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Research Article

Discontinuous Parabolic Problems with a Nonlocal Initial Condition

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We study parabolic differential equations with a discontinuous nonlinearity and subjected to a nonlocal initial condition. We are concerned with the existence of solutions in the weak sense. Our technique is based on the Green's function, integral representation of solutions, the method of upper and lower solutions, and fixed point theorems for multivalued operators.

1. Introduction

Let Ω be a an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. Let $Q_T = \Omega \times (0,T)$ and $\Gamma_T = \partial\Omega \times [0,T]$ where T is a positive real number. Then Γ_T is smooth and any point on Γ_T satisfies the inside (and outside) strong sphere property (see [1]). For $u : Q_T \to \mathbb{R}$ we denote its partial derivatives in the distributional sense (when they exist) by $D_t u = \partial u/\partial t$, $D_i u = \partial u/\partial x_i, D_i D_j u = \partial^2 u/\partial x_i \partial x_j,\ i,j = 1,\dots,N$.

In this paper, we study the following parabolic differential equation with a nonlocal initial condition

$$\begin{gathered}
D_t u + Lu = f(x,t,u), \quad (x,t) \in Q_T, \\
u(x,t) = 0, \quad (x,t) \in \Gamma_T, \\
u(x,0) = \int_0^T k(x,t,u(x,t))\,dt, \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is not necessarily continuous, but is such that for every fixed $u \in \mathbb{R}$ the function $(x,t) \to f(x,t,u)$ is measurable and $u \to f(x,t,u)$ is of bounded variations over compact interval in $\mathbb{R}$ and nondecreasing, and $k : Q_T \times \mathbb{R} \to \mathbb{R}$ is continuous; L is a strongly elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} D_i\big(a_{ij}(x,t)D_j u\big) + c(x,t)u. \tag{1.2}$$

Discontinuous parabolic problems have been studied by many authors, see for instance [2–5]. Parabolic problems with integral conditions appear in the modeling of concrete problems, such as heat conduction [6–10] and in thermoelasticity [11].

In order to investigate problem (1.1), we introduce some notations, function spaces, and notions from set-valued analysis.

Let $C(\overline{Q_T})$ denote the Banach space of all continuous functions $u : Q_T \to \mathbb{R}$, equipped with the norm $|u|_0 = \max_{(x,t)\in Q_T}|u(x,t)|$. Let $C^{2,1}(Q_T) = \{u : Q_T \to \mathbb{R};\ u(\cdot,t) \in C^2(\Omega)$ for each $t \in (0,T)$ and $u(x,\cdot) \in C^1(0,T)$ for each $x \in \Omega\}$. For $1 < p < +\infty$, we say that $u : Q_T \to \mathbb{R}$ is in $L^p(Q_T)$ if u is measurable and $\int_{Q_T} |u(x,t)|^p dx\,dt < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_{Q_T} |u(x,t)|^p dx\,dt\right)^{1/p}. \tag{1.3}$$

Let $J = [0,T]$ and let $H^1(\Omega)$ denote the Sobolev space of functions $z \in L^2(\Omega)$ having first generalized derivatives in $L^2(\Omega)$ and let $(H^1(\Omega))^*$ be its corresponding dual space. Then $H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))^*$ and they form an evolution triple with all embeddings being continuous, dense, and compact (see [2, 12]). The Bochner space $W = W^{2,2}(J, H^1(\Omega), (H^1(\Omega))^*)$ (see [13]) is the set of functions $u \in L^2(J; H_0^1(\Omega))$ with generalized derivative $du/dt \in L^2(J; (H^1(\Omega))^*)$. For $z \in W$, we define its norm by

$$\|z\|_W = \|z\|_{L^2(J;H_0^1(\Omega))} + \left\|\frac{dz}{dt}\right\|_{L^2(J;(H^1(\Omega))^*)}. \tag{1.4}$$

Then $(W, \|\cdot\|_W)$ is a separable reflexive Banach space. The embedding of $W_0 = W^{2,2}(J, H_0^1(\Omega), (H^1(\Omega))^*)$ into $C(J; L^2(\Omega))$ is continuous and the embedding $W_0 \subset L^2(Q_T)$ is compact.

