Research Article

On the Convergence Theorems for a Countable Family of Lipschitzian Pseudocontraction Mappings in Banach Spaces

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The purpose of this paper is to study the weak and strong convergence theorems of the implicit iteration process for a countable family of Lipschitzian pseudocontraction mappings in Banach spaces. The results presented in this paper extend and improve some recent results announced by some authors.

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|x\| = \|f\| \right\}, \quad x \in E. \tag{1.1}$$

It is well known that if E is smooth, that is, if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for all $x, y \in E$ with $\|x\| = \|y\| = 1$, then J is single valued.

Let $T : C \to C$ be a mapping. In the sequel, we denote $F(T)$ the set of fixed points of T. The strong convergence and weak convergence of any sequence are denoted by $\to$ and $\rightharpoonup$, respectively. For a given sequence $\{x_n\} \subset C$, we denote by $W_\omega(x_n)$ the weak ω-limit set defined by

$$W_\omega(x_n) = \{z \in C : \exists \{x_{n_i}\} \subset \{x_n\} \text{ s.t. } x_{n_i} \rightharpoonup z\}. \tag{1.3}$$

Definition 1.1. Let $T : C \to C$ be a mapping. T is said to be

(1) *L-Lipschitzian* if there exists an $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C, \tag{1.4}$$

(2) *pseudocontractive* [1, 2] if for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2, \tag{1.5}$$

and it is well known that condition (1.5) is equivalent to the following:

$$\|x - y\| \leq \|x - y + s[(I - Tx) - (I - Ty)]\|, \quad \forall s > 0, \ x, y \in C, \tag{1.6}$$

(3) *strongly pseudocontractive* if there exists a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that for any $x, y \in C$,

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \tag{1.7}$$

(4) *λ-strictly pseudocontractive in the terminology of Browder and Petryshyn (λ-strictly pseudocontractive, for short)* (see [3–5]) if there exists $\lambda > 0$ and $j(x - y) \in J(x - y)$ such that for any $x, y \in C$,

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2, \tag{1.8}$$

(5) *λ-demicontractive* if $F(T) \neq \emptyset$ and there exists a constant $\lambda > 0$ and $j(x - p) \in J(x - p)$ such that for any $x \in C, p \in F(T)$,

$$\langle Tx - p, j(x - p) \rangle \leq \|x - p\|^2 - \lambda\|x - Tx\|^2. \tag{1.9}$$

Remark 1.2. (1) From Definition 1.1, it is easy to see that each strongly pseudocontractive mapping and each strictly pseudocontractive mapping both are a special case of pseudo-contractive mapping. Furthermore, if T is a strictly pseudocontractive mapping with $F(T) \neq \emptyset$, then it is λ-demicontractive mapping.

(2) Each λ-strictly pseudocontractive mapping is $((1 + \lambda)/\lambda)$-lipschitzian and pseudocontractive.

Lemma 1.3 (see [6, 7]). *Let E be a real Banach space, let C be a nonempty closed convex subset of E, and let $T : C \rightarrow C$ be a continuous strongly pseudocontractive mapping. Then T has a unique fixed point in C.*

In 1974, Ishikawa [8] introduced an iterative method for finding a fixed point of Lipschitzian pseudocontractive mapping and proved the following.

Theorem 1.4 (see [8]). *Let C be a nonempty compact convex subset of a Hilbert space H, and let $T : C \rightarrow C$ be a Lipschitzian pseudocontractive mapping. For a fixed $x_0 \in C$, define a sequence $\{x_n\}$ by*

$$
\begin{aligned}
y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n,
\end{aligned}
\tag{1.10}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$;

(iii) $0 \leq \alpha_n \leq \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

It is natural to ask a question of whether or not the simple Mann iteration defined by $x_0 \in C$ and

$$
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1
\tag{1.11}
$$

can be used to obtain the same conclusion as of Theorem 1.4.

Recently, this question was resolved in the negative by Chidume and Matangadura [9]. They constructed an example of Lipschitzian pseudocontractive mapping defined on a compact convex subset of $\mathcal{R}^2$ that showed that Mann iteration sequence does not converge.

