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Review Article

An Overview of the Lower and Upper Solutions Method with Nonlinear Boundary Value Conditions

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The aim of this paper is to point out recent and classical results related with the existence of solutions of second-order problems coupled with nonlinear boundary value conditions.

1. Introduction

The first steps in the theory of lower and upper solutions have been given by Picard in 1890 [1] for Partial Differential Equations and, three years after, in [2] for Ordinary Differential Equations. In both cases, the existence of a solution is guaranteed from a monotone iterative technique. Existence of solutions for Cauchy equations have been proved by Perron in 1915 [3]. In 1927, Müller extended Perron's results to initial value systems in [4].

Dragoni [5] introduces in 1931 the notion of the method of lower and upper solutions for ordinary differential equations with Dirichlet boundary value conditions. In particular, by assuming stronger conditions than nowadays, the author considers the second-order boundary value problem

$$u''(t) = f\left(t, u(t), u'(t)\right), \quad t \in [a,b] \equiv I, \quad u(a) = A, \quad u(b) = B, \tag{1.1}$$

for $f : I \times \mathbb{R}^2 \to \mathbb{R}$ a continuous function and $A, B \in \mathbb{R}$.

The most usual form to define a lower solution is to consider a function $\alpha \in C^2(I)$ that satisfies the inequality

$$\alpha''(t) \geq f\left(t, \alpha(t), \alpha'(t)\right), \tag{1.2}$$

together with

$$\alpha(a) \le A, \qquad \alpha(b) \le B. \tag{1.3}$$

In the same way, an upper solution is a function $\beta \in C^2(I)$ that satisfies the reversed inequalities

$$\beta''(t) \le f(t, \beta(t), \beta'(t)), \tag{1.4}$$

$$\beta(a) \ge A, \quad \beta(b) \ge B. \tag{1.5}$$

When $\alpha \le \beta$ on I, the existence of a solution of the considered problem lying between α and β is proved.

In consequence, this method allows us to ensure the existence of a solution of the considered problem lying between the lower and the upper solution, that is, we have information about the existence and location of the solutions. So the problem of finding a solution of the considered problem is replaced by that of finding two well-ordered functions that satisfy some suitable inequalities.

Following these pioneering results, there have been a large number of works in which the method has been developed for different kinds of boundary value problems, thus first-, second- and higher-order ordinary differential equations with different type of boundary conditions such as, among others, periodic, mixed, Dirichlet, or Neumann conditions, have been considered. Also partial differential equations of first and second-order, have been treated in the literature.

In these situations, we have that for the Neumann problem

$$u''(t) = f(t, u(t), u'(t)), \quad u'(a) = A, \; u'(b) = B, \tag{1.6}$$

a lower solution α is a C^2-function that satisfies (1.2) coupled with the inequalities

$$\alpha'(a) \ge A, \qquad \alpha'(b) \le B. \tag{1.7}$$

$\beta \in C^2(I)$ is an upper solution of the Neumann problem if it satisfies (1.4) and

$$\beta'(a) \le A, \quad \beta'(b) \ge B. \tag{1.8}$$

Analogously, for the periodic problem

$$u''(t) = f(t, u(t), u'(t)), \quad u(a) = u(b), \; u'(a) = u'(b), \tag{1.9}$$

a lower solution α and an upper solution β are C^2-functions that satisfy (1.2) and (1.4), respectively, together with the inequalities

$$\begin{aligned} \alpha(a) &= \alpha(b), & \alpha'(a) &\ge \alpha'(b), \\ \beta(a) &= \beta(b), & \beta'(a) &\le \beta'(b). \end{aligned} \tag{1.10}$$

In the classical books of Bernfeld and Lakshmikantham [6] and Ladde et al. [7] the classical theory of the method of lower and upper solutions and the monotone iterative technique are given. This gives the solution as the limit of a monotone sequence formed by functions that solve linear problems related to the nonlinear equations considered. We refer the reader to the classical works of Mawhin [8–11] and the surveys in this field of De Coster and Habets [12–14] in which one can found historical and bibliographical references together with recent results and open problems.

It is important to point out that to derive the existence of a solution a growth condition on the nonlinear part of the equation with respect to the dependence on the first derivative is imposed. The most usual condition is the so-called Nagumo condition that was introduced for this author [15] in 1937. This condition imposes, roughly speaking, a quadratic growth in the dependence of the derivative. The most common form of presenting it is the following.