Now, we introduce some facts from set-valued analysis. For complete details, we refer the reader to the following books. [14–16]. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. We will denote the set of all subsets, of X having property ℓ by $P_\ell(X)$. For instance, $P_n(X)$ denotes the set of all nonempty subsets of X; $V \in P_{\text{cl}}(X)$ means V closed in X; when $\ell = b$ we have the bounded subsets of X, $\ell = \text{cv}$ for convex subsets, $\ell = \text{cp}$ for compact subsets and $\ell = \text{cp}, \text{cv}$ for compact and convex subsets. The domain of a multivalued map $R : X \to P_n(Y)$ is the set $\operatorname{dom} R = \{z \in X;\ R(z) \neq \emptyset\}$. R is convex (closed) valued if $R(z)$ is convex (closed) for each $z \in X$. R is bounded on bounded sets if $R(A) = \bigcup_{z\in A} R(z)$ is bounded in Y for all $A \in P_b(X)$ (i.e., $\sup_{z\in A}\{\sup\{\|y\|_Y; y \in R(z)\}\} < \infty$). R is called upper semicontinuous (u.s.c.) on X if for each $z \in X$ the set $R(z) \in P_{\text{cl}}(Y)$ is nonempty, and for each open subset Λ

of Y containing $R(z)$, there exists an open neighborhood Π of z such that $R(\Pi) \subset \Lambda$. In terms of sequences, R is usc if for each sequence $(z_n) \subset X$, $z_n \to z_0$, and B is a closed subset of Y such that $R(z_n) \cap B \neq \emptyset$ then $R(z_0) \cap B \neq \emptyset$.

The set-valued map R is called completely continuous if $R(A)$ is relatively compact in Y for every $A \in P_b(X)$. If R is completely continuous with nonempty compact values, then R is usc if and only if R has a closed graph (i.e., $z_n \to z$, $w_n \to w$, $w_n \in R(z_n) \Rightarrow w \in R(z)$). When $X \subset Y$ then R has a fixed point if there exists $z \in X$ such that $z \in R(z)$. A multivalued map $R : J \to P_{\mathrm{cl}}(X)$ is called measurable if for every $x \in X$, the function $\theta : J \to \mathbb{R}$ defined by $\theta(t) = \operatorname{dist}(x, R(t)) = \inf\{|x - z|; z \in R(t)\}$ is measurable. $\|R(z)\|_Y$ denotes $\sup\{\|y\|_Y; y \in R(z)\}$. The Kuratowski measure of noncompactness (see [15, page 113]) of $A \in P_b(X)$ is defined by

$$\alpha(A) = \inf\left\{\epsilon > 0;\ A \subset \bigcup_{i=1}^{m} A_i,\ \operatorname{diam}(A_i) \leq \epsilon\right\}. \tag{1.5}$$

The Kuratowski measure of noncompactness satisfies the following properties.

(i) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact;

(ii) $\alpha(A) = \alpha(\overline{A})$;

(iii) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;

(iv) $\alpha(cA) = |c|\alpha(A)$, $c \in \mathbb{R}$;

(v) $\alpha(\operatorname{conv} A) = \alpha(A)$, where $\operatorname{conv}(A)$ denotes the convex hull of A.

Definition 1.1 (see [17]). A function $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is called N-measurable on $\mathbb{R}$ if for every measurable function $u : Q_T \to \mathbb{R}$ the function $(x,t) \to f(x,t,u(x,t))$ is measurable.

Examples of N-measurable functions are Carathéodory functions, Baire measurable functions.

