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Research Article

System of General Variational Inequalities Involving Different Nonlinear Operators Related to Fixed Point Problems and Its Applications

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By using the projection methods, we suggest and analyze the iterative schemes for finding the approximation solvability of a system of general variational inequalities involving different nonlinear operators in the framework of Hilbert spaces. Moreover, such solutions are also fixed points of a Lipschitz mapping. Some interesting cases and examples of applying the main results are discussed and showed. The results presented in this paper are more general and include many previously known results as special cases.

1. Introduction

The originally variational inequality problem, introduced by Stampacchia [1], in the early sixties, has had a great impact and influence in the development of almost all branches of pure and applied sciences and has witnessed an explosive growth in theoretical advances, algorithmic development. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various algorithms for solving (generalized) variational inequalities and related optimization. It is well known that the variational inequality problems are equivalent to the fixed point problems. This alternative equivalent formulation is very important from the numerical analysis point of view and has played a significant part in several numerical methods for solving variational inequalities and complementarity; see [2, 3]. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. It is also worth noting that the projection methods have been applied widely to problems arising especially from complementarity, convex quadratic programming, and variational problems.

On the other hand, in 1985, Pang [4] studied the variational inequality problem on the product sets, by decomposing the original variational inequality into a system of variational inequalities, and discussed the convergence of method of decomposition for system of variational inequalities. Moreover, he showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem, and the general equilibrium programming problem, can be uniformly modelled as a variational inequality defined on the product sets. Later, it was noticed that variational inequality over product sets and the system of variational inequalities both are equivalent; see [4–7] for applications. Since then many authors, see, for example, [8–11], studied the existence theory of various classes of system of variational inequalities by exploiting fixed point theorems and minimax theorems. Recently, Verma [12] introduced a new system of nonlinear strongly monotone variational inequalities and studied the approximate solvability of this system based on a system of projection methods. Additional research on the approximate solvability of a system of nonlinear variational inequalities is according to Chang et al. [13], Cho et al. [14], Nie et al. [15], Noor [16], Petrot [17], Suantai and Petrot [18], Verma [19, 20], and others.

Motivated by the research works going on this field, in this paper, the methods for finding the common solutions of a system of general variational inequalities involving different nonlinear operators and fixed point problem are considered, via the projection method, in the framework of Hilbert spaces. Since the problems of a system of general variational inequalities and fixed point are both important, the results presented in this paper are useful and can be viewed as an improvement and extension of the previously known results appearing in the literature, which mainly improves the results of Chang et al. [13] and also extends the results of Huang and Noor [21], Verma [20] to some extent.

2. Preliminaries

Let C be a closed convex subset of real Hilbert H, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively.

We begin with some basic definitions and well-known results.

Definition 2.1. A nonlinear mapping $S : H \to H$ is said to be a *κ-Lipschitzian mapping* if there exists a positive constant κ such that

$$\|Sx - Sy\| \leq \kappa \|x - y\|, \quad \forall x, y \in H. \tag{2.1}$$

In the case $\kappa = 1$, the mapping S is known as a *nonexpansive mapping*. If S is a mapping, we will denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in H : Sx = x\}$.

Let C be a nonempty closed convex subset of H. It is well known that, for each $z \in H$, there exists a unique nearest point in C, denoted by $P_C z$, such that

$$\|z - P_C z\| \leq \|z - y\|, \quad \forall y \in C. \tag{2.2}$$

Such a mapping P_C is called the *metric projection* of H onto C. We know that P_C is nonexpansive. Furthermore, for all $z \in H$ and $u \in C$,

$$u = P_C z \Longleftrightarrow \langle u - z, w - u \rangle \geq 0, \quad \forall w \in C. \tag{2.3}$$

For the nonlinear operators $T, g : H \to H$, the *general variational inequality problem* (write GVI(T, g, C)) is to find $u \in H$ such that $g(u) \in C$ and

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall g(v) \in C. \tag{2.4}$$

The inequality of the type (2.4) was introduced by Noor [22]. It has been shown that a large class of unrelated odd-order and nonsymmetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, ecology, physical, mathematical, engineering, and physical sciences can be studied in the unified framework of the problem (2.4); see [22–24] and the references therein. We remark that, if the operator g is the identity operator, the problem (2.4) is nothing but the originally variational inequality problem, which was originally introduced and studied by Stampacchia [1].