Definition 1.1. We say that $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Nagumo condition if there is $h \in C(I)$ satisfying

$$|f(t,x,x')| \leq h(|x'|), \tag{1.11}$$

$$\int_{\lambda}^{\infty} \frac{s\,ds}{h(s)} = \infty, \tag{1.12}$$

with $\lambda(b-a) = \max\{|\beta(b)-\alpha(a)|,\ |\beta(a)-\alpha(b)|\}$.

The main importance of this condition is that it provides a priori bounds on the first derivative of all the possible solutions of the studied problem that lie between the lower and the upper solution. A careful proof of this property has been made in [6]. One can verify that in the proof the condition (1.12) can be replaced by the weaker one,

$$\int_{\lambda}^{\infty} \frac{s\,ds}{h(s)} > \max_{t\in I} \beta(t) - \min_{t\in I} \alpha(t). \tag{1.13}$$

The usual tool to derive an existence result consists in the construction of a modified problem that satisfies the two following properties.

(1) The nonlinear part of the modified equation is bounded.

(2) The nonlinear part of the modified equation coincides with the nonlinear part when the spatial variable is in $[\alpha, \beta]$.

When the Dirichlet problem (1.1) is studied, the usual truncated problem considered is

$$\begin{gathered} u''(t) = g(t,u(t),u'(t)) \equiv f(t,p(t,u(t)),q(u'(t))), \\ u(a) = A, \quad u(b) = B. \end{gathered} \tag{1.14}$$

Here

$$p(t,x) = \max\{\alpha(t), \{\min\{x, \beta(t)\}\}\}, \tag{1.15}$$

$$q(x) = \max\{-K, \{\min\{x, K\}\}\}, \tag{1.16}$$

with

$$\int_{\lambda}^{K} \frac{s\,ds}{h(s)} > \max_{t\in I} \beta(t) - \min_{t\in I} \alpha(t). \tag{1.17}$$

Notice that both p and q are continuous and bounded functions and, in consequence, if f is continuous, both properties are satisfied by g.

In the proof it is deduced that all the solutions u of the truncated problem (1.14) belong to the segment $[\alpha, \beta]$ and $|u'| \leq K$ on I. Notice that the constant K only depends on α, β and h. The existence of solutions is deduced from fixed point theory.

It is important to point out that the a priori bound is deduced for all solutions of the truncated problem. The boundary data is not used. This property is fundamental when more general situations are considered.

In 1954, Nagumo [16] constructed an example in which the existence of well-ordered lower and upper solutions is not sufficient to ensure the existence of solutions of a Dirichlet problem, that is, in general this growth condition cannot be removed for the Dirichlet case. An analogous result concerning the optimality of the Nagumo condition for periodic and Sturm-Liouville conditions has been showed recently by Habets and Pouso in [17].

In 1967, Kiguradze [18] proved that it is enough to consider a one-sided Nagumo condition (by eliminating the absolute value in (1.11)) to deduce existence results for Dirichlet problems. Similar results have been given in 1968 by Schrader [19].

Other classical assumptions that impose some growth conditions on the nonlinear part of the equation are given in 1939 by Tonelli [20]. In this situation, considering the Dirichlet problem (1.1) with $A = B = 0$, the following one-sided growth condition is assumed:

$$\begin{aligned} f(t,x,y) &\geq -\sigma_1|x| - \sigma_2|y| - \psi(t), \quad \text{if } x \geq 0, \\ f(t,x,y) &\leq \sigma_1|x| + \sigma_2|y| + \psi(t), \quad \text{if } x \leq 0, \end{aligned} \tag{1.18}$$

where $\psi \in L^1(I)$ and $\sigma_1,\ \sigma_2 \geq 0$ are sufficiently small numbers.

Different generalizations of these conditions have been developed by, among others, Epheser [21], Krasnoselskii [22], Kiguradze [23, 24], Mawhin [25], and Fabry and Habets [26].

In the case of f being a Carathéodory function, the arguments to deduce the existence result are not a direct translation from the continuous case. This is due to the fact that in the proof the properties are fulfilled at every point of the interval I. In this new situation the equalities and inequalities hold almost everywhere and, in consequence, the arguments must be directed to positive measurable sets. Thus, a suitable truncated problem is the following:

$$u''(t) = F(t, u(t), u'(t)) \equiv f\left(t, p(t, u(t)), q\left(\frac{d}{dt}p(t, u(t))\right)\right), \tag{1.19}$$

coupled with the corresponding boundary value conditions.

This truncation has been introduced by Gao and Wang in [27] for the periodic problem and improves a previous one given by Wang et al. in [28]. Notice that the function F is bounded in $L^1(I)$ and it is measurable because of the following result proved in [28, Lemma 2].

