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Research Article

A Method for a Solution of Equilibrium Problem and Fixed Point Problem of a Nonexpansive Semigroup in Hilbert's Spaces

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We introduce a viscosity approximation method for finding a common element of the set of solutions for an equilibrium problem involving a bifunction defined on a closed, convex subset and the set of fixed points for a nonexpansive semigroup on another one in Hilbert's spaces.

1. Introduction

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Denote the metric projection from $x \in H$ onto C by $P_C x$. Let $T : C \to C$ be a nonexpansive mapping on C, that is, $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$.

Let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on a closed convex subset C, that is,

(1) for each $s > 0$, $T(s)$ is a nonexpansive mapping on C,

(2) $T(0)x = x$ for all $x \in C$,

(3) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 > 0$,

(4) for each $x \in C$, the mapping $T(\cdot)x$ from $(0, \infty)$ into C is continuous.

Denote by $\mathcal{F} = \bigcap_{s>0} F(T(s))$. We know [1, 2] that $\mathcal{F}$ is a closed, convex subset in H and $\mathcal{F} \neq \emptyset$ if C is bounded.

The equilibrium problem is for a bifunction $G(u,v)$ defined on $C\times C$ to find $u^*\in C$ such that

$$G(u^*,v)\geq 0,\quad \forall v\in C. \tag{1.1}$$

Assume that the bifunction G satisfies the following set of standard properties:

(A1) $G(u,u)=0$, for all $u\in C$,

(A2) $G(u,v)+G(v,u)\leq 0$ for all $(u,v)\in C\times C$,

(A3) for every $u\in C$, $G(u,\cdot):C\to(-\infty,+\infty)$ is weakly lower semicontinuous and convex,

(A4) $\overline{\lim}_{t\to+0}G((1-t)u+tz,v)\leq G(u,v)$, for all $(u,z,v)\in C\times C\times C$.

Denote the set of solutions of (1.1) by $\mathrm{EP}(G)$. We also know [3] that $\mathrm{EP}(G)$ is a closed convex subset in H.

The problem studied in this paper is formulated as follows. Let C_1 and C_2 be closed convex subsets in H. Let $G(u,v)$ be a bifunction satisfying conditions (A1)–(A4) with C replaced by C_1 and let $\{T(s):s>0\}$ be a nonexpansive semigroup on C_2. Find an element

$$p\in \mathrm{EP}(G)\cap \mathcal{F}, \tag{1.2}$$

where $\mathrm{EP}(G)$ and $\mathcal{F}$ denote the set of solutions of an equilibrium problem involving by a bifunction $G(u,v)$ on $C_1\times C_1$ and the fixed point set of a nonexpansive semigroup $\{T(s):s>0\}$ on a closed convex subset C_2, respectively.

In the case that $C_1\equiv H$, $G(u,v)=0$, $C_2=C$, and $T(s)=T$, a nonexpansive mapping on C, for all $s>0$, (1.2) is the fixed point problem of a nonexpansive mapping. In 2000, Moudafi [4] proved the following strong convergence theorem.

Theorem 1.1. *Let C be a nonempty, closed, convex subset of a Hilbert space H and let T be a nonexpansive mapping on C such that $F(T)\neq\emptyset$. Let f be a contraction on C and let $\{x_k\}$ be a sequence generated by: $x_1\in C$ and*

$$x_{k+1}=\frac{\varepsilon_k}{1+\varepsilon_k}f(x_k)+\frac{1}{1+\varepsilon_k}Tx_k,\quad k\geq 1, \tag{1.3}$$

where $\{\varepsilon_k\}\in(0,1)$ satisfies

$$\lim_{k\to\infty}\varepsilon_k=0,\qquad \sum_{k=1}^{\infty}\varepsilon_k=\infty,\qquad \lim_{k\to\infty}\left|\frac{1}{\varepsilon_{k+1}}-\frac{1}{\varepsilon_k}\right|=0. \tag{1.4}$$

Then, $\{x_k\}$ converges strongly to $p\in F(T)$, where $p=P_{F(T)}f(p)$.

Such a method for approximation of fixed points is called the viscosity approximation method. It has been developed by Chen and Song [5] to find $p \in \mathcal{F}$, the set of fixed points for a semigroup $\{T(s) : s > 0\}$ on C. They proposed the following algorithm: $x_1 \in C$ and

$$x_{k+1} = \mu_k f(x_k) + (1 - \mu_k) \frac{1}{s_k} \int_0^{s_k} T(s) x_k ds, \quad k \geq 1, \tag{1.5}$$

where $f : C \to C$, is a contraction, $\{\mu_k\} \subset (0,1)$ and $\{s_k\}$ are sequences of positive real numbers satisfying the conditions: $\mu_k \to 0$, $\sum_{k=1}^{\infty} \mu_k = \infty$, and $s_k \to \infty$ as $k \to \infty$.