In 2007, Chidume et al. [10] proved a convergence theorem of the Mann iterations to a fixed point of a single strictly pseudocontractive mapping in Banach space. In 2010, Boonchari and Saejung [11] proved a convergence theorem of the Mann iterations to a fixed point of a countable family of λ-demicontractive mappings in Banach spaces.

On the other hand, in 2001, Xu and Ori [12] introduced the following implicit iteration process:

$$
\begin{aligned}
& x_0 \in K, \\
& x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad n \geq 1,
\end{aligned}
\tag{1.12}
$$

for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^{N}$ in a Hilbert space, where $T_n = T_{n(\bmod N)}$. In 2004, Osilike [13] extended the above sequence (1.12) from a class of nonexpansive mappings to more general class of strictly pseudocontractive mappings (cf. [14]). In 2006, Chen et al. [3] extended the results of Osilike [13] to more general Banach spaces.

In 2008, Zhou [5] extended the results of Chen et al. [3] from strictly pseudocontractive mapping extened to a finite family of Lipschitzian pseudocontractions $\{T_n\}_1^N$ and from q-uniformly smooth Banach spaces extended to uniformly convex Banach spaces with a Fréchet differentiable norm. Under suitable condition, he proved that the implicit iterative sequence (1.12) converges weakly to a common fixed point of $\{T_n\}_1^N$ (cf. [15, 16]).

Recently, Zhang [17] proved the weak convergence of implicit iteration process (1.12) for a countable family of Lipschitzian pseudocontractive mappings and strictly pseudocontractive semigroups in a general Banach space which extends and improves the corresponding results of Zhou [5], Chen et al. [3], Osilike [13], and Xu and Ori [12].

The purpose of this paper is to study the weak and strong convergence theorems of implicit iteration process (1.12) for a countable family of Lipschitzian pseudocontractive mappings and strictly pseudocontractive mappings in general Banach spaces. The result presented in this paper not only extend and improve the corresponding results of Zhou [5], Chen et al. [3], Osilike [13], Xu and Ori [12], and Zhang [17], but also replenish the corresponding results of Chidume et al. [10] and Boonchari and Saejung [11].

For this purpose, we recall some concepts and conclusions.

A Banach space E is said to be uniformly convex, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x, y \in E$ with $\|x\|, \|y\| \le 1$ and $\|x - y\| \ge \varepsilon$, $\|x + y\| \le 2(1 - \delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\|, \|y\| \le 1, \|x - y\| \ge \varepsilon \right\}, \quad \forall \varepsilon \in [0, 2]. \tag{1.13}$$

Lemma 1.5 (see [18]). *Let E be a uniformly convex Banach space with a modulus of convexity δ_E. Then $\delta_E : [0, 2] \to [0, 1]$ is continuous and increasing, $\delta_E(0) = 0$, $\delta_E(t) > 0$ for $t \in (0, 2]$, and*

$$\|cu + (1 - c)v\| \le 1 - 2\min\{c, 1 - c\}\delta_E(\|u - v\|), \tag{1.14}$$

for all $c \in [0, 1]$, and $u, v \in E$ with $\|u\|, \|v\| \le 1$.

A Banach space E is said to satisfy the *Opial condition* if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, the following inequality holds:

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \tag{1.15}$$

for any $y \in E$ with $y \ne x$. It is well known that each Hilbert space and l^p, $p > 1$ satisfy the Opial condition, while L^p does not unless $p = 2$.

Lemma 1.6 (see [5, 19]). *Let E be a real reflexive Banach space with the Opial condition. Let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a continuous pseudocontractive mapping. Then $I - T$ is demiclosed at zero; that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup y$ and $\|(I - T)x_n\| \to 0$, then $(I - T)y = 0$.*

2. Main Results

Lemma 2.1. *Let E be a smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $T_n : C \to C$ be L_n-Lipschitzian pseudocontractive mappings, $n = 1, 2, \ldots$ such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.12), and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\limsup_{n \to \infty} \alpha_n < 1$. Then the following conclusions hold:*

(i) *the sequence $\{x_n\}$ is well defined, and for each $p \in \mathcal{F}$, $\lim_{n \to \infty} \|x_n - p\|$ exists,*

(ii) $\lim_{n \to \infty} \|T_n x_n - x_n\| = 0$.