Lemma 1.2. *For any $u \in C^1(I)$, the two following properties hold:*

(a) $(d/dt)p(t,u(t))$ *exists for a.e.* $t \in I$;

(b) *If* $u,\ u_m \in C^1(I)$ *and* $u_m \overset{C^1(I)}{\longrightarrow} u$, *then* $\{(d/dt)p(t,u_m(t))\} \to (d/dt)p(t,u(t))$ *for a.e.* $t \in I$.

When a one-sided Lipschitz condition of the following type:

$$f(t,x,z) - f(t,y,z) \leq M\ (x-y), \quad \forall \alpha(t) \leq x \leq y \leq \beta(t), \tag{1.20}$$

is assumed on function f for some $M > 0$ and all $t \in I$ and $z \in \mathbb{R}$, it is possible to deduce the existence of extremal solutions in the sector $[\alpha,\beta]$ of the considered problem. By extremal solutions we mean the greatest and the smallest solutions in the set of all the solutions in $[\alpha,\beta]$. The deduction of such a property holds from an iterative technique that consists of solving related linear problems on u and using suitable maximum principles which are equivalent to the constant sign of the associated Green's function. One can find in [7] a complete development of this theory for different kinds of boundary value conditions.

It is important to note that there are many papers that have tried to get existence results under weaker assumptions on the definition of lower and upper solutions. In particular, Scorza Dragoni proves in 1938 [29], an existence result for the Dirichlet problem by assuming the existence of two C^1 functions $\alpha \leq \beta$ that satisfy

$$\begin{gathered}\alpha'(t) - \int^t f(s,\alpha(s),\alpha'(s))ds,\\ -\beta'(t) + \int^t f(s,\beta(s),\beta'(s))ds\end{gathered} \tag{1.21}$$

are nondecreasing in $t \in I$.

Kiguradze uses in [24] regular lower and upper solutions and explain that it is possible to get the same results for lower and upper solutions whose first derivatives are not absolutely continuous functions. Ponomarev considers in [30] two continuous functions $\alpha,\beta : I \to \mathbb{R}$ with right Dini derivatives $D_r\alpha$, absolute semicontinuous from below in I, and $D_r\beta$ absolute semicontinuous from above in I, that satisfy the following inequalities a.e. $t \in I$:

$$\begin{gathered}(D_r\alpha)'(t) \geq f(t,\alpha(t),D_r\alpha(t)),\\ (D_r\beta)'(t) \leq f(t,\beta(t),D_r\beta(t)).\end{gathered} \tag{1.22}$$

For further works in this direction see [31–36].

Cherpion et al. prove in [37] the existence of extremal solutions for the Dirichlet problem without assuming the condition (1.20). In fact, they consider a more general problem: the φ-laplacian equation. In this case, they define a concept of lower and upper solutions in which some kind of angles are allowed. The definitions are the following.

Definition 1.3. A function $\alpha \in C(I)$ is a lower solution of problem (1.1) (with $A = B = 0$), if $\alpha(a) \leq 0$, $\alpha(b) \leq 0$, and for any $t_0 \in (a,b)$, either $D_{-}\alpha(t_0) < D^{+}\alpha(t_0)$, or there exists an open interval $I_0 \subset I$ such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and, for a.e. $t \in I_0$,

$$\alpha''(t) \leq f\left(t, \alpha(t), \alpha'(t)\right). \tag{1.23}$$

Definition 1.4. A function $\beta \in C(I)$ is a lower solution of problem (1.1) (with $A = B = 0$), if $\beta(a) \geq 0$, $\beta(b) \geq 0$, and for any $t_0 \in (a,b)$, either $D^{-}\beta(t_0) > D_{+}\beta(t_0)$, or there exists an open interval $I_0 \subset I$ such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and, for a.e. $t \in I_0$,

$$\beta''(t) \leq f\left(t, \beta(t), \beta'(t)\right). \tag{1.24}$$

Here D^{+}, D_{+}, D^{-}, and D_{-} denote the usual Dini derivatives.

By means of a sophisticated argument, the authors construct a sequence of upper solutions that converges uniformly to the function defined at each point $t \in I$ as the minimum value attained by all the solutions of problem (1.1) in $[\alpha, \beta]$ at this point. Passing to the limit, they conclude that such function is a solution too. The construction of these upper solutions is valid only in the case that corners are allowed in the definition. The same idea is valid to get a maximal solution.

Similar results are deduced for the periodic boundary conditions in [12]. In this case the arguments follow from the finite intersection property of the set of solutions (see [38, 39]).

2. Two-Point Nonlinear Boundary Value Conditions

Two point nonlinear boundary conditions are considered with the aim of covering more complicated situations as, for instance, $u(b) = u^3(a)$ or $u'(b) - u'(a) = \arctan(u(a) + u(b))$.