Recently, Yao and Noor [6] proposed a new viscosity approximation method

$$x_{k+1} = \mu_k f(x_k) + \beta_k x_k + \gamma_k T(s_k) x_k, \quad k \geq 0, \; x_0 \in C, \tag{1.6}$$

where $\{\mu_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ are in $(0,1)$, $s_k \to \infty$, for finding $p \in \mathcal{F}$, when $\{T(s) : s > 0\}$ satisfies the uniformly asymptotically regularity condition

$$\lim_{s \to \infty} \sup_{x \in \widetilde{C}} \| T(t) T(s) x - T(s) x \| = 0, \tag{1.7}$$

uniformly in t, and $\widetilde{C}$ is any bounded subset of C. Further, Plubtieng and Pupaeng in [7] studied the following algorithm:

$$x_{k+1} = \mu_k f(x_k) + \beta_k x_k + (1 - \beta_k - \mu_k) \int_0^{s_k} T(s) x_k ds, \quad k \geq 0, \; x_0 \in C, \tag{1.8}$$

where $\{\mu_k\}$ and $\{\beta_k\}$ are in $[0,1]$ satisfying the following conditions: $\mu_k + \beta_k < 1$, $\lim_{k \to \infty} \mu_k = \lim_{k \to \infty} \beta_k = 0$, $\sum_{k \geq 1} \mu_k = \infty$, and $\{s_k\}$ is a positive divergent real sequence.

There were some methods proposed to solve equilibrium problem (1.1); see for instance [8–12]. In particular, Combettes and Histoaga [3] proposed several methods for solving the equilibrium problem.

In 2007, S. Takahashi and W. Takahashi [13] combinated the Moudafi's method with the Combettes and Histoaga's result in [3] to find an element $p \in \mathrm{EP}(G) \cap F(T)$. They proved the following strong convergence theorem.

Theorem 1.2. *Let C be a nonempty, closed, convex subset of a Hilbert space H, let T be a nonexpansive mapping on C and let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying (A1)–(A4) such that $\mathrm{EP}(G) \cap F(T) \neq \emptyset$. Let f be a contraction on C and let $\{x_k\}$ and $\{u_k\}$ be sequences generated by: $x_1 \in H$ and*

$$\begin{gathered} G(u_k, y) + \frac{1}{r_k} \langle u_k - x_k, y - u_k \rangle \geq 0, \quad \forall y \in C, \\ x_{k+1} = \mu_k f(x_k) + (1 - \mu_k) T u_k, \quad k \geq 1, \end{gathered} \tag{1.9}$$

where $\{\mu_k\} \in (0,1)$ *and* $\{r_k\} \subset (0,\infty)$ *satisfy*

$$\lim_{k\to\infty}\mu_k = 0, \qquad \sum_{k=1}^{\infty}\mu_k = \infty, \qquad \lim\inf_{k\to\infty} r_k > 0,$$
$$\sum_{k=1}^{\infty}\left|\mu_{k+1}-\mu_k\right| < \infty, \qquad \sum_{k=1}^{\infty}|r_{k+1}-r_k| < \infty. \tag{1.10}$$

Then, $\{x_k\}$ *and* $\{u_k\}$ *converge strongly to* $p \in \mathrm{EP}(G)\cap F(T)$*, where* $p = P_{\mathrm{EP}(G)\cap F(T)}f(p)$.

Very recently, Ceng and Wong in [14] combined algorithm (1.6) with the result in [3] to propose the following procudure:

$$G(u_k,y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \ge 0, \quad \forall y \in C,$$
$$x_{k+1} = \mu_k f(x_k) + \beta_k x_k + \gamma_k T(s_k)u_k, \quad k \ge 1, \tag{1.11}$$

for finding an element $p \in \mathrm{EP}(G)\cap \mathcal{F}$ in the case that $C_1 = C_2 = C$ under the uniformly asymptotic regularity condition on the nonexpansive semigroup $\{T(s) : s > 0\}$ on C.

In this paper, motivated by the above results, to solve (1.2), we introduce the following algorithm:

$$x_1 \in H, \quad \text{any element},$$
$$u_k \in C_1 : G(u_k,y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \ge 0, \quad \forall y \in C_1, \tag{1.12}$$
$$x_{k+1} = \mu_k f(u_k) + \beta_k x_k + \gamma_k T_k P_{C_2} u_k, \quad k \ge 1,$$

where f is a contraction on H, that is, $f : H \to H$ and $\|f(x) - f(y)\| \le a\|x - y\|$, for all $x, y \in H, 0 \le a < 1$,

$$T_k x = \frac{1}{s_k}\int_0^{s_k} T(s)x\,ds, \tag{1.13}$$

for all $x \in C_2$, $\{\mu_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ be the sequences in (0,1), and $\{r_k\}$, $\{s_k\}$ are the sequences in $(0,\infty)$ satisfy the following conditions:

(i) $\mu_k + \beta_k + \gamma_k = 1$,

(ii) $\lim_{k\to\infty}\mu_k = 0$, $\sum_{k\ge1}\mu_k = \infty$,

(iii) $0 < \lim\inf_{k\to\infty}\beta_k \le \lim\sup_{k\to\infty}\beta_k < 1$,

(iv) $\lim_{k\to\infty}s_k = \infty$ with bounded $\sup_{k\ge1}|s_k - s_{k+1}|$,

(v) $\lim\inf_{k\to\infty}r_k > 0$ and $\lim_{k\to\infty}|r_k - r_{k+1}| = 0$.