Proof. (i) For a fixed $u \in C$ and for each $n \geq 1$, define a mapping $S_n : C \to C$ by

$$S_n x = \alpha_n u + (1 - \alpha_n) T_n x, \quad x \in C. \tag{2.1}$$

It is easy to see that $S_n : C \to C$ is a continuous and strongly pseudocontractive mapping. By Lemma 1.3, there exists a unique fixed $x_n \in C$ such that

$$x_n = \alpha_n u + (1 - \alpha_n) T_n x_n. \tag{2.2}$$

This shows that the sequence $\{x_n\}$ is well defined.

Since E is smooth, the normalized duality mapping $J : E \to E^*$ is single valued. For each $p \in \mathcal{F}$, we have

$$
\begin{aligned}
\|x_n - p\|^2 &= \langle x_n - p, J(x_n - p) \rangle \\
&= \alpha_n \langle x_{n-1} - p, J(x_n - p) \rangle + (1 - \alpha_n) \langle T_n x_n - p, J(x_n - p) \rangle \\
&\leq \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1 - \alpha_n) \|x_n - p\|^2, \quad \forall n \geq 1.
\end{aligned}
\tag{2.3}
$$

This implies that

$$\|x_n - p\| \leq \|x_{n-1} - p\| \quad \forall n \geq 1. \tag{2.4}$$

Consequently, the limit $\lim_{n \to \infty} \|x_n - p\|$ exists.

(ii) By virtue of (1.6) and (1.12), we have

$$
\begin{aligned}
\|x_n - p\| &\leq \left\| x_n - p + \frac{1 - \alpha_n}{2\alpha_n}(x_n - T_n x_n) \right\| \\
&= \left\| x_n - p + \frac{1 - \alpha_n}{2}(x_{n-1} - T_n x_n) \right\| \\
&= \left\| \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n - p + \frac{1 - \alpha_n}{2}(x_{n-1} - T_n x_n) \right\| \\
&= \left\| \frac{x_{n-1} + x_n}{2} - p \right\| \\
&= \|x_{n-1} - p\| \cdot \left\| \frac{x_{n-1} - p}{2\|x_{n-1} - p\|} + \frac{x_n - p}{2\|x_{n-1} - p\|} \right\|.
\end{aligned}
\tag{2.5}
$$

Let $u = (x_{n-1} - p)/\|x_{n-1} - p\|$ and $v = (x_n - p)/\|x_{n-1} - p\|$. Then, we know that $\|u\| = 1$, $\|v\| \leq 1$ from (2.4). It follows from (2.5) and Lemma 1.5 that

$$\|x_n - p\| \leq \|x_{n-1} - p\| \left\{ 1 - \delta_E \left(\frac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|} \right) \right\}. \tag{2.6}$$

Therefore, we have

$$\|x_{n-1} - p\| \delta_E \left(\frac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|} \right) \leq \|x_{n-1} - p\| - \|x_n - p\|. \tag{2.7}$$

This implies that

$$\sum_{n=1}^{\infty} \|x_{n-1} - p\| \delta_E \left(\frac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|} \right) \leq \|x_0 - p\|. \tag{2.8}$$

Let $\lim_{n \to \infty} \|x_n - p\| = r$. If $r = 0$, then the conclusion of Lemma 2.1 is proved. If $r > 0$, then it follows from the property of the modulus of convexity δ_E that $\|x_{n-1} - x_n\| \to 0$ $(n \to \infty)$. Therefore, from (1.12) and the assumption $\limsup_{n \to \infty} \alpha_n < 1$, we have that

$$\|x_{n-1} - T_n x_n\| = \frac{1}{1 - \alpha_n} \|x_n - x_{n-1}\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{2.9}$$

This together with (1.12) implies that

$$\lim_{n \to \infty} \|x_n - T_n x_n\| = \lim_{n \to \infty} \alpha_n \|x_{n-1} - T_n x_n\| = 0. \tag{2.10}$$

