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Research Article

Variational-Like Inclusions and Resolvent Equations Involving Infinite Family of Set-Valued Mappings

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We study variational-like inclusions involving infinite family of set-valued mappings and their equivalence with resolvent equations. It is established that variational-like inclusions in real Banach spaces are equivalent to fixed point problems. This equivalence is used to suggest an iterative algorithm for solving resolvent equations. Some examples are constructed.

1. Introduction

The important generalization of variational inequalities, called variational inclusions, have been extensively studied and generalized in different directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, economics, finance and applied sciences, and so forth; see, for example [1–7] and references theirin. The resolvent operator technique for solving variational inequalities and variational inclusions is interesting and important. The resolvent operator technique is used to establish an equivalence between variational inequalities and resolvent equations. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving various classes of variational inequalities (inclusions) and related optimization problems.

In this paper, we established a relationship between variational-like inclusions and resolvent equations. We propose an iterative algorithm for computing the approximate solutions which converge to exact solution of considered resolvent equations. Some examples are constructed.

2. Formulation and Preliminaries

Throughout the paper, unless otherwise specified, we assume that E is a real Banach space with its norm $\|\cdot\|$, E^* is the topological dual of E, $\langle\cdot,\cdot\rangle$ is the pairing between E and E^*, d is the metric induced by the norm $\|\cdot\|$, 2^E (resp., $CB(E)$) is the family of nonempty (resp., nonempty closed and bounded) subsets of E, and $\mathscr{H}(\cdot,\cdot)$ is the Housdorff metric on $CB(E)$ defined by

$$\mathscr{H}(P,Q)=\max\left\{\sup_{x\in P}d(x,Q),\sup_{y\in Q}d(P,y)\right\}, \tag{2.1}$$

where $d(x,Q) = \inf_{y\in Q}d(x,y)$ and $d(P,y) = \inf_{x\in P}d(x,y)$. The normalized duality mapping $\mathcal{J}:E\to 2^{E^*}$ is defined by

$$\mathcal{J}(x)=\left\{f\in E^*:\langle x,f\rangle=\|x\|\cdot\|f\|,\ \|f\|=\|x\|,\ \forall x\in E\right\}. \tag{2.2}$$

Definition 2.1. Let E be a real Banach space. Let $\eta:E\times E\to E$; $g,A:E\to E$ be the single-valued mapping, and let $M:E\to 2^E$ be a set-valued mapping. Then,

(i) the mapping g is said to be accretive if

$$\langle g(x)-g(y),j(x-y)\rangle\geq 0,\quad \forall x,y\in E, \tag{2.3}$$

(ii) the mapping g is said to be strictly accretive if

$$\langle g(x)-g(y),j(x-y)\rangle\geq 0,\quad \forall x,y\in E, \tag{2.4}$$

and the equality hold if and only if $x=y$,

(iii) the mapping g is said to be k-strongly accretive ($k\in(0,1)$) if for any $x,y\in E$, there exists $j(x-y)\in\mathcal{J}(x-y)$ such that

$$\langle g(x)-g(y),j(x-y)\rangle\geq k\|x-y\|^2, \tag{2.5}$$

(iv) the mapping A is said to be r-strongly η-accretive, if there exists a constant $r>0$ such that

$$\langle A(x)-A(y),j(\eta(x,y))\rangle\geq r\|x-y\|^2,\quad \forall x,y\in E, \tag{2.6}$$

(v) the mapping M is said to be m-relaxed η-accretive, if there exists a constant $m>0$ such that

$$\langle u-v,j(\eta(x,y))\rangle\geq -m\|x-y\|^2,\quad \forall x,y\in E,\ u\in M(x),\ v\in M(y). \tag{2.7}$$

Definition 2.2. Let $A : E \to E$, $\eta : E \times E \to E$ be the single-valued mappings. Then, a set-valued mapping $M : E \to 2^E$ is called (A, η)-accretive if M is m-relaxed η-accretive and $(A + \rho M)(E) = E$, for every $\rho > 0$.