The strong convergence of (1.12)-(1.13) and its corollaries are showed in the next section.

2. Main Results

We formulate the following facts needed in the proof of our results.

Lemma 2.1. *Let H be a real Hilbert space H. There holds the following identity:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

Lemma 2.2 (see [15]). *Let C be a nonempty, closed, convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that $\|z-x\| \leq \|y-x\|$, for all $y \in C$, and $z \in P_C x$ if and only if $\langle z-x, y-z\rangle \geq 0$ for all $y \in C$.*

Lemma 2.3 (see [16]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following condition:*

$$a_{k+1} \leq (1-b_k)a_k + b_k c_k, \tag{2.2}$$

where $\{b_k\}$ and $\{c_k\}$ are sequences of real numbers such that $b_k \in [0,1]$, $\sum_{k=1}^{\infty} b_k = \infty$, and $\limsup_{k\to\infty} c_k \leq 0$. Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.4 (see [9]). *Let C be a nonempty, closed, convex subset of H and G be a bifunction of $C\times C$ into $(-\infty, +\infty)$ satisfying the conditions (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z, v) + \frac{1}{r}\langle z-x, v-z\rangle \geq 0, \quad \forall v \in C. \tag{2.3}$$

Lemma 2.5 (see [9]). *Assume that $G : C \times C \to (-\infty, +\infty)$ satisfies the conditions (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : G(z, v) + \frac{1}{r}\langle z-x, v-z\rangle \geq 0, \forall v \in C \right\}. \tag{2.4}$$

Then, the following statements hold:

(i) *T_r is single-valued,*

(ii) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x-y\rangle, \tag{2.5}$$

(iii) *$F(T_r) = \mathrm{EP}(G)$,*

(iv) *$\mathrm{EP}(G)$ is closed and convex.*

Lemma 2.6 (see [17]). *Let C be a nonempty bounded closed convex subset in a real Hilbert space H and let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C. Then, for any $h > 0$,*

$$\lim \sup_{t\to\infty} \sup_{y\in C} \left\| T(h)\left(\frac{1}{t}\int_0^t T(s)y ds\right) - \frac{1}{t}\int_0^t T(s)y ds \right\| = 0. \tag{2.6}$$

Lemma 2.7 (Demiclosedness Principle [18]). *If C is a closed convex subset of H, T is a nonexpansive mapping on C, $\{x_k\}$ is a sequence in C such that $x_k \rightharpoonup x \in C$ and $x_k - Tx_k \to 0$, then $x - Tx = 0$.*

Lemma 2.8 (see [19]). *Let $\{x_k\}$ and $\{z_k\}$ be bounded sequences in a Banach space E and $\{\beta_k\}$ be a sequence in* $[0,1]$ *with* $0 < \liminf_{k\to\infty}\beta_k \le \limsup_{k\to\infty}\beta_k < 1$. *Suppose $x_{k+1} = \beta_k x_k + (1-\beta_k)z_k$ for all $k \ge 1$ and* $\limsup_{k\to\infty}\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\| \le 0$. *Then,* $\lim_{k\to\infty}\|z_k - x_k\| = 0$.

Now, we are in a position to prove the following result.

Theorem 2.9. *Let C_1 and C_2 be two nonempty, closed, convex subsets in a real Hilbert space H. Let G be a bifunction from $C_1 \times C_1$ to $(-\infty, +\infty)$ satisfying conditions (A1)–(A4) with C replaced by C_1, let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C_2 such that* $\mathrm{EP}(G)\cap \mathcal{F} \neq \emptyset$ *and let f be a contraction of H into itself. Then, $\{x_k\}$ and $\{u_k\}$ generated by* (1.12)-(1.13) *converge strongly to* $p \in \mathrm{EP}(G)\cap\mathcal{F}$, *where* $p = P_{\mathrm{EP(G)}\cap\mathcal{F}} f(p)$.

Proof. Let $Q = P_{\mathrm{EP(G)}\cap\mathcal{F}}$. Then, Qf is a contraction of H into itself. In fact, from $\|f(x) - f(y)\| \le a\|x - y\|$ for all $x, y \in H$ and the nonexpansive property of P_C for a closed convex subset C in H, it implies that

$$\|Qf(x) - Qf(y)\| \le \|f(x) - f(y)\| \le a\|x - y\|. \tag{2.7}$$

Hence, Qf is a contraction of H into itself. Since H is complete, there exists a unique element $p \in H$ such that $p = Qf(p)$. Such a p is an element of $C_1 \cap C_2$, because $\mathrm{EP(G)} \cap \mathcal{F} \neq \emptyset$.