Applying (2.3), one can obtain the following result.

Lemma 2.2. *Let C be a closed convex set in H such that $C \subset g(H)$. Then $u \in H$ is a solution of the problem* (2.4) *if and only if $g(u) = P_C[g(u) - \rho Tu]$, where $\rho > 0$ is a constant.*

It is clear, in view of Lemma 2.2, that the variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation is suggest in the study of the variational inequalities and related optimization problems.

Let $T_i, g_i : H \to H$ be nonlinear operator, and let r_i be a fixed positive real number, for each $i = 1, 2, 3$. Set $\Xi = \{T_1, T_2, T_3\}$ and $\Lambda = \{g_1, g_2, g_3\}$. The *system of general variational inequalities involving three different nonlinear operators* generated by r_1, r_2, and r_3 is defined as follows.

Find $(x^*, y^*, z^*) \in H \times H \times H$ such that

$$\begin{aligned} &\langle r_1 T_1 y^* + g_1(x^*) - g_1(y^*), g_1(x) - g_1(x^*) \rangle \geq 0, \quad \forall g_1(x) \in C, \\ &\langle r_2 T_2 z^* + g_2(y^*) - g_2(z^*), g_2(x) - g_2(y^*) \rangle \geq 0, \quad \forall g_2(x) \in C, \\ &\langle r_3 T_3 x^* + g_3(z^*) - g_3(x^*), g_3(x) - g_3(z^*) \rangle \geq 0, \quad \forall g_3(x) \in C. \end{aligned} \tag{2.5}$$

We denote by SGVID(Ξ, Λ, C) the set of all solutions (x^*, y^*, z^*) of the problem (2.5).

By using (2.3), we see that the problem (2.5) is equivalent to the following projection problem:

$$\begin{aligned} g_1(x^*) &= P_C[g_1(y^*) - r_1 T_1 y^*], \\ g_2(y^*) &= P_C[g_2(z^*) - r_2 T_2 z^*], \\ g_3(z^*) &= P_C[g_3(x^*) - r_3 T_3 x^*], \end{aligned} \tag{2.6}$$

provided $C \subset g_i(H)$ for each $i = 1, 2, 3$.

We now discuss several special cases of the problem (2.5).

(i) If $g_1 = g_2 = g_3 = g$, then the system (2.5) reduces to the problem of finding $(x^*, y^*, z^*) \in H \times H \times H$ such that

$$\begin{aligned}
&\langle r_1 T_1 y^* + g(x^*) - g(y^*), g(x) - g(x^*) \rangle \geq 0, \quad \forall g(x) \in C, \\
&\langle r_2 T_2 z^* + g(y^*) - g(z^*), g(x) - g(y^*) \rangle \geq 0, \quad \forall g(x) \in C, \\
&\langle r_3 T_3 x^* + g(z^*) - g(x^*), g(x) - g(z^*) \rangle \geq 0, \quad \forall g(x) \in C.
\end{aligned} \tag{2.7}$$

We denote by SGVID(Ξ, g, C) the set of all solutions (x^*, y^*, z^*) of the problem (2.7).

(ii) If $T_1 = T_2 = T_3 = T$, then the system (2.7) reduces to the following *system of general variational inequalities* , (write SGVI(T, g, C), for shot): find $x^*, y^*, z^* \in H$ such that

$$\begin{aligned}
&\langle r_1 T y^* + g(x^*) - g(y^*), g(x) - g(x^*) \rangle \geq 0, \quad \forall g(x) \in C, \\
&\langle r_2 T z^* + g(y^*) - g(z^*), g(x) - g(y^*) \rangle \geq 0, \quad \forall g(x) \in C, \\
&\langle r_3 T x^* + g(z^*) - g(x^*), g(x) - g(z^*) \rangle \geq 0, \quad \forall g(x) \in C.
\end{aligned} \tag{2.8}$$