Proposition 2.3 (see [8, 9]). *Let E be a real Banach space, and let $\mathcal{J} : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in \mathcal{J}(x + y). \tag{2.8}$$

Definition 2.4. Let $A : E \to E$, $W : E \times E \to E$, and let $N : E^\infty = E \times E \times E \cdots \to E$ be the mappings. Then,

(i) the mapping A is said to be Lipschitz continuous with constant λ_A if

$$\|A(x) - A(y)\| \le \lambda_A \|x - y\|, \quad \forall x, y \in E, \tag{2.9}$$

(ii) the mapping W is said to be Lipschitz continuous in the first argument with constant λ_{W_1} if

$$\|W(x_1, \cdot) - W(x_2, \cdot)\| \le \lambda_{W_1} \|x_1 - x_2\|, \quad \forall x_1, x_2 \in E. \tag{2.10}$$

Similarly, we can define Lipschitz continuity in the second argument.

(iii) the mapping N is said to be Lipschitz continuous in the ith argument with constant β_i if

$$\|N(\cdot, \ldots, x_i, \ldots) - N(\cdot, \ldots, y_i, \ldots)\| \le \beta_i \|x_i - y_i\|, \quad \forall x_i, y_i \in E. \tag{2.11}$$

Definition 2.5. Let $A : E \to E$ be a strictly η-accretive mapping, and let $M : E \to 2^E$ be an (A, η)-accretive mapping. Then, the resolvent operator $J_{\eta,M}^{\rho,A} : E \to E$ is defined by

$$J_{\eta,M}^{\rho,A}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in E. \tag{2.12}$$

Proposition 2.6 (see [10]). *Let E be a real Banach space, and let $\eta : E \times E \to E$ be τ-Lipschitz continuous; let $A : E \to E$ be an r-strongly η-accretive mapping, and let $M : E \to 2^E$ be an (A, η)-accretive mapping. Then the resolvent operator $J_{\eta,M}^{\rho,A} : E \to E$ is $\tau/(r - \rho m)$-Lipschitz continuous, that is,*

$$\left\|J_{\eta,M}^{\rho,A}(x) - J_{\eta,M}^{\rho,A}(y)\right\| \le \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in E, \tag{2.13}$$

where $\rho \in (0, r/m)$ is a constant.

Example 2.7. Let $E = \mathbb{R}$, $A(x) = \sqrt{x}$, $M(y) = \sqrt{y}$, and $\eta(x, y) = (\sqrt{x} - \sqrt{y})$ for all $x, y \geq 0 \in E$. Then, M is η-accretive.

Example 2.8. Let $M(\cdot, \cdot) : E \times E \to 2^E$ be r-strongly η-accretive in the first argument. Then, M is m-relaxed η-accretive for $m \in (1, r + r^2)$, for $r > 0.6180$.

Let $T_i : E \to CB(E)$, $i = 1, 2, \ldots, \infty$ be an infinite family of set-valued mappings, and let $N : E^\infty = E \times E \times E \cdots \to E$ be a nonlinear mapping. Let $\eta, W : E \times E \to E; A, g, m : E \to E$ be single-valued mappings, let and B, C, D:$E \to CB(E)$ be set-valued mappings. Suppose that $M(\cdot, \cdot) : E \times E \to 2^E$ is (A, η)-accretive mapping in the first argument. We consider the following problem.

Find $u \in E$, $w_i \in T_i(u)$, $i = 1, 2, \ldots, \infty$, $a \in B(u)$, $x \in C(u)$, and $y \in D(u)$ such that

$$0 \in N(w_1, w_2, \ldots) - W(x, y) + m(a) + M(g(u) - m(a), u). \tag{2.14}$$

The problem (2.14) is called variational-like inclusions problem.

Special Cases

(i) If $W = 0$, $m = 0$, then problem (2.14) reduces to the problem of finding $u \in E$, $w_i \in T_i(u)$, $i = 1, 2, \ldots, \infty$ such that

$$0 \in N(w_1, w_2, \ldots) + M(g(u), u). \tag{2.15}$$

Problem (2.15) is introduced and studied by Wang [11].

(ii) If $W = 0$, $m = 0$, $N(\cdot, \ldots) = N(\cdot, \cdot)$, then problem (2.14) reduces to a problem considered by Chang, et al. [12, 13] that is, find $u \in H$, $w_1 \in T_1(u)$, $w_2 \in T_2(u)$ such that

$$0 \in N(w_1, w_2) + M(g(u), u). \tag{2.16}$$

It is now clear that for a suitable choice of maps involved in the formulation of problem (2.14), we can drive many known variational inclusions considered and studied in the literature.