By Lemma 2.4, $\{u_k\}$ and $\{x_k\}$ are well defined. For each $u \in \mathrm{EP}(G) \cap \mathcal{F}$, by putting $u_k = T_{r_k}x_k$ and using (ii) and (iii) in Lemma 2.5, we have that

$$\|u_k - u\| = \|T_{r_k}x_k - T_{r_k}u\| \le \|x_k - u\|. \tag{2.8}$$

Put $M_u = \max\{\|x_1 - u\|, (1/(1-a))\|f(u) - u\|\}$. Clearly, $\|x_1 - u\| \le M_u$. Suppose that $\|x_k - u\| \le M_u$. Then, we have, from the nonexpansive property of $T_kP_{C_2}$, condition (i) and (2.8), that

$$\begin{aligned}
\|x_{k+1} - u\| &= \|\mu_k(f(u_k) - u) + \beta_k(x_k - u) + \gamma_k(T_kP_{C_2}u_k - u)\| \\
&\le \mu_k\|f(u_k) - u\| + \beta_k\|x_k - u\| + \gamma_k\|T_kP_{C_2}u_k - T_kP_{C_2}u\| \\
&\le \mu_k(\|f(u_k) - f(u)\| + \|f(u) - u\|) + \beta_k\|x_k - u\| + \gamma_k\|u_k - u\| \\
&\le \mu_k(a\|u_k - u\| + \|f(u) - u\|) + (1 - \mu_k)\|x_k - u\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\mu_k(1-a)\right)\|x_k-u\|+\mu_k(1-a)\frac{1}{1-a}\|f(u)-u\| \\
&\leq \left(1-\mu_k(1-a)\right)M_u+\mu_k(1-a)M_u=M_u.
\end{aligned}
\tag{2.9}
$$

So, $\|x_k-u\|\leq M_u$ for all $k\geq 1$ and hence $\{x_k\}$ is bounded. Therefore, $\{u_k\}$, $\{T_kP_{C_2}u_k\}$, and $\{f(u_k)\}$ are also bounded.

Next, we show that $\|x_{k+1}-x_k\|\to 0$ as $k\to\infty$. For this purpose, we define a sequence $\{x_k\}$ by

$$
x_{k+1}=\beta_k x_k+\left(1-\beta_k\right)z_k. \tag{2.10}
$$

Then, we observe that

$$
\begin{aligned}
z_{k+1}-z_k &= \frac{\mu_{k+1}f(u_{k+1})+\gamma_{k+1}T_{k+1}P_{C_2}u_{k+1}}{1-\beta_{k+1}} \\
&\quad -\frac{\mu_k f(u_k)+\gamma_k T_kP_{C_2}u_k}{1-\beta_k} \\
&= \frac{\mu_{k+1}}{1-\beta_{k+1}}f(u_{k+1})-\frac{\mu_k}{1-\beta_k}f(u_k) \\
&\quad +\frac{\gamma_{k+1}}{1-\beta_{k+1}}(T_{k+1}P_{C_2}u_{k+1}-T_{k+1}P_{C_2}u_k) \\
&\quad +\frac{\gamma_{k+1}}{1-\beta_{k+1}}T_{k+1}P_{C_2}u_k-\frac{\gamma_k}{1-\beta_k}T_kP_{C_2}u_k \\
&= \frac{\mu_{k+1}}{1-\beta_{k+1}}f(u_{k+1})-\frac{\mu_k}{1-\beta_k}f(u_k) \\
&\quad +\frac{\gamma_{k+1}}{1-\beta_{k+1}}(T_{k+1}P_{C_2}u_{k+1}-T_{k+1}P_{C_2}u_k)+T_{k+1}P_{C_2}u_k \\
&\quad -\frac{\mu_{k+1}}{1-\beta_{k+1}}T_{k+1}P_{C_2}u_k-T_kP_{C_2}u_k+\frac{\mu_k}{1-\beta_k}T_kP_{C_2}u_k,
\end{aligned}
\tag{2.11}
$$

and, hence,

$$
\begin{aligned}
\|z_{k+1}-z_k\|-\|x_{k+1}-x_k\| &\leq \frac{\mu_{k+1}}{1-\beta_{k+1}}\left(\|f(u_{k+1})\|+\|T_{k+1}P_{C_2}u_k\|\right) \\
&\quad +\frac{\mu_k}{1-\beta_k}\left(\|f(u_k)\|+\|T_kP_{C_2}u_k\|\right)\frac{\gamma_{k+1}}{1-\beta_{k+1}}\|u_{k+1}-u_k\| \\
&\quad +\|T_{k+1}P_{C_2}u_k-T_kP_{C_2}u_k\|-\|x_{k+1}-x_k\|.
\end{aligned}
\tag{2.12}
$$

Now, we estimate the value $\|u_{k+1} - u_k\|$ by using $u_k = T_{r_k} x_k$ and $u_{k+1} = T_{r_{k+1}} x_{k+1}$. We have from (2.4) that

$$G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq 0, \quad \forall y \in C_1, \tag{2.13}$$

$$G(u_{k+1}, y) + \frac{1}{r_{k+1}}\langle u_{k+1} - x_{k+1}, y - u_{k+1}\rangle \geq 0, \quad \forall y \in C_1. \tag{2.14}$$

Putting $y = u_{k+1}$ in (2.13) and $y = u_k$ in (2.14), adding the one to the other obtained result and using (A2), we obtain that

$$\left\langle \frac{u_k - x_k}{r_k} - \frac{u_{k+1} - x_{k+1}}{r_{k+1}}, u_{k+1} - u_k \right\rangle \geq 0 \tag{2.15}$$

and, hence,

$$\left\langle u_k - u_{k+1} + u_{k+1} - x_k - \frac{r_k}{r_{k+1}}(u_{k+1} - x_{k+1}), u_{k+1} - u_k \right\rangle \geq 0. \tag{2.16}$$