In general, the framework of linear boundary conditions cannot be directly translated to this new situation. For instance, as we have noticed in the previous section, to ensure existence results for linear boundary conditions, we make use of the fixed point theory. So, in the case of a Dirichlet problem, the set of solutions of (1.1) coincide with the set of the fixed points of the operator $T : C^1(I) \to C^1(I)$, defined by

$$Tu(t) = \int_a^b G(t,s) f\left(s, u(s), u'(s)\right) ds + \frac{1}{b-a}\ (A(b-t) + B\ (t-a)), \tag{2.1}$$

where

$$G(t,s) = \frac{1}{b-a} \begin{cases} (a-s)(b-t), & \text{if } a \leq s \leq t \leq b, \\ (a-t)(b-s), & \text{if } a \leq t < s \leq b. \end{cases} \tag{2.2}$$

is the Green's function related to the linear problem

$$u''(t) = 0, \quad \forall t \in I, \quad u(a) = u(b) = 0. \tag{2.3}$$

It is obvious that when nonlinear boundary value conditions are treated, the operator whose fixed points are the solutions of the considered problem must be modified.

Moreover the truncations that have to be made in the nonlinear part of the problem (1.14), for the continuous case, and in (1.19), for the Carathéodory one, must be extended to the nonlinear boundary conditions. This new truncation on the boundary conditions must satisfy similar properties to the ones of the nonlinear part of the equation, that is,

(1) the modified nonlinear boundary value conditions must be bounded,

(2) the modified nonlinear boundary value conditions coincide with the nonmodified ones in $[\alpha,\beta]$.

So, to deduce existence results for this new situation, it is necessary to make use of the qualitative properties of continuity and monotonicity of the functions that define the nonlinear boundary value conditions.

Perhaps the first work that considers nonlinear boundary value conditions coupled with lower and upper solutions is due to Bebernes and Fraker [40] in 1971. In this work, the equation $u'' = f(t,u,u')$ coupled with the boundary conditions $(0,u(0),u'(0)) \in S_1$ and $(1,u(1),u'(1)) \in S_2$ is considered. Here, S_1 is compact and connected and S_2 is closed and connected. Under some additional conditions on the two sets, that include as a particular case $u(0) = 0$; $u'(1) = -L_1\, u(1)$, the existence result is deduced under the assumption that a pair of well-ordered lower and upper solutions exist and a Nagumo condition is satisfied.

Later Bernfeld and Lakshmikantham [6] studied the boundary conditions $g(u(a),u'(a)) = 0 = z(u(a),u'(a))$; with g and z monotone nonincreasing in the second variable.

Erbe considers in [41] the three types of boundary value conditions:

$$\begin{gathered} g\big(u(a),u(b),u'(a),u'(b)\big) = 0, \qquad z(u(a)) = u(b), \\ x\big(u(a),u'(a)\big) = 0 = y\big(u(a),u(b),u'(a),u'(b)\big), \\ r\big(u(b),u'(b)\big) = 0 = w\big(u(a),u(b),u'(a),u'(b)\big). \end{gathered} \tag{2.4}$$

Here functions g,z,x,y,r and w satisfy suitable monotonicity conditions. Such monotonicity properties include, as particular cases, the periodic problem in the first situation and the separated conditions in the second and third cases.

The proofs follow from the study of the Dirichlet problem (1.1) with $A \in [\alpha(a),\beta(a)]$ and $B \in [\alpha(b),\beta(b)]$. From the monotonicity assumptions it is proved, by a similar argument to the shooting method, that there is at least a pair A, B for which the boundary conditions hold.

Mawhin studies in [11] the nonlinear separated boundary conditions

$$g_a\big(u(a),u'(a)\big) = g_b\big(u(b),u'(b)\big) = 0, \tag{2.5}$$

with $g_a(x,\cdot)$ and $g_b(y,\cdot)$ two nondecreasing functions in $\mathbb{R}$ for all $x \in [\alpha(a),\beta(a)]$ and $y \in [\alpha(b),\beta(b)]$.

In this case, he constructs the modified problem

$$\begin{gathered}
u''(t) = f\bigl(t, p(t, u(t)), u'(t)\bigr) + h\bigl(\left|u'(t)\right|\bigr)\ \bigl(u(t) - p(t, u(t))\bigr), \\
u(a) = g_a\bigl(p(a, u(a)), u'(a)\bigr) + p(a, u(a)), \\
u(b) = -g_b\bigl(p(b, u(b)), u'(b)\bigr) + p(b, u(b)),
\end{gathered} \tag{2.6}$$

with h defined in (1.12) and p in (1.15).