(iii) If $g = I$ (:= the identity operator), then, from the problem (2.7), we have the following *system of variational inequalities involving three different nonlinear operators* (write SVID(Ξ, C), for shot): find $(x^*, y^*, z^*) \in H \times H \times H$ such that

$$\begin{aligned}
&\langle r_1 T_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle r_2 T_2 z^* + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle r_3 T_3 x^* + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C.
\end{aligned} \tag{2.9}$$

(iv) If $T_1 = T_2 = T_3 = T$, then, from the problem (2.9), we have the following *system of variational inequalities* (write SVI(T, C), for shot): find $(x^*, y^*, z^*) \in H \times H \times H$ such that

$$\begin{aligned}
&\langle r_1 T y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle r_2 T z^* + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle r_3 T x^* + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C.
\end{aligned} \tag{2.10}$$

(v) If $r_3 = 0$, then the problem (2.10) reduces to the following problem: find $(x^*, y^*) \in H \times H$ such that

$$\begin{aligned}
&\langle r_1 T y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\
&\langle r_2 T x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C.
\end{aligned} \tag{2.11}$$

The problem (2.10) has been introduced and studied by Verma [20].

(vi) If $r_2 = 0$, then the problem (2.11) reduces to the following problem: find $x^* \in H$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{2.12}$$

which is, in fact, the originally variational inequality problem, introduced by Stampacchia [1].

This shows that, roughly speaking, for suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems. Consequently, the class of system of general variational inequalities involving three different nonlinear operators problems is more general and has had a great impact and influence in the development of several branches of pure, applied, and engineering sciences. For the recent applications, numerical methods, and formulations of variational inequalities, see [1–27] and the references therein.

Now we recall the definition of a class of mappings.

Definition 2.3. The mapping $T : H \to H$ is said to be *ν-strongly monotone* if there exists a constant $\nu > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \nu \|x - y\|^2, \quad \forall x, y \in H. \tag{2.13}$$

In order to prove our main result, the next lemma is very useful.

Lemma 2.4 (see [28]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n) a_n + b_n + c_n, \quad \forall n \geq n_0, \tag{2.14}$$

where n_0 is a nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\Sigma_{n=1}^{\infty} \lambda_n = \infty$, $b_n = \circ(\lambda_n)$, and $\Sigma_{n=1}^{\infty} c_n < \infty$, then $\lim_{n \to \infty} a_n = 0$.

Denotation. Let $\Omega \subset H \times H \times H$. In what follows, we will put the symbol $\Omega_1 := \{x \in H : (x, y, z) \in \Omega\}$.

3. Main Results

We begin with some observations which are related to the problem (2.5).

Remark 3.1. If $(x^*, y^*, z^*) \in \mathrm{SGVID}(\Xi, \Lambda, C)$, by (2.6), we see that

$$x^* = x^* - g_1(x^*) + P_C[g_1(y^*) - r_1 T_1 y^*], \tag{3.1}$$

provided $C \subset g_1(H)$. Consequently, if S is a Lipschitz mapping such that $x^* \in F(S)$, then it follows that

$$x^* = S(x^*) = S(x^* - g_1(x^*) + P_C[g_1(y^*) - r_1 T_1 y^*]). \tag{3.2}$$

The formulation (3.2) is used to suggest the following iterative method for finding common elements of two different sets, which are the solutions set of the problem (2.5) and the set of fixed points of a Lipschitz mapping. Of course, since we hope to use the formulation (3.2) as an initial idea for constructing the iterative algorithm, hence, from now on, we will assume that $g_i : H \to H$ satisfies a condition $C \subset g_i(H)$ for each $i = 1, 2, 3$. Now, in view of the formulations (2.6) and (3.2), we suggest the following algorithm.

Algorithm 1. Let r_1, r_2, and r_3 be fixed positive real numbers. For arbitrary chosen initial $x_0 \in \mathscr{H}$, compute the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ such that

$$\begin{aligned} g_3(z_n) &= P_C[g_3(x_n) - r_3T_3x_n], \\ g_2(y_n) &= P_C[g_2(z_n) - r_2T_2z_n], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S(x_n - g_1(x_n) + P_C[g_1(y_n) - r_1T_1y_n]), \end{aligned} \tag{3.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $S : H \to H$ is a mapping.