In connection with problem (2.14), we consider the following resolvent equation problem.

Find $z, u \in E$, $w_i \in T_i(u)$, $i = 1, 2, \ldots, \infty$; $a \in B(u)$, $x \in C(u)$, $y \in D(u)$ such that

$$N(w_1, w_2, \ldots) - W(x, y) + m(a) + \rho^{-1} R^{\rho, A}_{\eta, M(\cdot, u)}(z) = 0, \tag{2.17}$$

where ρ is a constant and $R^{\rho, A}_{\eta, M(\cdot, u)} = I - A(J^{\rho, A}_{\eta, M(\cdot, u)})$, where $A[J^{\rho, A}_{\eta, M(\cdot, u)}(z)] = [A(J^{\rho, A}_{\eta, M(\cdot, u)})](z)$ and I is the identity mapping. Equation (2.17) is called the resolvent equation problem.

In support of problem (2.17), we have the following example.

Example 2.9. Let us suppose that $E = \mathbb{R}$, $T_i(u) = [-i, i]$, $i = 1, 2, \ldots, \infty$, $C(u) = \{\pi/2\}$, $B(u) = [0, 1]$, and $D(u) = \{1\}$.

We define for $w_i \in T_i(u)$, $i = 1, 2, \ldots, \infty$, $a \in B(u)$, $x \in C(u)$ and $y \in D(u)$.

(i) $N(w_1, w_2, \ldots) = \min\{-1, \sin w_1, \sin w_2, \ldots\}$,

(ii) $m(a) = \sin^{-1} a + \cos^{-1} a$,

(iii) $W(x, y) = xy$,

(iv) $A(x) = x - 1$, for all $x \in \mathbb{R}$,

(v) $M(\cdot, x) = 1$, for all $x \in \mathbb{R}$,

Then, for $\rho = 1$, it is easy to check that the resolvent equation problem (2.17) is satisfied.

3. An Iterative Algorithm and Convergence Result

We mention the following equivalence between the problem (2.14) and a fixed point problem which can be easily proved by using the definition of resolvent operator.

Lemma 3.1. *Let* $(u, a, x, y, (w_1, w_2, \ldots))$ *where* $u \in E$, $w_i \in T_i(u)$, $i = 1, 2, \ldots, \infty$, $a \in B(u)$, $x \in C(u)$, *and* $y \in D(u)$, *is a solution of* (2.14) *if and only if it is a solution of the following equation:*

$$g(u) = m(a) + J^{\rho,A}_{\eta,M(\cdot,u)}\left[A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right]. \tag{3.1}$$

Now, we show that the problem (2.14) is equivalent to a resolvent equation problem.

Lemma 3.2. *Let* $u \in E$, $w_i \in T_i(u)$, $i = 1, 2, \ldots, \infty$, $a \in B(u)$, $x \in C(u)$, $y \in D(u)$, *then the following are equivalent:*

(i) $(u, a, x, y, (w_1, w_2, \ldots))$ *is a solution of variational inclusion problem* (2.14),

(ii) $(z, u, a, x, y, (w_1, w_2, \ldots))$ *is a solution of the problem* (2.17),

where

$$\begin{aligned} z &= A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}, \\ g(u) &= m(a) + J^{\rho,A}_{\eta,M(\cdot,u)}\left[A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right]. \end{aligned} \tag{3.2}$$

Proof. Let $(u, a, x, y, (w_1, w_2, \ldots))$ be a solution of the problem (2.14), then by Lemma 3.1, it is a solution of the problem

$$g(u) = m(a) + J^{\rho,A}_{\eta,M(\cdot,u)}\left[A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right], \tag{3.3}$$

using the fact that

$$\begin{aligned}
R^{\rho,A}_{\eta,M(\cdot,u)} &= I - A\left(J^{\rho,A}_{\eta,M(\cdot,u)}\right), \\
R^{\rho,A}_{\eta,M(\cdot,u)}(z) &= R^{\rho,A}_{\eta,M(\cdot,u)}\left[A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right] \\
&= \left(I - A\left(J^{\rho,A}_{\eta,M(\cdot,u)}\right)\right)\left[A(g(u) - m(a)) - \{\rho N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right] \\
&= A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\} \\
&\quad - A\left(J^{\rho,A}_{\eta,M(\cdot,u)}\left[A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right]\right) \\
&= A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\} \\
&\quad - A(g(u) - m(a)),
\end{aligned} \tag{3.4}$$