Without loss of generality, let us assume that there exists a real number b such that $r_k > b > 0$ for all $k \geq 1$. Then, we have

$$\begin{aligned} \|u_{k+1} - u_k\|^2 &\leq \left\langle x_{k+1} - x_k + \left(1 - \frac{r_k}{r_{k+1}}\right)(u_{k+1} - x_{k+1}), u_{k+1} - u_k \right\rangle \\ &\leq \left(\|x_{k+1} - x_k\| + \left|1 - \frac{r_k}{r_{k+1}}\right| \|u_{k+1} - x_{k+1}\| \right) \|u_{k+1} - u_k\| \end{aligned} \tag{2.17}$$

and, hence,

$$\begin{aligned} \|u_{k+1} - u_k\| &\leq \|x_{k+1} - x_k\| + \frac{1}{r_{k+1}} |r_{k+1} - r_k| \|u_{k+1} - x_{k+1}\| \\ &\leq \|x_{k+1} - x_k\| + \frac{2M_u}{b} |r_{k+1} - r_k|. \end{aligned} \tag{2.18}$$

On the other hand,

$$\begin{aligned} &\|T_k P_{C_2} u_k - T_{k+1} P_{C_2} u_k\| \\ &\quad = \left\| \frac{1}{s_k} \int_0^{s_k} T(s) P_{C_2} u_k ds - \frac{1}{s_{k+1}} \int_0^{s_{k+1}} T(s) P_{C_2} u_k ds \right\| \\ &\quad = \left\| \frac{1}{s_k} \int_0^{s_k} [T(s) P_{C_2} u_k - T(s) P_{C_2} u] ds - \frac{1}{s_{k+1}} \int_0^{s_{k+1}} [T(s) P_{C_2} u_k - T(s) P_{C_2} u] ds \right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \left(\frac{1}{s_k} - \frac{1}{s_{k+1}} \right) \int_0^{s_{k+1}} [T(s)P_{C_2}u_k - T(s)P_{C_2}u]ds + \frac{1}{s_k} \int_{s_{k+1}}^{s_k} [T(s)P_{C_2}u_k - T(s)P_{C_2}u]ds \right\| \\
&\le \left| \frac{1}{s_k} - \frac{1}{s_{k+1}} \right| s_{k+1} M_u + \frac{|s_k - s_{k+1}|}{s_k} M_u \\
&\le \frac{\sup_{k\ge 1}|s_{k+1} - s_k|}{s_k} 2M_u.
\end{aligned}
\tag{2.19}
$$

So, we get from (2.10), (2.12), (2.18), (2.19), and the nonexpansive property of $T_{k+1}P_{C_2}$ that

$$
\begin{aligned}
\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\| \le{}& \frac{\mu_{k+1}}{1 - \beta_{k+1}} \left(\|f(u_{k+1})\| + \|T_{k+1}P_{C_2}u_k\| \right) \\
&+ \frac{\mu_k}{1 - \beta_k} \left(\|f(u_k)\| + \|T_k P_{C_2}u_k\| \right) \\
&+ \frac{\gamma_{k+1} 2M_u}{(1 - \beta_{k+1})b} |r_{k+1} - r_k| + \frac{\sup_{k\ge 1}|s_{k+1} - s_k|}{s_k} 2M_u.
\end{aligned}
\tag{2.20}
$$

So,

$$
\limsup_{k\to\infty} \|z_{k+1} - z_k\| - \|x_{k+1} - x_k\| \le 0, \tag{2.21}
$$

and by Lemma 2.8, we have

$$
\lim_{k\to\infty} \|z_k - x_k\| = 0. \tag{2.22}
$$

Consequently, it follows from (2.10) and condition (iii) that

$$
\lim_{k\to\infty} \|x_{k+1} - x_k\| = \lim_{k\to\infty} (1 - \beta_k) \|z_k - x_k\| = 0. \tag{2.23}
$$

By (2.18), (2.23), and

$$
\lim_{k\to\infty} |r_k - r_{k+1}| = 0, \tag{2.24}
$$

we also obtain

$$
\lim_{k\to\infty} \|u_{k+1} - u_k\| = 0. \tag{2.25}
$$

We have, for every $u \in \mathrm{EP}(G) \cap \mathcal{F}$, from (iii) in Lemma 2.5, that

$$\begin{aligned}
\|u_k - u\|^2 &= \|T_{r_k} x_k - T_{r_k} u\|^2 \\
&\leq \langle T_{r_k} x_k - T_{r_k} u, x_k - u \rangle \\
&= \langle u_k - u, x_k - u \rangle \\
&= \frac{1}{2}\left[\|u_k - u\|^2 + \|x_k - u\|^2 - \|u_k - x_k\|^2\right]
\end{aligned} \tag{2.26}$$