In what follows, if $T : H \to H$ is a ν-strongly monotone and μ-Lipschitz continuous mapping, then we define a function $\Phi_T : [0, +\infty) \to (-\infty, +\infty)$, associated with such a mapping T, by

$$\Phi_T(r) = \sqrt{1 - 2r\nu + r^2\mu^2}, \quad \forall r \in [0, +\infty). \tag{3.4}$$

We now state and prove the main results of this paper.

Theorem 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let $T_i : H \to H$ be ν_i-strongly monotone and μ_i-Lipschitz mapping, and let $g_i : H \to H$ be λ_i-strongly monotone and δ_i-Lipschitz mapping for $i = 1, 2, 3$. Let $S : H \to H$ be a τ-Lipschitz mapping such that $(\mathrm{SGVID}(\Xi, \Lambda, C))_1 \cap F(S) \neq \emptyset$. Put*

$$p_i = \sqrt{1 + \delta_i^2 - 2\lambda_i} \tag{3.5}$$

for each $i = 1, 2, 3$. If

(i) $p_i \in [0, (\mu_i - \sqrt{\mu_i^2 - \nu_i^2})/2\mu_i) \cup [(\mu_i + \sqrt{\mu_i^2 - \nu_i^2})/2\mu_i, 1)$, *for each $i = 1, 2, 3$,*

(ii) $|r_i - \nu_i/\mu_i^2| < \sqrt{\nu_i^2 - \mu_i^2(4p_i)(1 - p_i)}/\mu_i^2$, *for each $i = 1, 2, 3$,*

(iii) $\tau \prod_{i=1}^{3}((\Phi_{T_i}(r_i) + p_i)/(1 - p_i)) < 1$,

(iv) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by Algorithm 1 converge strongly to x^*, y^*, and z^*, respectively, such that $(x^*, y^*, z^*) \in \mathrm{SGVID}(\Xi, \Lambda, C)$ and $x^* \in F(S)$.

Proof. Let $(x^*, y^*, z^*) \in \text{SGVID}(\Xi, \Lambda, C)$ be such that $x^* \in F(S)$. By (2.6) and (3.2), we have

$$\begin{aligned} g_3(z^*) &= P_C\left[g_3(x^*) - r_3T_3x^*\right], \\ g_2(y^*) &= P_C\left[g_2(z^*) - r_2T_2z^*\right], \\ x^* &= (1-\alpha_n)x^* + \alpha_n S\left\{x^* - g_1(x^*) + P_C\left[g_1(y^*) - r_1T_1y^*\right]\right\}. \end{aligned} \tag{3.6}$$

Consequently, by (3.3), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n S\left(x_n - g_1(x_n) + P_C\left[g_1(y_n) - r_1T_1y_n\right]\right) - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\big\|S\left(x_n - g_1(x_n) + P_C\left[g_1(y_n) - r_1T_1y_n\right]\right) \\ &\qquad - S\left(x^* - g_1(x^*) + P_C\left[g_1(y^*) - r_1T_1y^*\right]\right)\big\| \\ &\le (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n\tau\left\{\|x_n - x^* - \left[g_1(x_n) - g_1(x^*)\right]\| + \big\|y_n - y^* - \left[g_1(y_n) - g_1(y^*)\right]\big\|\right. \\ &\qquad \left. + \big\|y_n - y^* - r_1\left[T_1y_n - T_1y^*\right]\big\|\right\}. \end{aligned} \tag{3.7}$$

By the assumption that T_1 is ν_1-strongly monotone and μ_1-Lipschitz mapping, we obtain

$$\begin{aligned} \big\|y_n - y^* - r_1[T_1y_n - T_1y^*]\big\|^2 &= \|y_n - y^*\|^2 - 2r_1\langle y_n - y^*, T_1y_n - T_1y^*\rangle + r_1^2\|T_1y_n - T_1y^*\|^2 \\ &\le \|y_n - y^*\|^2 - 2r_1\nu_1\|y_n - y^*\|^2 + r_1^2\mu_1^2\|y_n - y^*\|^2 \\ &= \left(1 - 2r_1\nu_1 + r_1^2\mu_1^2\right)\|y_n - y^*\|^2 \\ &= \left(\Phi_{T_1}(r_1)\right)^2\|y_n - y^*\|^2. \end{aligned} \tag{3.8}$$