which implies that

$$N(w_1, w_2, \ldots) - W(x, y) + m(a) + \rho^{-1} R^{\rho,A}_{\eta,M(\cdot,u)}(z) = 0, \tag{3.5}$$

with

$$z = A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}, \tag{3.6}$$

that is, $(z, u, a, x, y, (w_1, w_2, \ldots))$ is a solution of problem (2.17).

Conversely, let $(z, u, a, x, y, (w_1, w_2, \ldots))$ be a solution of problem (2.17), then

$$\begin{aligned}
\rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\} &= -R^{\rho,A}_{\eta,M(\cdot,u)}(z), \\
\rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\} &= A\left[J^{\rho,A}_{\eta,M(\cdot,u)}(z)\right] - z,
\end{aligned} \tag{3.7}$$

from (3.2) and (3.7), we have

$$\begin{aligned}
&\rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\} \\
&\quad = A\left[J^{\rho,A}_{\eta,M(\cdot,u)}\left(A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right)\right] \\
&\qquad - \left[A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right],
\end{aligned} \tag{3.8}$$

which implies that

$$g(u) = m(a) + J^{\rho,A}_{\eta,M(\cdot,u)}\left[A\left(g(u) - m(a)\right) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\}\right], \tag{3.9}$$

that is, $(u, a, x, y, (w_1, w_2, \ldots))$ is a solution of (2.14). □

We now invoke Lemmas 3.1 and 3.2 to suggest the following iterative algorithm for solving resolvent equation problem (2.17).

Algorithm 3.3. For a given $z_0, u_0 \in E$, $w_i^0 \in T_i(u_0)$, $i = 1, 2, \ldots, \infty$, $a_0 \in B(u_0)$, $x_0 \in C(u_0)$, and $y_0 \in D(u_0)$. Let

$$z_1 = A\left(g(u_0) - m(a_0)\right) - \rho\left\{N\left(w_1^0, w_2^0, \ldots\right) - W\left(x_0, y_0\right) + m(a_0)\right\}. \tag{3.10}$$

Take $z_1, u_1 \in E$ such that

$$g(u_1) = m(a_1) + J^{\rho,A}_{\eta,M(\cdot,u_1)}(z_1). \tag{3.11}$$

Since for each $i, w_i^0 \in T_i(u_0)$, $a_0 \in B(u_0)$, $x_0 \in C(u_0)$, and $y_0 \in D(u_0)$ by Nadler's theorem [14] there exist $w_i^1 \in T_i(u_1)$, $a_1 \in B(u_1)$, $x_1 \in C(u_1)$, and $y_1 \in D(u_1)$ such that

$$\begin{aligned} \left\|w_i^0 - w_i^1\right\| &\leq \mathscr{H}(T_i(u_0), T_i(u_1)), \\ \|a_0 - a_1\| &\leq \mathscr{H}(B(u_0), B(u_1)), \\ \|x_0 - x_1\| &\leq \mathscr{H}(C(u_0), C(u_1)), \\ \|y_0 - y_1\| &\leq \mathscr{H}(D(u_0), D(u_1)), \end{aligned} \tag{3.12}$$

where $\mathscr{H}$ is the Housdorff metric on $CB(E)$.

Let

$$z_2 = A\left(g(u_1) - m(a_1)\right) - \rho\left\{N\left(w_1^1, w_2^1, \ldots\right) - W\left(x_1, y_1\right) + m(a_1)\right\}, \tag{3.13}$$

and take any $u_2 \in E$ such that

$$g(u_2) = m(a_2) + J^{\rho,A}_{\eta,M(\cdot,u_2)}(z_2). \tag{3.14}$$

Continuing the above process inductively, we obtain the following.