Let $g(x,t,u) = \liminf_{z \to u} f(x,t,u)$ and $h(x,t,u) = \limsup_{z \to u} f(x,t,u)$. Then (see [17, Proposition 1]) the function $u \to g(x,t,u)$ is lower semicontinuous, that is, for every $(x,t) \in Q_T$ the set $\{u : g(x,t,u) > r\}$ is open for any $r \in \mathbb{R}$, and the function $u \to h(x,t,u)$ is upper semicontinuous, that is, for every $(x,t) \in Q_T$, the set $\{u : h(x,t,u) < r\}$ is open for any $r \in \mathbb{R}$. Moreover, the functions $u \to g(x,t,u)$ and $u \to h(x,t,u)$ are nondecreasing.

Definition 1.2. The multivalued function F defined by $F(x,t,u) = [g(x,t,u), h(x,t,u)]$ for all $(x,t) \in Q_T$ is called N-measurable on $\mathbb{R}$ if both functions g and h are N-measurable on $\mathbb{R}$.

Definition 1.3. The operator $N_F : L^2(Q_T) \to L^2(Q_T)$ defined by

$$N_F(u) = \left\{q \in L^2(Q_T);\ g(x,t,u) \leq q(x,t) \leq h(x,t,u), (x,t) \in Q_T\right\} \tag{1.6}$$

is called the Nemitskii operator of the multifunction F.

Since F is an N-measurable and upper semicontinuous multivalued function with compact and convex values, we have the following properties for the operator N_F (see [17, Corollary 1.1]).

Lemma 1.4. *N_F is N-measurable, compact and convex-valued, upper semicontinuous and maps bounded sets into precompact sets.*

We will consider solutions of problem (1.1) as solutions of the following parabolic problem with multivalued right-hand side:

$$\begin{gathered} D_t u + Lu \in F(x,t,u), \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ u(x,0) = \int_0^T k(x,t,u(x,t))dt, \quad x \in \Omega, \end{gathered} \tag{1.7}$$

where $F(x,t,u) = [g(x,t,u), h(x,t,u)]$ for all $(x,t) \in Q_T$. As pointed out in [15, Example 1.3 page 5], this is the most general upper semicontinuous set-valued map with compact and convex values in $\mathbb{R}$.

Theorem 1.5 (see [18]). *Let E be a Banach space and $\Upsilon : E \to P_{cp,cv}(E)$ a condensing map. If the set $S := \{z \in E; \lambda z \in \Upsilon(z)$ for some $\lambda > 1\}$ is bounded, then Υ has a fixed point.*

We remark that a compact map is the simplest example of a condensing map.

2. The Linear Problem

We will assume throughout this paper that the functions $a_{ij}, c : Q_T \longrightarrow \mathbb{R}$ are Hölder continuous, $a_{ij} = a_{ji}$ and moreover, there exist positive numbers λ_0, and λ_1 such that

$$\lambda_0 \|\xi\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \lambda_1 \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^N, \ \forall (x,t) \in Q_T. \tag{2.1}$$

Given a continuous function $u_0 : \Omega \to \mathbb{R}$, the linear parabolic problem

$$\begin{gathered} D_t u + Lu = f(x,t) \quad (x,t) \in Q_T, \\ u(x,t) = 0 \quad (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{gathered} \tag{2.2}$$

is well known and completely solved (see the books [1, 19, 20]).

The linear homogeneous problem

$$\begin{gathered} D_t u + Lu = 0, \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ u(x,0) = 0, \quad x \in \Omega \end{gathered} \tag{2.3}$$

has only the trivial solution. There exists a unique function, $G(x,t;y,s)$, called Green's function corresponding to the linear homogeneous problem. This function satisfies the following (see [1, 20]):

(i) $D_t G + LG = \delta(t-s)\delta(x-y), s < t, x, y \in \Omega$;

(ii) $G(x,t;y,s) = 0, s > t,\ x, y \in \Omega$;

(iii) $G(x,t;y,s) = 0, (x,t), (y,s) \in \Gamma_T$;

(iv) $G(x,t;y,s) > 0$ for $(x,t) \in Q_T$;