This completes the proof of Lemma 2.1.   □

Theorem 2.2. *Let E be a smooth and uniformly convex Banach space satisfying the Opial condition, and let C be a nonempty closed convex subset of E. Let $T_n : C \to C$ be L_n-Lipschitzian pseudocontractive mappings, $n = 1, 2, \ldots$ such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $L := \sup_{n \geq 1} L_n < \infty$. Let $\{x_n\}$ be the sequence defined by (1.12), and let $\{\alpha_n\}$ be a sequence in (0, 1). If the following conditions are satisfied:*

   (i) $\limsup_{n \to \infty} \alpha_n < 1$;

   (ii) *for each $m \geq 1$, $\lim_{n \to \infty} \sup_{x \in D} \|T_m T_n x - T_n x\| = 0$, where*

$$D = \{ x \in E : \|x\| \leq \gamma \}, \quad \gamma = \sup_{n \geq 1} \|x_n\|, \tag{2.11}$$

then $\{x_n\}$ converges weakly to a point $u \in \mathcal{F}$.

Proof. From Lemma 2.1, we know that $\lim_{n\to\infty}\|x_n - p\|$ exists, $\lim_{n\to\infty}\|T_n x_n - x_n\| = 0$, and $\{x_n\}$ is bounded. Now, we prove that for each $m \geq 1$,

$$\lim_{n\to\infty}\|T_m x_n - x_n\| = 0. \tag{2.12}$$

In fact, for each $m \geq 1$, we have

$$\begin{aligned}
\|T_m x_n - x_n\| &\leq \|T_m x_n - T_m T_n x_n\| + \|T_m T_n x_n - T_n x_n\| \\
&\quad + \|T_n x_n - x_n\| \\
&\leq (1 + L_m)\|T_n x_n - x_n\| + \|T_m T_n x_n - T_n x_n\| \\
&\leq (1 + L)\|T_n x_n - x_n\| + \sup_{x\in D}\|T_m T_n x - T_n x\|,
\end{aligned} \tag{2.13}$$

where $L = \sup_{n\geq 1} L_n < \infty$. By using condition (ii) and (2.10), we have

$$\lim_{n\to\infty}\|T_m x_n - x_n\| = 0, \quad \text{for each } m \geq 1. \tag{2.14}$$

The conclusion (2.12) is proved.

Finally, we prove that $\{x_n\}$ converges weakly to a point $u \in \mathcal{F}$. Since E is uniformly convex, it is reflexive. Again since $\{x_n\} \subset C$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup u \in W_\omega(x_n)$. Hence, from (2.12), for any $m \geq 1$, we have

$$\|T_m x_{n_i} - x_{n_i}\| \longrightarrow 0 \quad (\text{as } n_i \longrightarrow \infty). \tag{2.15}$$

By virtue of Lemma 1.6, $u \in F(T_m)$, for all $m \geq 1$. This implies that

$$u \in \bigcap_{n\geq 1} F(T_n) \cap W_\omega(x_n). \tag{2.16}$$

Next, we prove that $W_\omega(x_n)$ is a singleton. Supposing the contrary, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup q \in W_\omega(x_n)$ and $q \neq u$. By the same method as above we can also prove that

$$q \in \bigcap_{n\geq 1} F(T_n) \cap W_\omega(x_n). \tag{2.17}$$

Taking $p = u$ and $p = q$ in (2.4), then we know that the following limits:

$$\lim_{n\to\infty}\|x_n - u\|, \qquad \lim_{n\to\infty}\|x_n - q\| \tag{2.18}$$

exist. Since E satisfies the Opial condition, we have

$$
\begin{aligned}
\lim_{n \to \infty} \|x_n - u\| = \limsup_{n \to \infty} \|x_{n_i} - u\| &< \limsup_{n_i \to \infty} \|x_{n_i} - q\| \\
&= \lim_{n \to \infty} \|x_n - q\| = \limsup_{n_j \to \infty} \|x_{n_j} - q\| \\
&< \limsup_{n_j \to \infty} \|x_{n_j} - u\| = \lim_{n \to \infty} \|x_n - u\|.
\end{aligned}
\tag{2.19}
$$

This is a contradiction, which shows that $q = u$. Hence

$$
W_\omega(x_n) = \{u\} \subset \mathcal{F} := \bigcap_{n \geq 1} F(T_n).
\tag{2.20}
$$

This implies that the sequence $\{x_n\}$ converges weakly to u. This completes the proof of Theorem 2.2. $\qquad\square$

Next we establish a weak convergence theorem for a countable family of strictly pseudocontractive mappings.