For any $z_0, u_0 \in E$, $w_i^0 \in T_i(u_0)$, $i = 1, 2, \ldots, \infty$, $a_0 \in B(u_0)$, $x_0 \in C(u_0)$, and $y_0 \in D(u_0)$. Compute the sequences $\{z_n\}, \{u_n\}, \{w_i^n\}$, $i = 1, 2, \ldots, \infty$, $\{a_0\}, \{x_0\}, \{y_0\}$ by the following iterative schemes:

$$\text{(i)}\quad g(u_n) = m(a_n) + J^{\rho,A}_{\eta,M(\cdot,u_n)}(z_n), \tag{3.15}$$

$$\text{(ii)}\quad a_n \in B(u_n), \quad \|a_n - a_{n+1}\| \le \mathscr{H}(B(u_n), B(u_{n+1})), \tag{3.16}$$

$$\text{(iii)}\quad x_n \in C(u_n), \quad \|x_n - x_{n+1}\| \le \mathscr{H}(C(u_n), C(u_{n+1})), \tag{3.17}$$

$$\text{(iv)}\quad y_n \in D(u_n), \quad \|y_n - y_{n+1}\| \le \mathscr{H}(D(u_n), D(u_{n+1})), \tag{3.18}$$

$$\text{(v)}\quad \text{for each } i, w_i^n \in T_i(u_n), \quad \left\|w_i^n - w_i^{n+1}\right\| \le \mathscr{H}(T_i(u_n), T_i(u_{n+1})), \tag{3.19}$$

$$\text{(vi)}\quad z_{n+1} = A\big(g(u_{\text{n}}) - m(a_n)\big) - \rho\big\{N\big(w_1^n, w_2^n, \ldots\big) - W\big(x_n, y_n\big) + m(a_n)\big\}, \tag{3.20}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

Theorem 3.4. *Let E be a real Banach space. Let $T_i, B, C, D : E \to CB(E)$ be $\mathscr{H}$-Lipschitz continuous mapping with constants $\delta_i, \alpha, t, \gamma$, respectively. Let $N = E^\infty = E \times E \times E \cdots \to E$ be Lipschitz continuous with constant β_i, let $A, g, m : E \to E$ be Lipschitz continuous with constants $\lambda_A, \lambda_g, \lambda_m$, respectively, and let A be r-strongly η-accretive mapping. Suppose that $\eta, W : E \times E \to E$ are mappings such that η is Lipschitz continuous with constant τ and W is Lipschitz continuous in both the argument with constant λ_{W_1} and λ_{W_2}, respectively. Let $M : E \times E \to 2^E$ be (A, η)-accretive mapping in the first argument such that the following holds for $\mu > 0$:*

$$\left\|J^{\rho,\eta}_{M(\cdot,u_n)}(z_n) - J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_n)\right\| \le \mu\|u_n - u_{n-1}\|. \tag{3.21}$$

Suppose there exists a $\rho > 0$ such that

$$\begin{aligned} &\lambda_A\lambda_g + \lambda_m\alpha(\lambda_A + \rho) + \rho\sum_{i=1}^{\infty}\beta_i\delta_i + \rho\big(\lambda_{W_1}t + \lambda_{W_2}\gamma\big) \\ &\quad < \frac{(r - \rho m)}{\tau}\sqrt{1 - \big(\lambda_m^2\alpha^2 + \mu^2 - 2k\big)}, \quad m < \frac{r}{\rho},\ \lambda_m^2\alpha^2 < 1 + 2k - \mu^2. \end{aligned} \tag{3.22}$$

Then, there exist $z, u, \in E$, $a \in B(E)$, and $x \in C(E)$, $y \in D(E)$, and $w_i \in T_i(u)$ that satisfy resolvent equation problem (2.17). The iterative sequences $\{z_n\}, \{u_n\}, \{a_n\}$ $\{x_n\}, \{y_n\}$, and $\{w_i^n\}$, $i = 1, 2, \ldots, \infty$, $n = 0, 1, \ldots$ generated by Algorithm 3.3 converge strongly to z, u, a, x, y, w_i, respectively.