(v) $G,\ D_t G,\ D_i G$, and $D_i D_j G$ are continuous functions of $(x,t), (y,s) \in Q_T,\ t-s > 0$;

(vi) $|G(x,t;y,s)| \leq C(t-s)^{-N/2}\exp(-a\|x-y\|^2_{\mathbb{R}^n}/(t-s))$, for some positive constants C, a (see [19]);

(vii) for any Hölder continuous function $f\colon Q_T \to \mathbb{R}$, the function $u : Q_T \to \mathbb{R}$, given for $(x,t) \in Q_T$ by $u(x,t) = \int_\Omega G(x,t;y,0)\ \ u_0(y)dy + \int_0^t \int_\Omega G(x,t;y,s) f(y,s) dy\, ds$,is the unique classical solution, that is, $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$, of the nonhomogeneous problem (2.2).

It is clear from property (vi) above that $G \in L^2(Q_T \times Q_T)$. Also, the integral representation in (vii) implies that the function $(x,t) \to \int_\Omega G(x,t;y,0)dy$ is continuous. Let $\gamma_0 = \max_{(x,t)\in Q_T} \int_\Omega G(x,t;y,0)dy$.

Lemma 2.1. *If $f \in L^2(Q_T)$, then* (2.2) *has a unique weak solution $u \in L^2(Q_T)$. Moreover, there exists a positive constant M, depending only on u_0, γ_0, T, and Ω, such that*

$$|u|_{L^2(Q_T)} \leq M + |G|_{L^2(Q_T\times Q_T)} |f|_{L^2(Q_T)}. \tag{2.4}$$

Proof. Consider the following representation (see property (vii) above):

$$u(x,t) = \int_\Omega G(x,t;y,0)u_0(y)dy + \int_0^t \int_\Omega G(x,t;y,s) f(y,s) dy\, ds, \quad (x,t) \in Q_T. \tag{2.5}$$

Define an operator $\mathbf{G} : L^2(Q_T) \to L^2(Q_T)$ by

$$\mathbf{G}f(x,t) = \int_0^t \int_\Omega G(x,t;y,s) f(y,s) dy\, ds, \quad (x,t) \in Q_T. \tag{2.6}$$

Then $\mathbf{G}$ is a bounded linear operator with

$$|\mathbf{G}f|_{L^2(Q_T)} \leq |G|_{L^2(Q_T\times Q_T)} |f|_{L^2(Q_T)}. \tag{2.7}$$

Then for each $(x,t) \in Q_T$,

$$u(x,t) = \int_\Omega G(x,t;y,0)u_0(y)dy + \mathbf{G}f(x,t). \tag{2.8}$$

This implies that for each $(x,t) \in Q_T$

$$|u(x,t)| \leq \gamma_0 |u_0|_0 + \left|\mathbf{G}f(x,t)\right|. \tag{2.9}$$

Minkowski's inequality leads to

$$|u|_{L^2(Q_T)} \leq M + |G|_{L^2(Q_T \times Q_T)} \left|f\right|_{L^2(Q_T)}. \tag{2.10}$$

□

3. Problem with a Discontinuous Nonlinearity

In this section, we investigate the multivalued problem (1.7). We define the notion of a weak solution.

Definition 3.1. A solution of (1.7) is a function $u \in W_0$ such that

(i) there exists $w \in L^2(Q_T)$ with $g(x,t,u) \leq w(x,t) \leq h(x,t,u)$, $(x,t) \in Q_T$;

(ii) $D_t u + Lu = w(x,t)$, $(x,t) \in Q_T$;

(iii) $u(x,0) = \int_0^T k(x,t,u(x,t))dt$, $x \in \Omega$.

We introduce the notion of lower and upper solutions of problem (1.7).