Virzhbitskiĭ and Sadyrbaev consider in [42] the conditions

$$\bigl(u(0), u'(0)\bigr) \in \Gamma, \qquad g\bigl(u(0), u'(0), u(1), u'(1)\bigr) = 0, \tag{2.7}$$

where Γ is a continuously parametrized curve in $\mathbb{R}^2$ and g is a continuous function. The proof is based on reducing the problem to another one with divided boundary conditions and applying the Bol'-Brauer theorem.

Fabry and Habets treat in [26] the two types of boundary value conditions

$$g\bigl(u(a), u'(a), u'(b)\bigr) = 0, \qquad z(u(a)) = u(b), \tag{2.8}$$

$$w\bigl(u(a), u'(a), u(b), u'(b)\bigr) = 0 = r\bigl(u(a), u(b), u'(a), u'(b)\bigr), \tag{2.9}$$

with g, z, w, and r monotone functions in some of their variables.

The first case covers the periodic case and the second one separated boundary conditions.

In the proofs, a more general definition of lower and upper solutions is used. In particular, they replace the definitions (1.2) and (1.4) by the following ones.

$\alpha, \beta : I \to \mathbb{R}$ are continuous functions with right Dini derivatives $D^+\alpha$ and $D^+\beta$ continuous from the right and left Dini derivatives $D^-\alpha$ and $D^-\beta$ such that

(1) for all $t \in I$ it is satisfied that $\alpha(t) \le \beta(t)$, $D^-\alpha(t) \le D^+\alpha(t)$ and $D^-\beta(t) \ge D^+\beta(t)$;

(2) the functions

$$\begin{gathered}
D^+\alpha(t) - \int^t f(s, \alpha(s), D^+\alpha(s))ds, \\
-D^+\beta(t) + \int^t f\bigl(s, \beta(s), D^+\beta(s)\bigr)ds
\end{gathered} \tag{2.10}$$

are nondecreasing in t.

It is clear that if α and β are C^2-functions, this definition reduces to (1.2) and (1.4). Moreover, they assume a more general condition than the Nagumo one.

To deduce existence results for (2.8) they consider a variant of the truncated problem (1.14) (by adding the term tanh $(u(t) - p(t, u(t)))$ coupled with the following nonconstant

Dirichlet boundary conditions:

$$u(a) = p\big(a, u(a) + g\big(u(a), u'(a), u'(b)\big)\big), \qquad u(b) = z(u(a)), \tag{2.11}$$

with p defined in (1.15).

When the conditions (2.9) are studied, the authors consider

$$\begin{aligned} u(a) &= p\big(a, u(a) + w\big(u(a), u'(a), u(b), u'(b)\big)\big), \\ u(b) &= p\big(b, u(a) + r\big(u(a), u'(a), u(b), u'(b)\big)\big). \end{aligned} \tag{2.12}$$

The proofs follow from oscillation theory and boundedness of the boundary conditions.

In [43], by using degree theory, Rachůnková proves the existence of at least two different solutions with boundary conditions

$$g_1\big(u(a), u'(a)\big) = 0 = g_2\big(u(b), u'(b)\big). \tag{2.13}$$

Here, g_1 and g_2 satisfy some suitable monotonicity conditions that cover as a particular case the separated ones.

In all of the previous works, f is considered a continuous function.

For f being a Carathéodory function Sadyrbaev studies in [44, 45] the first-order system $u' = f(t, u, v)$, $v' = g(t, u, v)$, coupled with boundary value conditions $(u(i), v(i)) \in S_i,\ i = 0, 1$, with $S_1, S_2 \subset \mathbb{R}^2$ some suitable sets.

Lepin et al. generalize in [31, 34, 35] some of the results proved by Erbe in [41].

Adje generalizes in [46] the results obtained by Fabry and Habets in [26] for problem (2.8) and proves the existence of solution by considering the boundary value conditions

$$L_1\big(u(a), u(b), u'(a), u'(b)\big) = 0 = L_2(u(a), u(b)), \tag{2.14}$$

and f is a L^p-Carathéodory function.

Franco and O'Regan, by avoiding some monotonicity assumptions on the boundary data, introduce in [47] a new definition of coupled lower and upper solutions for the boundary value conditions (2.9). In this case, the definition of such functions concerns both of the functions together. Under this definition they cover, under the same notation, periodic, antiperiodic, and Dirichlet boundary value conditions. Moreover they introduce a new concept of Nagumo condition as follows.