Proof. From Algorithm 3.3, we have

$$\begin{aligned}\|z_{n+1}-z_n\| &= \Big\|A(g(u_n)-m(a_n))-\rho\{N(w_1^n,w_2^n,\ldots)-W(x_n,y_n)+m(a_n)\}\\ &\quad -\Big[A(g(u_{n-1})-m(a_{n-1}))\\ &\quad -\rho\Big\{N\Big(w_1^{n-1},w_2^{n-1},\ldots\Big)-W(x_{n-1},y_{n-1})+m(a_{n-1})\Big\}\Big]\Big\|\\ &\le \|A(g(u_n)-m(a_n))-(A(g(u_{n-1})-m(a_{n-1}))\|\\ &\quad +\rho\Big\|N(w_1^n,w_2^n,\ldots)-N\Big(w_1^{n-1},w_2^{n-1},\ldots\Big)\Big\|\\ &\quad +\rho\|W(x_n,y_n)-W(x_{n-1},y_{n-1})\|+\rho\|m(a_n)-m(a_{n-1})\|.\end{aligned} \tag{3.23}$$

By using the Lipschitz continuty of A, g, and m with constants λ_A, λ_g, and λ_m, respectively, and by Algorithm 3.3, we have

$$\begin{aligned}&\|A(g(u_n)-m(a_n))-(A(g(u_{n-1})-m(a_{n-1})))\|\\ &\quad\le \lambda_A\|g(u_n)-g(u_{n-1})\|+\lambda_A\|m(a_n)-m(a_{n-1})\|\\ &\quad\le \lambda_A\lambda_g\|u_n-u_{n-1}\|+\lambda_A\lambda_m\|a_n-a_{n-1}\|\\ &\quad\le \lambda_A\lambda_g\|u_n-u_{n-1}\|+\lambda_A\lambda_m\mathscr{H}(B(u_n),B(u_{n-1}))\\ &\quad\le \lambda_A\lambda_g\|u_n-u_{n-1}\|+\lambda_A\lambda_m\alpha\|u_n-u_{n-1}\|\\ &\quad= (\lambda_A\lambda_g+\lambda_A\lambda_m\alpha)\|u_n-u_{n-1}\|.\end{aligned} \tag{3.24}$$

Since N is Lipschitz continuous in all the arguments with constant β_i, $i = 1,2,\ldots$, respectively, and using $\mathscr{H}$-Lipschitz continuity of T_i's with constant δ_i, we have

$$\begin{aligned}&\Big\|N(w_1^n,w_2^n,\ldots)-N\Big(w_1^{n-1},w_2^{n-1},\ldots\Big)\Big\|\\ &\quad= \Big\|N(w_1^n,w_2^n,\ldots)-N\Big(w_1^{n-1},w_2^n,\ldots\Big)+N\Big(w_1^{n-1},w_2^n,\ldots\Big)+\cdots\Big\|\\ &\quad\le \Big\|N(w_1^n,w_2^n,\ldots)-N\Big(w_1^{n-1},w_2^n,\ldots\Big)\Big\|\\ &\qquad+\Big\|N\Big(w_1^{n-1},w_2^n,\ldots\Big)-N\Big(w_1^{n-1},w_2^{n-1},\ldots\Big)\Big\|+\cdots\\ &\quad\le \beta_1\Big\|w_1^n-w_1^{n-1}\Big\|+\beta_2\Big\|w_2^n-w_2^{n-1}\Big\|+\cdots\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{\infty}\beta_i \left\| w_i^n - w_i^{n-1} \right\| \\
&\leq \sum_{i=1}^{\infty}\beta_i \mathscr{H}(T_i(u_n), T_i(u_{n-1})) \\
&\leq \sum_{i=1}^{\infty}\beta_i \delta_i \|u_n - u_{n-1}\|, \quad n = 0, 1, 2, \ldots .
\end{aligned} \tag{3.25}$$

Since W is a Lipschitz continuous in both the arguments with constant $\lambda_{W_1}, \lambda_{W_2}$ respectively, and C and D are $\mathscr{H}$-Lipschitz continuous with constant t and γ, respectively, we have

$$\begin{aligned}
\left\| W(x_n, y_n) - W(x_{n-1}, y_{n-1}) \right\| &\leq \lambda_{W_2} \left\| y_n - y_{n-1} \right\| + \lambda_{W_1} \|x_n - x_{n-1}\| \\
&\leq \lambda_{W_2}\gamma \|u_n - u_{n-1}\| + \lambda_{W_1} t \|u_n - u_{n-1}\| \\
&= \left(\lambda_{W_1} t + \lambda_{W_2}\gamma\right) \|u_n - u_{n-1}\|.
\end{aligned} \tag{3.26}$$