and, hence,

$$\|u_k - u\|^2 \leq \|x_k - u\|^2 - \|u_k - x_k\|^2. \tag{2.27}$$

Therefore, from the convexity of $\| \cdot \|^2$ and condition (i), we have

$$\begin{aligned}
\|x_{k+1} - u\|^2 &\leq \mu_k \left\| f(u_k) - u \right\|^2 + \beta_k \|x_k - u\|^2 + \gamma_k \|T_k P_{C_2} u_k - u\|^2 \\
&\leq \mu_k \left\| f(u_k) - u \right\|^2 + \beta_k \|x_k - u\|^2 + \gamma_k \|u_k - u\|^2 \\
&\leq \mu_k \left\| f(u_k) - u \right\|^2 + \beta_k \|x_k - u\|^2 + \gamma_k \left(\|x_k - u\|^2 - \|u_k - x_k\|^2\right) \\
&\leq \mu_k \left\| f(u_k) - u \right\|^2 + (1 - \mu_k)\|x_k - u\|^2 - \gamma_k \|u_k - x_k\|^2 \\
&\leq \mu_k \left\| f(u_k) - u \right\| + \|x_k - u\|^2 - \gamma_k \|u_k - x_k\|^2
\end{aligned} \tag{2.28}$$

and, hence,

$$\begin{aligned}
\gamma_k \|u_k - x_k\|^2 &\leq \mu_k \| f(u_k) - u\| + \|x_k - u\|^2 - \|x_{k+1} - u\|^2 \\
&\leq \mu_k \| f(u_k) - u\| + 2M_u \|x_k - x_{k+1}\|.
\end{aligned} \tag{2.29}$$

Without loss of generality, we assume that $0 < \beta^* \leq \beta_k \leq \widetilde{\beta} < 1$ for all $k \geq 1$. Then, for sufficiently large k,

$$0 \leq \left(1 - \widetilde{\beta} - \mu_k\right) \|u_k - x_k\|^2 \leq \mu_k \| f(u_k) - u\| + 2M_u \|x_k - x_{k+1}\|. \tag{2.30}$$

So, we have

$$\lim_{k \to \infty} \|u_k - x_k\| = 0. \tag{2.31}$$

Further, since $x_{k+1} = \mu_k f(u_k) + \beta_k x_k + \gamma_k T_k P_{C_2} u_k$, by condition (i), (2.19) and

$$\begin{aligned} x_{k+1} - T_{k+1} P_{C_2} u_{k+1} &= \mu_k f(u_k) + \beta_k x_k + \gamma_k T_k P_{C_2} u_k \\ &\quad - (\mu_k + \beta_k + \gamma_k) T_k P_{C_2} u_k + T_k P_{C_2} u_k - T_{k+1} P_{C_2} u_{k+1} \\ &= \mu_k \left(f(u_k) - T_k P_{C_2} u_k \right) + \beta_k (x_k - T_k P_{C_2} u_k) \\ &\quad + T_k P_{C_2} u_k - T_{k+1} P_{C_2} u_{k+1}, \end{aligned} \tag{2.32}$$

we obtain that

$$\begin{aligned} \| x_{k+1} - T_{k+1} P_{C_2} u_{k+1} \| &\leq \mu_k \| f(u_k) - T_k P_{C_2} u_k \| + \beta_k \| x_k - T_k P_{C_2} u_k \| \\ &\quad + \| u_{k+1} - u_k \| + \frac{\sup_{k \geq 1} |s_{k+1} - s_k|}{s_k} 2 M_u. \end{aligned} \tag{2.33}$$

Then, from (2.25), (2.33) and the conditions on $\{\mu_k\}$ and $\{s_k\}$, it implies that

$$\left(1 - \widetilde{\beta}\right) \limsup_{k \to \infty} \| x_k - T_k P_{C_2} u_k \| \leq 0, \tag{2.34}$$

and so

$$\limsup_{k \to \infty} \| x_k - T_k P_{C_2} u_k \| \leq 0. \tag{2.35}$$

Since

$$\| T_k P_{C_2} u_k - u_k \| \leq \| T_k P_{C_2} u_k - x_k \| + \| x_k - u_k \|, \tag{2.36}$$

we obtain from (2.31) that

$$\lim_{k \to \infty} \| T_k P_{C_2} u_k - u_k \| = 0. \tag{2.37}$$

Next, we show that

$$\limsup_{k \to \infty} \langle f(p) - p, x_k - p \rangle \leq 0. \tag{2.38}$$

We choose a subsequence $\{u_{k_i}\}$ of the sequence $\{u_k\}$ such that

$$\limsup_{k \to \infty} \langle f(p) - p, x_k - p \rangle = \lim_{i \to \infty} \langle f(p) - p, x_{k_i} - p \rangle. \tag{2.39}$$

As $\{u_k\}$ is bounded, there exists a subsequence $\{u_{k_j}\}$ of the sequence $\{u_{k_i}\}$ which converges weakly to z. From (2.37), we also have that $\{T_{k_j} P_{C_2} u_{k_j}\}$ converges weakly to z. Since $\{u_k\} \subset C_1$ and $\{T_k P_{C_2} u_k\} \subset C_2$ and C_1, C_2 are two closed convex subsets in H, we have that $z \in C_1 \cap C_2$.