Definition 2.1. One says that f satisfies a Nagumo condition relative to the interval $[\alpha, \beta]$, where α is a lower solution and β is an upper solution if for

$$r_0 = \frac{\max\{|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|\}}{b - a} \tag{2.15}$$

there exists a constant M such that

$$M > \max\left\{ r_0, \sup_{t\in I}\left|\alpha'(t)\right|, \sup_{t\in I}\left|\beta'(t)\right| \right\} \tag{2.16}$$

and a continuous function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\begin{gathered}\left|f(t,u,v)\right| \le \psi(|v|), \quad t \in I, \quad \alpha(t) \le u \le \beta(t), \quad v \in \mathbb{R}, \\ \int_{r_0}^{M} \frac{1}{\psi(s)} ds > b - a.\end{gathered} \tag{2.17}$$

3. φ-Laplacian Problems and Functional Boundary Conditions

A more general framework of the second-order general equation $u''(t) = f(t, u(t), u'(t))$ is given by the so-called p-laplacian equation. This kind of problems follow the expression

$$\left(\varphi_p\left(u'(t)\right)\right)' = f\left(t, u(t), u'(t)\right), \quad t \in I, \tag{3.1}$$

where $\varphi_p(x) = x|x|^{p-2}$ for some $p > 1$. This type of equations appears in the study of nonNewtonian fluid mechanics [48, 49].

As far as the author is aware, the first reference in which this problem has been studied in combination with the method of lower and upper solutions is due to De Coster in [50], who considers (3.1) (without dependence on u') coupled with Dirichlet conditions. Moreover, she treat, a more general operator φ that includes, as a particular case, the p-laplacian operator. To be concise, operator φ conserves the two main qualitative properties of operator φ_p:

(1) φ is a strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, such that $\varphi(\mathbb{R}) = \mathbb{R}$;

(2) $\varphi(0) = 0$.

As consequence, after this work authors considered the φ-laplacian equation

$$\left(\varphi\left(u'(t)\right)\right)' = f\left(t, u(t), u'(t)\right), \quad t \in I, \tag{3.2}$$

with φ an operator that satisfies the above mentioned properties.

After this paper, the method of lower and upper solutions has been applied to φ-laplacian problems with Mixed boundary conditions in [51] and for Neumann and periodic boundary conditions in [52].

In this case, the definition of a lower and an upper solution, for f being a Carathéodory function, is the direct translation to this case for φ the identity.

Definition 3.1. A function $\alpha \in C^1(I)$ is said to be a lower solution for (3.2) if $\varphi(\alpha') \in W^{1,1}(I)$ and

$$\left(\varphi\left(\alpha'(t)\right)\right)' \ge f\left(t, \alpha(t), \alpha'(t)\right), \quad \text{for a.e. } t \in I. \tag{3.3}$$

A function $\beta \in C^1(I)$ is an upper solution for (3.2) if $\varphi(\beta') \in W^{1,1}(I)$ and

$$\left(\varphi(\beta'(t))\right)' \le f\left(t, \beta(t), \beta'(t)\right), \quad \text{for a.e. } t \in I. \tag{3.4}$$

We will say that $u \in C^1(I)$ is a solution of (3.2) if it is both a lower and an upper solution.

Of course, some additional assumptions are needed depending on the considered boundary conditions, that is, we assume (1.3) and (1.5) for the Dirichlet case, (1.7) and (1.8) for the Neumann case, or (1.10) for the periodic case.

The definition of a Nagumo condition, see [52], is the direct adaptation of the one used by Adje in [46]. Note that such condition does not depend on the boundary data of the problem.

Definition 3.2. One says that the function f satisfies a Nagumo condition with respect to continuous functions α and β, with $\alpha \le \beta$, if there exist $k \in L^p(I)$, $1 \le p \le \infty$, and a continuous function $\theta : [0,\infty) \to [0,\infty)$, such that

$$\left|f(t,u,v)\right| \le k(t)\theta(|v|), \quad \text{on } \Omega, \tag{3.5}$$

where $\Omega = \{(t,u,v) \in \mathbb{R}^3 : t \in I,\ \alpha(t) \le u \le \beta(t),\ v \in \mathbb{R}\}$.