Combining (3.24), (3.25), and (3.26) with (3.23), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| \leq{}& \left(\lambda_A \lambda_g + \lambda_A \lambda_m \alpha\right) \|u_n - u_{n-1}\| + \rho \sum_{i=1}^{\infty} \beta_i \delta_i \|u_n - u_{n-1}\| \\
&+ \rho\left(\lambda_{W_1} t + \lambda_{W_2}\gamma\right) \|u_n - u_{n-1}\| + \rho \lambda_m \alpha \|u_n - u_{n-1}\| \\
={}& \left[\left(\lambda_A \lambda_g + \lambda_m \alpha(\lambda_A + \rho)\right) + \rho \sum_{i=1}^{\infty} \beta_i \delta_i + \rho\left(\lambda_{W_1} t + \lambda_{W_2}\gamma\right) \right] \|u_n - u_{n-1}\|.
\end{aligned} \tag{3.27}$$

By using Proposition 2.3 and k-strong accretiveness of g, we have

$$\begin{aligned}
\|u_n - u_{n-1}\|^2 ={}& \Big\| m(a_n) + J^{\rho,\eta}_{M(\cdot,u_n)}(z_n) - m(a_{n-1}) \\
&- J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_{n-1}) - \left[g(u_n) - u_n - \left(g(u_{n-1}) - u_{n-1}\right) \right] \Big\|^2 \\
\leq{}& \|m(a_n) - m(a_{n-1})\|^2 + \left\| J^{\rho,\eta}_{M(\cdot,u_n)}(z_n) - J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_{n-1}) \right\|^2 \\
&- 2\left\langle g(u_n) - u_n - \left(g(u_{n-1}) - u_{n-1}\right), j(u_n - u_{n-1}) \right\rangle \\
\leq{}& \|m(a_n) - m(a_{n-1})\|^2 \\
&+ \left\| J^{\rho,\eta}_{M(\cdot,u_n)}(z_n) - J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_n) + J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_n) - J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_{n-1}) \right\|^2 \\
&- 2\left\langle g(u_n) - u_n - \left(g(u_{n-1}) - u_{n-1}\right), j(u_n - u_{n-1}) \right\rangle,
\end{aligned}$$

$$
\begin{aligned}
\|u_n - u_{n-1}\|^2 \le{}& \lambda_m^2\alpha^2\|u_n - u_{n-1}\|^2 + \left\| J^{\rho,\eta}_{M(\cdot,u_n)}(z_n) - J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_n)\right\|^2 \\
&+ \left\| J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_n) - J^{\rho,\eta}_{M(\cdot,u_{n-1})}(z_{n-1})\right\|^2 \\
&- 2\left\langle g(u_n) - u_n - \left(g(u_{n-1}) - u_{n-1}\right), j(u_n - u_{n-1})\right\rangle \\
\le{}& \lambda_m^2\alpha^2\|u_n - u_{n-1}\|^2 + \mu^2\|u_n - u_{n-1}\|^2 \\
&+ \left(\frac{\tau}{r-\rho m}\right)^2\|z_n - z_{n-1}\|^2 - 2k\|u_n - u_{n-1}\|^2, \\
\|u_n - u_{n-1}\|^2 \le{}& \left(\lambda_m^2\alpha^2 + \mu^2 - 2k\right)\|u_n - u_{n-1}\|^2 + \left(\frac{\tau}{r-\rho m}\right)^2\|z_n - z_{n-1}\|^2, \\
\|u_n - u_{n-1}\|^2 \le{}& \frac{\left(\tau/(r-\rho m)\right)^2}{\left[1 - (\lambda_m^2\alpha^2 + \mu^2 - 2k)\right]}\|z_n - z_{n-1}\|^2, \\
\|u_n - u_{n-1}\| \le{}& \frac{\tau}{(r-\rho m)\sqrt{\left[1 - (\lambda_m^2\alpha^2 + \mu^2 - 2k)\right]}}\|z_n - z_{n-1}\|.
\end{aligned}
\tag{3.28}
$$