Theorem 2.3. *Let E be a smooth and reflexive Banach space satisfying the Opial condition, and let C be a nonempty closed convex subset of E. Let $T_n : C \to C$, $n = 1, 2, \ldots$ be a λ_n-strictly pseudocontractive mapping with $\lambda := \inf_{n \geq 1} \lambda_n > 0$ and $\mathcal{F} := \bigcap_{n \geq 1} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.12), and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. If the following conditions are satisfied:*

(i) $\limsup_{n \to \infty} \alpha_n < 1$;

(ii) $\lim_{n \to \infty} (\lambda_n / \alpha_n) = K$, *where K is a positive constant;*

(iii) *for each $m \geq 1$, $\lim_{n \to \infty} \sup_{x \in D} \|T_m T_n x - T_n x\| = 0$, where*

$$
D = \{x \in E : \|x\| \leq \gamma\}, \quad \gamma = \sup_{n \geq 1} \|x_n\|,
\tag{2.21}
$$

then $\{x_n\}$ converges weakly to a point $u \in \mathcal{F}$.

Proof. It follows from (1.8) and (1.12) that for any given $p \in \mathcal{F}$,

$$
\begin{aligned}
\|x_n - p\|^2 &= \langle x_n - p, J(x_n - p) \rangle \\
&= \alpha_n \langle x_{n-1} - p, J(x_n - p) \rangle + (1 - \alpha_n) \langle T_n x_n - p, J(x_n - p) \rangle \\
&\leq \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1 - \alpha_n) \|x_n - p\|^2 \\
&\quad - \lambda_n (1 - \alpha_n) \|x_n - T_n x_n\|^2,
\end{aligned}
\tag{2.22}
$$

which implies that

$$
\|x_n - p\|^2 \leq \|x_{n-1} - p\| \|x_n - p\| - \frac{\lambda_n}{\alpha_n} (1 - \alpha_n) \|x_n - T_n x_n\|^2.
\tag{2.23}
$$

This shows that

$$\|x_n - p\| \leq \|x_{n-1} - p\|, \quad \forall n \geq 1. \tag{2.24}$$

Therefore the limit $\lim_{n \to \infty} \|x_n - p\|$ exists and so $\{\|x_n\|\}$ is bounded. Denote $\beta = \sup_{n \geq 0} \|x_n - p\|$. From (2.23), we have

$$\frac{\lambda_n}{\alpha_n}(1 - \alpha_n)\|x_n - T_n x_n\|^2 \leq \beta\{\|x_{n-1} - p\| - \|x_n - p\|\}. \tag{2.25}$$

Letting $n \to \infty$ and taking the limit on the both sides of (2.25) and by using condition (i) and condition (ii), we have

$$\lim_{n \to \infty} \|x_n - T_n x_n\| = 0. \tag{2.26}$$

Furthermore by the assumption that for each $n \geq 1$, $T_n : C \to C$ is λ_n-strictly pseudocontractive. From Remark 1.2-(2), it follows that T_n is $(1 + \lambda_n)/\lambda_n$-Lipschitzian and pseudocontractive. Therefore for each $n \geq 1$, T_n is $(1 + 1/\lambda)$-Lipschitzian and pseudocontractive, where $\lambda = \inf_{n \geq 1} \lambda_n$. By the same method as given in the proof of Theorem 2.2, from (2.26) and condition (iii), we can prove that $\{x_n\}$ converges weakly to some point $u \in \mathcal{F}$. This completes the proof of Theorem 2.3. □

Remark 2.4. Theorems 2.2 and 2.3 extend and improve the corresponding results of Chen et al. [3], Osilike [13], Xu and Ori [12], Zhou [5], and Zhang [17].