Furthermore

$$\begin{gathered}
\int_{-\infty}^{\varphi(-\nu)} \frac{\left|\varphi^{-1}(u)\right|^{(p-1)/p}}{\theta\left(\left|\varphi^{-1}(u)\right|\right)} du > \mu^{(p-1)/p} \|k\|_p \\
\int_{\varphi(\nu)}^{\infty} \frac{\left|\varphi^{-1}(u)\right|^{(p-1)/p}}{\theta\left(\left|\varphi^{-1}(u)\right|\right)} du > \mu^{(p-1)/p} \|k\|_p,
\end{gathered} \tag{3.6}$$

where

$$\begin{gathered}
\mu = \max_{t\in I} \beta(t) - \min_{t\in I} \alpha(t), \\
\nu = \frac{\max\left\{\left|\alpha(a) - \beta(b)\right|, \left|\alpha(b) - \beta(a)\right|\right\}}{b-a}, \\
\|k\|_p = \begin{cases} \sup\limits_{t\in I} |k(t)|, & \text{if } p = \infty, \\ \left[\displaystyle\int_a^b |k(t)|^p dt\right]^{1/p}, & \text{if } 1 \le p < \infty. \end{cases}
\end{gathered} \tag{3.7}$$

If $p = \infty$, we replace $(p-1)/p = 1$.

It is obvious that, to construct an operator whose fixed points coincide with the solutions of the considered problem, there is no possibility of constructing a Green's function, so no operator analogous to (2.1) can be given. In this case, one can see [52] that the solutions of (3.2) coupled with Dirichlet boundary conditions are the fixed points of the operator $L : C^1(I) \to C^1(I)$, defined as

$$Lu(t) = A + \int_a^t \varphi^{-1}\left(\tau + \int_a^r f(s, u(s), u'(s))ds\right)dr, \tag{3.8}$$

where τ is the unique solution of the equation

$$B - A = \int_a^b \varphi^{-1}\left(\tau + \int_a^r f(s, u(s), u'(s))ds\right)dr. \tag{3.9}$$

In the previous mentioned papers, the existence of solutions lying between a pair of well-ordered lower and upper solutions was shown. In [37], the existence of extremal solutions for the Dirichlet problem is proved. As we have noted earlier, in that paper, a new definition of lower and upper solutions with corners is used that allows one to construct a sequence of upper solutions over the function minimum of the solutions in $[\alpha, \beta]$.

Two point nonlinear boundary value conditions have been treated in [53]. In this case $u(b) = z(u(a))$ and $g(u(a), u'(a), u'(b)) = 0$ have been studied. The monotonicity assumptions on z and g cover the periodic conditions. The proof follows from a truncated problem on f as in the nonlinear part of (1.19) and a truncated boundary conditions as in (2.11).

Lepin et al. study in [54] the more general equation $(\varphi(t, u, u'))' = f(t, u, u')$ together with the two point nonlinear boundary conditions

$$H_1(u(a), u(b), u'(a), u'(b)) = h_1, \qquad H_2(u(a), u(b), u'(a), u'(b)) = h_2. \tag{3.10}$$

The existence results are on the basis of the monotone properties of functions H_1 and H_2.

Functional boundary conditions allow us to consider dependence on some intermediate points of the interval of definition. This is the case of the multipoint boundary conditions:

$$u(a) = \sum_{i=1}^{m} a_i u(\tau_i), \qquad u(b) = \sum_{i=1}^{n} b_i u(\eta_i), \tag{3.11}$$

where $a < \tau_1 < \tau_2 < \cdots < \tau_m < b$, $a < \eta_1 < \eta_2 < \cdots < \eta_n < b$, and a_i, $b_j \in \mathbb{R}$ have the same sign for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Many other type of boundary conditions can be treated, is the case, for instance, of the following:

$$\begin{gathered} u(a) = \max_{t \in J} u(t), \quad J \subset I, \\ u(b) = \int_J u^l(s)ds, \quad l \in \mathbb{N}. \end{gathered} \tag{3.12}$$

In this situation, it is necessary to consider functions that are not only defined at the extremes of the interval, but also in its interior. Such kind of problems have been treated in [55]. There the authors consider nonlinear functional boundary conditions of the form

$$L_1\left(u(a), u(b), u'(a), u'(b), u\right) = 0 = L_2(u(a), u(b)), \tag{3.13}$$

with L_1 and L_2 a continuous functions that satisfy certain monotonicity conditions which include, as particular cases, the periodic ones.

More precisely, it is considered the equation

$$\left[\varphi(u'(t))\right]' = r\left(u'(t)\right) f\left(t, u(t), u'(t)\right), \quad \text{for a.e. } t \in I, \tag{3.14}$$

with $r : \mathbb{R} \to (0, \infty)$ and $1/r$ locally bounded (possibly discontinuous) function and $r \circ \varphi^{-1}$ measurable.