Using (3.28), (3.27) becomes

$$
\|z_{n+1} - z_n\| \le \frac{\left[\lambda_A\lambda_g + \lambda_m\alpha(\lambda_A + \rho) + \rho\sum_{i=1}^{\infty}\beta_i\delta_i + \rho\left(\lambda_{W_1}t + \lambda_{W_2}\gamma\right)\right]\tau}{(r-\rho m)\sqrt{1 - (\lambda_m^2\alpha^2 + \mu^2 - 2k)}}\|z_n - z_{n-1}\|,
\tag{3.29}
$$

$$
\text{that is, } \|z_{n+1} - z_n\| \le \theta\|z_n - z_{n-1}\|,
$$

where

$$
\theta = \frac{\left[\lambda_A\lambda_g + \lambda_m\alpha(\lambda_A + \rho) + \rho\sum_{i=1}^{\infty}\beta_i\delta_i + \rho\left(\lambda_{W_1}t + \lambda_{W_2}\gamma\right)\right]\tau}{(r-\rho m)\sqrt{1 - (\lambda_m^2\alpha^2 + \mu^2 - 2k)}}.
\tag{3.30}
$$

From (3.22), we have $\theta < 1$, and consequently $\{z_n\}$ is a Cauchy sequence in E. Since E is a Banach space, there exists $z \in E$ such that $z_n \to z$. From (3.28), we know that $\{u_n\}$ is also a Cauchy sequence in E. Therefore, there exists $u \in E$ such that $u_n \to u$. Since the mappings T_i's, B, C and D are $\mathscr{H}$-Lipschitz continuous, it follows from (3.16)–(3.19) of Algorithm 3.3 that $\{a_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{w_i^n\}$ are also Cauchy sequences. We can assume that $w_i^n \to w_i$, $a_n \to a$, $x_n \to x$, and $y_n \to y$.

Now, we prove that $w_i \in T_i(u)$. In fact, since $w_i^n \in T_i(u_i)$ and

$$\begin{aligned} d(w_i, T_i(u)) &\le \|w_i - w_i^n\| + d(w_i^n, T_i(u)) \\ &\le \|w_i - w_i^n\| + \max\left\{ \sup_{q_2 \in T_i(u_n)} d(q_2, T_i(u)), \sup_{q_1 \in T_i(u)} d(T_i(u), q_1) \right\} \\ &= \|w_i - w_i^n\| + \mathscr{H}(T_i(u_n), T_i(u)) \\ &\le \|w_i - w_i^n\| + \delta_i \|u_n - u_{n-1}\| \longrightarrow 0 \quad (n \longrightarrow \infty), \end{aligned} \tag{3.31}$$

which implies that $d(w_i, T_i(u)) = 0$. As $T_i(u) \in CB(E)$, we have $w_i \in T_i(u), i = 1, 2, \ldots \infty$.

Finally, by the continuity of A, g, m, N, and W and by Algorithm 3.3, it follows that

$$\begin{aligned} z_{n+1} &= A(g(u_n) - m(a_n)) - \rho\{N(w_1^n, w_2^n, \ldots) - W(x_n, y_n) + m(a)\}, \\ &\longrightarrow z = A(g(u) - m(a)) - \rho\{N(w_1, w_2, \ldots) - W(x, y) + m(a)\} \quad (n \longrightarrow \infty), \\ J_{M(\cdot, u_n)}^{\rho,\eta}(z_n) &= g(u_n) - m(a_n) \longrightarrow g(u) - m(a) = J_{M(\cdot,u)}^{\rho,\eta}(z) \quad (n \longrightarrow \infty). \end{aligned} \tag{3.32}$$

From (3.32), and Lemma 3.2, it follows that

$$\begin{gathered} N(w_1, w_2, \ldots) - W(x, y) + m(a) + \rho^{-1}\left[z - A\left(J_{M(\cdot,u)}^{\rho,\eta}(z)\right)\right] = 0, \\ N(w_1, w_2, \ldots) - W(x, y) + m(a) + \rho^{-1} R_{M(\cdot,u)}^{\rho,\eta}(z) = 0, \end{gathered} \tag{3.33}$$

that is, $(z, u, a, x, y, (w_1, w_2, \ldots))$ is a solution of resolvent equation poblem (2.17). □

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