J. R. L. Webb, "Positive solutions of some three point boundary value problems via fixed point index theory," *Nonlinear Analysis*, vol. 47, no. 7, pp. 4319–4332, 2001.

[22] E. Zeidler, *Nonlinear Analysis and Its Applications I: Fixed-Point Theorems*, Springer, New York, NY, USA, 1993.

[23] Z. He, "Double positive solutions of three-point boundary value problems for p-Laplacian difference equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 24, no. 2, pp. 305–315, 2005.

[24] Y. Li and L. Lu, "Existence of positive solutions of p-Laplacian difference equations," *Applied Mathematics Letters*, vol. 19, no. 10, pp. 1019–1023, 2006.

[25] Y. Liu and W. Ge, "Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 551–561, 2003.

[26] H.-R. Sun and W.-T. Li, "Existence theory for positive solutions to one-dimensional p-Laplacian boundary value problems on time scales," *Journal of Differential Equations*, vol. 240, no. 2, pp. 217–248, 2007.

[27] H.-R. Sun and W.-T. Li, "Positive solutions of p-Laplacian m-point boundary value problems on time scales," *Taiwanese Journal of Mathematics*, vol. 12, no. 1, pp. 105–127, 2008.