Definition 3.2. $\underline{U} \in W_0$ is a weak lower solution of (1.7) if

(i) $D_t\underline{U} + L\underline{U} \leq g(x,t,\underline{U})$, $(x,t) \in Q_T$;

(ii) $\underline{U}(x,t) \leq 0$, $(x,t) \in \Gamma_T$;

(iii) $\underline{U}(x,0) \leq \int_0^T k(x,t,\underline{U}(x,t))dt$, $x \in \Omega$.

Definition 3.3. $\overline{U} \in W_0$ is a weak upper solution of (1.7) if

(j) $D_t\overline{U} + L\overline{U} \geq h(x,t,\overline{U})$, $(x,t) \in Q_T$;

(jj) $\overline{U}(x,t) \geq 0$, $(x,t) \in \Gamma_T$;

(jjj) $\overline{U}(x,0) \geq \int_0^T k(x,t,\overline{U}(x,t))dt$, $x \in \Omega$.

We will assume that the function $f : Q_T \times \mathbb{R} \to \mathbb{R}$, generating the multivalued function F, is N-measurable on $\mathbb{R}$, which implies that F is an N-measurable, upper semicontinuous multivalued function with nonempty, compact, and convex values. In addition, we will need the following assumptions:

(H1) there exists $p \in L^2(Q_T)$ such that $|f(x,t,u)| \leq p(x,t)$, $(x,t) \in Q_T$;

(H2) there exist a lower solution $\underline{U}$ and an upper solution $\overline{U}$ of (1.7) such that $\underline{U} \leq \overline{U}$;

(H3) $k : Q_T \times \mathbb{R} \to \mathbb{R}$ is continuous, and $u \mapsto k(x,t,u)$ is nondecreasing with $k(x,t,0) = 0$.

We state and prove our main result.

Theorem 3.4. *Assume that (H1), (H2), and (H3) are satisfied. Then the multivalued problem* (1.7) *has at least one solution* $u \in [\underline{U}, \overline{U}]$.

Proof. First, it is clear that the operator $\delta : L^2(Q_T) \to [\underline{U}, \overline{U}]$ defined by

$$\delta(u) = \max\left\{\underline{U}, \min\left(u, \overline{U}\right)\right\} \tag{3.1}$$

is continuous and uniformly bounded. Consider the modified problem

$$\begin{gathered} D_t u + Lu \in F(x,t,\delta(u)), \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ u(x,0) = \int_0^T k(x,t,\delta(u)(x,t))dt, \quad x \in \Omega. \end{gathered} \tag{3.2}$$

We show that possible solutions of (3.2) are a priori bounded. Let $u \in L^2(Q_T)$ be a solution of (3.2). It follows from the definition and the representation (2.5) that for each $(x,t) \in Q_T$,

$$u(x,t) = \int_0^T \int_\Omega G(x,t;y,0)k(y,s,\delta(u)(y,s))dy\,ds + \int_0^t \int_\Omega G(x,t;y,s)w(y,s)dy\,ds, \tag{3.3}$$

where $w \in L^2(Q_T)$ with $g(x,t,\delta(u)) \leq w(x,t) \leq h(x,t,\delta(u))$, $(x,t) \in Q_T$. Since k is continuous and δ is uniformly bounded there exists $m_k > 0$ such that $|k(x,t,\delta(u))| \leq m_k$. Also, assumption (H1) implies that $|w(x,t)| \leq p(x,t)$. The relation (3.3) together with Lemma 2.1 yields

$$|u|_{L^2(Q_T)} \leq C := M_1 + |G|_{L^2(Q_T \times Q_T)} |p|_{L^2(Q_T)}, \tag{3.4}$$

where M_1 depends only on m_k, T, γ_0. Let $V = \{u \in L^2(Q_T); |u|_{L^2(Q_T)} \leq C\}$.