First, we prove that $z \in \mathrm{EP}(G)$. From (2.4) it follows that

$$G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq 0, \quad \forall y \in C_1, \tag{2.40}$$

and, hence, by using condition (A2), we get

$$\frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq G(y, u_k), \quad \forall y \in C_1. \tag{2.41}$$

Therefore,

$$\left\langle \frac{u_{k_j} - x_{k_j}}{r_{k_j}}, y - u_{k_j} \right\rangle \geq G\left(y, u_{k_j}\right), \quad \forall y \in C_1. \tag{2.42}$$

This together with condition (A3) and (2.31) imply that

$$0 \geq G(y, z), \quad \forall y \in C_1. \tag{2.43}$$

So, $G(z, y) \geq 0$ for all $y \in C_1$. It means that $z \in \mathrm{EP}(G)$.

Next we show that $z \in \mathcal{F}$. Since $T_k P_{C_2} u_k \in C_2$, we have

$$\begin{aligned} \|T_k P_{C_2} u_k - P_{C_2} u_k\| &= \|P_{C_2} T_k P_{C_2} u_k - P_{C_2} u_k\| \\ &\leq \|T_k P_{C_2} u_k - u_k\|, \end{aligned} \tag{2.44}$$

and, hence, from (2.31) it follows that

$$\lim_{k \to \infty} \|T_k P_{C_2} u_k - P_{C_2} u_k\| = 0. \tag{2.45}$$

Thus, (2.37) together with (2.45) imply

$$\lim_{k \to \infty} \|u_k - P_{C_2} u_k\| = 0. \tag{2.46}$$

Therefore, $\{P_{C_2} u_{k_j}\}$ also converges weakly to z, as $j \to \infty$.

On the other hand, for each $h > 0$, we have that

$$\begin{aligned}\|T(h)P_{C_2}u_k - P_{C_2}u_k\| &\leq \left\|T(h)P_{C_2}u_k - T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds\right)\right\| \\ &\quad + \left\|T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds\right) - \frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds\right\| \\ &\quad + \left\|\frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds - P_{C_2}u_k\right\| \\ &\leq 2\left\|\frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds - P_{C_2}u_k\right\| \\ &\quad + \left\|T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds\right) - \frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds\right\|.\end{aligned} \tag{2.47}$$

Let $C_2^0 = \{x \in C_2 : \|x - p\| \leq M_p\}$. Since $p = P_{\mathcal{F}\cap EQ(G)} f(p) \in C_2$, we have from (2.33) that

$$\|P_{C_2}u_k - p\| = \|P_{C_2}u_k - P_{C_2}p\| \leq \|u_k - p\| \leq \|x_k - p\| \leq M_p. \tag{2.48}$$

So, C_2^0 is a nonempty bounded closed convex subset. It is easy to verify that $\{T(s) : s > 0\}$ is a nonexpansive semigroup on C_2^0. By Lemma 2.6, we get

$$\lim_{k\to\infty}\left\|T(h)\left(\frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds\right) - \frac{1}{s_k}\int_0^{s_k} T(s)P_{C_2}u_k ds\right\| = 0, \tag{2.49}$$

for every fixed $h > 0$, and hence, by (2.45)–(2.47), we obtain

$$\lim_{n\to\infty}\|T(h)P_{C_2}u_k - u_k\| = 0 \tag{2.50}$$

for each $h > 0$. By Lemma 2.7, $z \in F(T(h)P_{C_2}) = F(T(h))$ for all $h > 0$, because $F(TP_C) = F(T)$ for any mapping $T : C \to C$. It means that $z \in \mathcal{F}$. Therefore, $z \in \mathcal{F} \cap \mathrm{EP}(G)$. Since $p = P_{\mathrm{EP}(G)\cap\mathcal{F}} f(p)$, we have from Lemma 2.2 that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(p) - p, x_k - p\rangle &= \lim_{i\to\infty}\langle f(p) - p, x_{k_i} - p\rangle \\ &= \langle f(p) - p, z - p\rangle \leq 0.\end{aligned} \tag{2.51}$$

So, (2.38) is proved. Further, since $x_{k+1} - p = \mu_k(f(u_k) - p) + \beta_k(x_k - p) + \gamma_k(T_k P_{C_2} u_k - p)$, by using Lemma 2.1, we have that

$$\begin{aligned}
\|x_{k+1} - p\|^2 &\le \|\beta_k(x_k - p) + \gamma_k(T_k P_{C_2} u_k - p)\|^2 + 2\mu_k\langle f(u_k) - p, x_{k+1} - p\rangle \\
&\le (\beta_k\|x_k - p\| + \gamma_k\|u_k - p\|)^2 + 2\mu_k\langle f(u_k) - f(p), x_{k+1} - p\rangle \\
&\quad + 2\mu_k\langle f(p) - p, x_{k+1} - p\rangle \\
&\le (1-\mu_k)^2\|x_k - p\|^2 + 2\mu_k a\|u_k - p\|\|x_{k+1} - p\| \\
&\quad + 2\mu_k\langle f(p) - p, x_{k+1} - p\rangle \\
&\le (1-\mu_k)^2\|x_k - p\|^2 + \mu_k a\left[\|u_k - p\|^2 + \|x_{k+1} - p\|^2\right] \\
&\quad + 2\mu_k\langle f(p) - p, x_{k+1} - p\rangle.
\end{aligned} \tag{2.52}$$