Next we establish a strong convergence theorem for a countable family of Lipschitzian pseudocontractive mappings.

Theorem 2.5. *Let E be a smooth and uniformly convex Banach space satisfying the Opial condition, and let C be a nonempty closed convex subset of E. Let $T_n : C \to C$ be L_n-Lipschitzian pseudocontractive mappings, $n = 1, 2, \ldots$ such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $L := \sup_{n \geq 1} L_n < \infty$. Let $\{x_n\}$ be the sequence defined by (1.12) and $\{\alpha_n\}$ be a sequence in $(0, 1)$. If the following conditions are satisfied:*

(i) $\limsup_{n \to \infty} \alpha_n < 1$;

(ii) for each $m \geq 1$, $\lim_{n \to \infty} \sup_{x \in D} \|T_m T_n x - T_n x\| = 0$, where

$$D = \{x \in E : \|x\| \leq \gamma\}, \quad \gamma = \sup_{n \geq 1} \|x_n\|, \tag{2.27}$$

(iii) there exists a compact subset $K \subset C$ such that for each $m \geq 1$, $T_m(C) \subset K$,

then $\{x_n\}$ converges strongly to a point $u \in \mathcal{F}$.

Proof. Since $\{x_n\} \subset C$, by condition (iii), for each $m \geq 1$, $T_m(\{x_n\}) \subset K$. Since K is compact, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\lim_{n_i \to \infty} T_m x_{n_i} \longrightarrow u \in C. \tag{2.28}$$

Hence from (2.12), we have that $\lim_{n_i \to \infty} x_{n_i} = u$. Therefore, we have

$$\|T_m u - u\| = \lim_{n_i \to \infty} \{\|T_m u - T_m x_{n_i}\| + \|T_m x_{n_i} - x_{n_i}\| + \|x_{n_i} - u\|\}$$

$$\leq \lim_{n_i \to \infty} \{(1 + L_m)\|x_{n_i} - u\| + \|T_m x_{n_i} - x_{n_i}\|\} \qquad (2.29)$$

$$= 0.$$

This implies that $u \in F(T_m)$, for all $m \geq 1$, that is, $u \in \mathcal{F}$ and $x_{n_i} \to u$. From Lemma 2.1(i), it follows that $\{x_n\}$ converges strongly to a point $u \in \mathcal{F}$. This completes the proof of Theorem 2.5. $\qquad \square$

Theorem 2.6. *Let E be a smooth and reflexive Banach space satisfying the Opial condition, and let C be a nonempty closed convex subset of E. Let $T_n : C \to C$, $n = 1, 2, \ldots$ be a λ_n-strictly pseudocontractive mapping with $\lambda := \inf_{n \geq 1} \lambda_n > 0$ and $\mathcal{F} := \bigcap_{n \geq 1} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.12), and let $\{\alpha_n\}$ be a sequence in $(0, 1)$. If the following conditions are satisfied:*

(i) $\limsup_{n \to \infty} \alpha_n < 1$;

(ii) $\lim_{n \to \infty}(\lambda_n / \alpha_n) = K$, *where K is a positive constant;*

(iii) *for each $m \geq 1$, $\lim_{n \to \infty} \sup_{x \in D} \|T_m T_n x - T_n x\| = 0$, where*

$$D = \{x \in E : \|x\| \leq \gamma\}, \quad \gamma = \sup_{n \geq 1} \|x_n\|, \qquad (2.30)$$

(iv) *there exists a compact subset $K \subset C$ such that for each $m \geq 1$, $T_m(C) \subset K$,*

then $\{x_n\}$ converges strongly to a point $u \in \mathcal{F}$.

Remark 2.7. Theorems 2.5 and 2.6 improve and extend the corresponding results of Boonchari and Saejung [11], Chidume et al. [10], Chen et al. [3], Osilike [13], Xu and Ori [12], Zhang [17], and Zhou [5].

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