It is clear that solutions of (3.2) are fixed point of the multivalued operator $\mathcal{F} : L^2(Q_T) \to L^2(Q_T)$, defined by

$$\mathcal{F}u = \mathbf{k}(u) + \mathbf{G}N_F(u). \tag{3.5}$$

Here, $\mathbf{k} : L^2(Q_T) \to L^2(Q_T)$ is a single-valued operator defined by

$$\mathbf{k}(u)(x,t) = \int_0^T \int_\Omega G(x,t;y,0)k(y,s,\delta(u(y,s)))dy\,ds, \tag{3.6}$$

and $\mathbf{G}N_F : L^2(Q_T) \to L^2(Q_T)$ is a multivalued operator defined by

$$\mathbf{G}N_F(u)(x,t) = \int_0^t \int_\Omega G(x,t;y,s)N_F(\delta(u(y,s)))dy\,ds. \tag{3.7}$$

Claim 1. $\overline{\mathbf{k}(V)}$ is compact in $L^2(Q_T)$. Since the function k is continuous and the operator δ is uniformly bounded there exists $m_k > 0$ such that $|k(x,t,\delta(u))| \leq m_k$. Also, $G(x,t;y,0)$ is continuous and has no singularity for $t > 0$. It follows that the operator $\mathbf{k}$ is continuous and there exists ρ, depending only on T and Ω, such that $\|\mathbf{k}(u)\|_{W_0} \leq \rho T \gamma_0 m_k$, so that $\mathbf{k}(V)$ is uniformly bounded in W_0. Since the embedding $W_0 \subset L^2(Q_T)$ is compact it follows that $\mathbf{k}(V)$ is compact in $L^2(Q_T)$.

Claim 2. $\overline{\mathbf{G}N_F(V)}$ is also compact in $L^2(Q_T)$. This follows from the continuity of the Green's function and the properties of the Nemitski operator N_F. See Lemma 1.4.

Claim 3. $\alpha(\mathcal{F}(V)) = 0$, that is, it is a condensing multifunction.We have $\alpha(\mathcal{F}(V)) = \alpha(\mathbf{k}(V) + \mathbf{G}N_F(V)) \leq \alpha(\mathbf{k}(V)) + \alpha(\mathbf{G}N_F(V)) = 0$.

Also Lemma 1.4 implies that N_F has nonempty, compact, convex values. Since $\mathbf{k}$ is single-valued, the operator $\mathcal{F}$ has nonempty compact and convex values. We show that $\mathcal{F}$ has a closed graph. Let $v_n \to v^*, h_n \in \mathcal{F}(v_n)$, and $h_n \to h^*$. We show that $h^* \in \mathcal{F}(v^*)$. Now, $h_n \in \mathcal{F}(v_n)$ implies that $h_n - \mathbf{k}(v_n) \in \mathbf{G}N_F(v_n)$. It is clear that $h_n - \mathbf{k}(v_n) \to h^* - \mathbf{k}(v^*)$ in $L^2(Q_T)$. We can use the last part of Lemma 4.1 in [13] to conclude that $h^* - \mathbf{k}(v^*) \in \mathbf{G}N_F(v^*)$, which, in turn, implies that $h^* \in \mathbf{k}(v^*) + \mathbf{G}N_F(v^*) = \mathcal{F}(v^*)$. This will imply that $\mathcal{F}$ is upper semicontinuous.

Therefore, $\mathcal{F} : L^2(Q_T) \to P_{\mathrm{cp,cv}}(L^2(Q_T))$ is condensing. İt remains to show that the set $\{z \in L^2(Q_T);\ \ \lambda z \in \mathcal{F}(z)$ for some $\lambda > 1\}$ is bounded; but this is a consequence of inequality (3.4). Theorem 1.5 implies that the operator $\mathcal{F}$ has a fixed point $z \in \overline{V}$, which is a solution of (3.2).