This with (2.8) implies that

$$\begin{aligned}
\|x_{k+1} - p\|^2 &\le \frac{(1-\mu_k)^2 + \mu_k a}{1 - \mu_k a}\|x_k - p\|^2 + \frac{2\mu_k}{1-\mu_k a}\langle f(p) - p, x_{k+1} - p\rangle \\
&= \frac{1 - 2\mu_k + \mu_k a}{1 - \mu_k a}\|x_k - p\|^2 + \frac{\mu_k^2}{1 - \mu_k a}\|x_k - p\|^2 \\
&\quad + \frac{2\mu_k}{1 - \mu_k a}\langle f(p) - p, x_{k+1} - p\rangle \\
&= \left(1 - \frac{2(1-a)\mu_k}{1 - \mu_k a}\right)\|x_k - p\|^2 + \frac{2(1-a)\mu_k}{1-\mu_k a} \\
&\quad \times \left[\frac{\mu_k M_p^2}{2(1-a)} + \frac{1}{1-a}\langle f(p) - p, x_{k+1} - p\rangle\right] \\
&= (1 - b_k)\|x_k - p\|^2 + b_k c_k,
\end{aligned} \tag{2.53}$$

where

$$b_k = \frac{2(1-a)\mu_k}{1-\mu_k a}, \qquad c_k = \left[\frac{\mu_k M_p^2}{2(1-a)} + \frac{1}{1-a}\langle f(p) - p, x_{k+1} - p\rangle\right]. \tag{2.54}$$

Using Lemma 2.3, we get

$$\lim_{k\to\infty}\|x_k - p\| = 0. \tag{2.55}$$

From (2.33) it follows that $u_k \to p$ as $k \to \infty$. This completes the proof. □

Remarks. (a) Note that the following parameters $\mu_k = 1/(3+k)$, $\beta_k = \mu_k + 1/4$, $\gamma_k = -2\mu_k + 3/4$, $r_k = \mu_k + a_0$ for any fixed number $a_0 > 0$, and $s_k = (b_0 k + c_0)$ with $b_0, c_0 > 0$ for all $k \geq 1$ satisfy all conditions in Theorem 2.9.

(b) If $T(s) = T$ for all $s > 0$ and $C_1 = C_2 = C$, then we have the following corollary.

Corollary 2.10. *Let C be a nonempty, closed, convex subsets in a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying conditions (A1)–(A4), let T be a nonexpansive mapping on C such that $\mathrm{EP}(G) \cap F(T) \neq \emptyset$ and let f be a contraction of H into itself. Let $\{x_k\}$ and $\{u_k\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{gathered} u_k \in C, \quad G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k \rangle \geq 0, \quad \forall y \in C, \\ x_{k+1} = \mu_k f(u_k) + \beta_k x_k + \gamma_k T u_k, \quad k \geq 1, \end{gathered} \tag{2.56}$$

where $\{\mu_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, and $\{r_k\}$ satisfy conditions (i)–(v). Then, $\{x_k\}$ and $\{u_k\}$ converge strongly to $p \in \mathrm{EP}(G) \cap F(T)$, where $p = P_{\mathrm{EP}(G) \cap F(T)} f(p)$.

Proof. From the proof of the theorem, $\|T_k P_{C_2} u_{k-1} - T_{k-1} P_{C_2} u_{k-1}\| = \|T u_{k-1} - T u_{k-1}\| = 0$ in (2.12). □

(c) In the case that $C_1 = C_2 = C$, a closed convex subset in H, $G(u, v) = 0$ for all $(u, v) \in C \times C$, we have the following result.

Corollary 2.11. *Let C be a nonempty, closed, convex subsets in a real Hilbert space H. Let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} \neq \emptyset$ and let f be a contraction of H into itself. Let $\{x_k\}$ and $\{u_k\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{gathered} u_k = P_C x_k, \\ x_{k+1} = \mu_k f(u_k) + \beta_k x_k + \gamma_k T_k u_k, \quad k \geq 1, \end{gathered} \tag{2.57}$$

where $T_k x$ is defined by (1.13) *for all $x \in C$ and $\{\mu_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, and $\{s_k\}$ satisfy conditions (i)–(v). Then, the sequences $\{x_k\}$ and $\{u_k\}$ converge strongly to $p \in \mathcal{F}$, where $p = P_{\mathcal{F}} f(p)$.*

Proof. By Lemma 2.2, $u_k = P_C x_k$ if and only if

$$\langle u_k - x_k, y - u_k \rangle \geq 0, \quad \forall y \in C. \tag{2.58}$$

Clearly, in addition, if f is a contraction of C into itself and $x_1 \in C$, then we obtain the algoritm

$$x_{k+1} = \mu_k f(x_k) + \beta_k x_k + \gamma_k T_k x_k, \quad k \geq 1, \tag{2.59}$$

where T_k is defined by (1.13) and $\{\mu_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, and $\{s_k\}$ satisfy conditions (i)–(v). This algorithm is different from Yao and Noor's algorithm (1.6), in which $T_k x = T(s_k)x$ for all $x \in C$. It likes completely the Plubtieng and Punpaeng's algorithm (1.8), but converges under a new condition on $\{\beta_k\}$. □

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