The discontinuity on u' can be eliminated by the use of the transformation $\tilde{\varphi} = Q \circ \varphi$, where Q is given by

$$Q(x) = \int_0^x \frac{ds}{r\left(\varphi^{-1}(s)\right)}. \tag{3.15}$$

In this case, the studied problem is translated to the usual φ-laplacian

$$\left(\tilde{\varphi}(u')\right)' = f\left(t, u, u'\right), \quad \text{for a.e. } t \in I, \tag{3.16}$$

and the existence of extremal solutions lying between a pair of well-ordered lower and upper solutions is obtained. The results follows from an appropriate truncated problem and the extension to this case of the arguments used in [37] to get the extremal solutions.

4. General Functional Equations

In this last section, we mention some kind of problems that model different real phenomena that, as we will see, can be presented under the same formulation.

(1) We consider the classical *self-adjoint equation*

$$\left(r(t)u'\right)'(t) = f\left(t, u(t), u'(t)\right), \quad \text{a.e. } t \in I. \tag{4.1}$$

with $r : \mathbb{R} \to (0, \infty)$ such that $r \in L^1(\mathbb{R})$.

We can formulate the previous equation in the form

$$\frac{d}{d\,t}\varphi\left(t, u'(t)\right) = f\left(t, u(t), u'(t)\right), \quad \text{a.e. } t \in I, \tag{4.2}$$

with

$$\varphi(t,x) = r(t)\ x. \tag{4.3}$$

(2) We refer to the usual *diffusion equation*

$$(k(u)u')'(t) = f(t,u(t),u'(t)), \quad \text{a.e. } t \in I, \tag{4.4}$$

with $k : \mathbb{R} \to (0,\infty)$ such that $k \in C(\mathbb{R})$ and $(k \circ u)\ u' \in W^{1,1}(I)$.

By defining

$$\varphi(x,y) = k(x)\ y, \tag{4.5}$$

it is possible to rewrite this equation as

$$\frac{d}{dt}\varphi(u(t),u'(t)) = f(t,u(t),u'(t)), \quad \text{a.e. } t \in I. \tag{4.6}$$

(3) We study a higher-order differential equation, for instance, the following third-order problem:

$$\begin{gathered} u'''(t) = f(t,u(t),u'(t),u''(t)), \quad \text{a.e. } t \in I, \\ u(a) = 0, \\ u'(a) = u'(b) = 0. \end{gathered} \tag{4.7}$$

By means of the change of variables

$$u(t) = \int_a^t v(s)ds, \tag{4.8}$$

we arrive at the following equivalent second-order Dirichlet functional equation:

$$\begin{gathered} v''(t) = f\left(t, \int_a^t v(s)ds, v(t), v'(t)\right) \equiv g(t,v,v(t),v'(t)), \quad \text{a.e. } t \in I, \\ v(a) = v(b) = 0. \end{gathered} \tag{4.9}$$

(4) Consider the following third-order problem:

$$\begin{gathered} u'''(t) = f(t, u'(t), u''(t)), \quad \text{a.e. } t \in I, \\ u(a) = u(b) = 0, \\ u'(a) = 0. \end{gathered} \tag{4.10}$$

Using the same change of variable as above, we arrive at the following second-order differential equation with functional boundary conditions:

$$\begin{gathered} v''(t) = f(t, v(t), v'(t)) \quad \text{a.e. } t \in I, \\ v(a) = 0, \\ \int_a^b v(s)\, ds = 0. \end{gathered} \tag{4.11}$$

In order to include under the same formulation all the previous problems, the following equation is considered in [56]:

$$\frac{d}{dt}\varphi(t, u, u(t), u'(t)) = f(t, u, u(t), u'(t)), \quad \text{a.e. } t \in I, \tag{4.12}$$

coupled with the nonlinear functional boundary conditions (3.13).

The definitions of the lower and upper solutions cover all the usual cases but the Nagumo condition does not generalize the case of φ depending only on the first derivative given in Definition 3.2. This gap has been covered in the definition given in [57], where (4.12) is considered coupled with the boundary conditions

$$L_1(u(a), u'(a), u) = 0 = L_2(u(b), u'(b), u), \tag{4.13}$$

covering in this case the Sturm-Liouville and the multipoint boundary conditions as particular cases.

5. Final Remarks

It is important to note that in some of the previous results some kind of discontinuities on the spatial variable are assumed. In this case, some techniques developed by Heikkilä and Lakshmikantham in [58] are used.

There is large bibliography on papers related with lower and upper solutions with nonlinear boundary value conditions for first- and higher-order equations.

Problems with impulses, difference equations, and partial differential equations have been studied under this point of view for an important number of researchers.

Some theories as the Thompson's notion of compatibility [59] or the Frigon's tube-solutions [60] give some generalizations of the concept of lower and upper solutions that ensure the existence of solutions of nonlinear boundary problems under weaker assumptions.

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