We, now, show that $z \in [\underline{U}, \overline{U}]$. We prove that $z \geq \underline{U}$. It follows from the definition of a solution of (3.2) that there exists $w \in L^2(Q_T)$ with $g(x,t,\delta(z)) \leq w(x,t) \leq h(x,t,\delta(z)), (x,t) \in Q_T$, such that

$$z(x,t) = \int_0^T \int_\Omega G(x,t;y,0) k(y,s,\delta(z)(y,s))\, dy\, ds + \int_0^t \int_\Omega G(x,t;y,s) w(y,s)\, dy\, ds. \quad (3.8)$$

On the other hand, $\underline{U}$ satisfies

$$\underline{U}(x,t) \leq \int_0^T \int_\Omega G(x,t;y,0) k(y,s,\underline{U}(y,s))\, dy\, dt + \int_0^t \int_\Omega G(x,t;y,s) g(y,s,\underline{U}(y,s))\, dy\, ds. \quad (3.9)$$

Let $\phi(x,t) = z(x,t) - \underline{U}(x,t)$ for each $(x,t) \in Q_T$. Then

$$\begin{aligned} \phi(x,t) \geq & \int_0^T \int_\Omega G(x,t;y,0) \left[k(y,s,\delta(z)(y,s)) - k(y,s,\underline{U}(y,s))\right] dy\, ds \\ & + \int_0^t \int_\Omega G(x,t;y,s) \left[w(y,s) - g(y,s,\underline{U}(y,s))\right] dy\, ds. \end{aligned} \quad (3.10)$$

Since $\delta(z) \geq \underline{U}$ and the functions $u \to k(y,s,u)$ and $u \to g(y,s,u)$ are nondecreasing, it follows that $\phi(x,t) \geq 0$, so that $z(x,t) \geq \underline{U}(x,t)$ for a.e. $(x,t) \in Q_T$. We can show in a similar way that $z(x,t) \leq \overline{U}(x,t)$ for a.e. $(x,t) \in Q_T$. In this case $\delta(z) = z$, and (3.2) reduces to (1.7). Therefore, problem (1.7) has a solution, and consequently, (1.1) has a solution. □

4. Example

Consider the problem

$$
\begin{aligned}
D_t u + Lu \in F(x,t,u) &= [-1,1], \quad (x,t) \in Q_T, \\
u(x,t) &= 0, \quad (x,t) \in \Gamma_T, \\
u(x,0) &= \mu \int_0^T u(x,t)dt, \quad x \in \Omega.
\end{aligned}
\tag{4.1}
$$

Let $\xi(x,t) = \int_0^t \int_\Omega G(x,t;y,s)dy\,ds = -\eta(x,t)$. It is clear that ξ is a classical solution of the problem

$$
\begin{aligned}
D_t u + Lu &= 1, \quad (x,t) \in Q_T, \\
u(x,t) &= 0, \quad (x,t) \in \Gamma_T, \\
u(x,0) &= 0, \quad x \in \Omega,
\end{aligned}
\tag{4.2}
$$

and η is a classical solution of the problem

$$
\begin{aligned}
D_t u + Lu &= -1, \quad (x,t) \in Q_T, \\
u(x,t) &= 0, \quad (x,t) \in \Gamma_T, \\
u(x,0) &= 0, \quad x \in \Omega.
\end{aligned}
\tag{4.3}
$$

Let $\overline{U}(x,t) = \xi(x,t) + a(x,t)$, where a is a solution of the problem $D_t u + Lu = 0$, $u = 0$ on Γ_T, and $u(x,0) = 1$. Then $a(x,t) = \int_\Omega G(x,t;y,0)dy$ and $\overline{U}$ is an upper solution of problem (4.1) provided that $\mu \sup_{x\in\Omega} \int_0^T (\xi(x,t) + a(x,t))dt < 1$.

Similarly, let b be a solution of $D_t u + Lu = 0$, $u = 0$ on Γ_T, and $u(x,0) = -1$.Then $b(x,t) = -a(x,t)$ and $\underline{U}(x,t) = \eta(x,t) + b(x,t)$ is a lower solution of problem (4.1) provided that $1 + \mu \inf_{x\in\Omega} \int_0^T (\eta(x,t) + b(x,t))dt \geq 0$.

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