Notice that

$$\begin{aligned} \|y_n - y^*\| &= \big\|y_n - y^* - \left[g_2(y_n) - g_2(y^*)\right] + \left[g_2(y_n) - g_2(y^*)\right]\big\| \\ &\le \big\|y_n - y^* - \left[g_2(y_n) - g_2(y^*)\right]\big\| + \|g_2(y_n) - g_2(y^*)\big\|. \end{aligned} \tag{3.9}$$

Now we consider,

$$\begin{aligned} \big\|y_n - y^* - [g_2(y_n) - g_2(y^*)]\big\|^2 &= \|y_n - y^*\|^2 - 2\langle y_n - y^*, g_2y_n - g_2y^*\rangle + \|g_2y_n - g_2y^*\|^2 \\ &\le \|y_n - y^*\|^2 - 2\left\{\lambda_2\|y_n - y^*\|^2\right\} + \delta_2^2\|y_n - y^*\|^2 \\ &= \left(1 - 2\lambda_2 + \delta_2^2\right)\|y_n - y^*\|^2 \\ &= (p_2)^2\|y_n - y^*\|^2, \end{aligned} \tag{3.10}$$

since g_2 is λ_2-strongly monotone and δ_2-Lipschitz mapping. And

$$\begin{aligned}\|g_2(y_n) - g_2(y^*)\| &= \left\|P_C\left[g_2(z_n) - r_2T_2z_n\right] - P_C\left[g_2(z^*) - r_2T_2z^*\right]\right\| \\ &\le \left\|g_2(z_n) - g_2(z^*) - r_2[T_2z_n - T_2z^*]\right\| \\ &\le \left\|z_n - z^* - \left[g_2(z_n) - g_2(z^*)\right]\right\| + \|z_n - z^* - r_2[T_2z_n - T_2z^*]\|.\end{aligned} \tag{3.11}$$

By the assumptions of T_2 and g_2, using the same lines as obtained in (3.8) and (3.10), we know that

$$\|z_n - z^* - r_2[T_2z_n - T_2z^*]\|^2 \le (\Phi_{T_2}(r_2))^2\|z_n - z^*\|^2, \tag{3.12}$$

$$\left\|z_n - z^* - [g_2(z_n) - g_2(z^*)]\right\|^2 \le (p_2)^2\|z_n - z^*\|^2, \tag{3.13}$$

respectively.

Substituting (3.12) and (3.13) into (3.11), we have

$$\|g_2(y_n) - g_2(y^*)\| \le \left(\Phi_{T_2}(r_2) + p_2\right)\|z_n - z^*\|. \tag{3.14}$$

Combining (3.9), (3.10), and (3.14) yields that

$$\|y_n - y^*\| \le p_2\|y_n - y^*\| + \left(\Phi_{T_2}(r_2) + p_2\right)\|z_n - z^*\|. \tag{3.15}$$

Observe that,

$$\begin{aligned}\|z_n - z^*\| &= \left\|z_n - z^* - \left[g_3(z_n) - g_3(z^*)\right] + \left[g_3(z_n) - g_3(z^*)\right]\right\| \\ &\le \left\|z_n - z^* - \left[g_3(z_n) - g_3(z^*)\right]\right\| + \left\|g_3(z_n) - g_3(z^*)\right\|,\end{aligned} \tag{3.16}$$

$$\left\|g_3(z_n) - g_3(z^*)\right\| \le \left\|x_n - x^* - \left[g_3(x_n) - g_3(x^*)\right]\right\| + \|x_n - x^* - r_3[T_3x_n - T_3x^*]\|. \tag{3.17}$$

Using the assumptions of T_3 and g_3, we know that

$$\|x_n - x^* - r_3[T_3x_n - T_3x^*]\|^2 \le (\Phi_{T_3}(r_3))^2\|x_n - x^*\|^2, \tag{3.18}$$

$$\left\|x_n - x^* - [g_3(x_n) - g_3(x^*)]\right\|^2 \le (p_3)^2\|x_n - x^*\|^2, \tag{3.19}$$

$$\left\|z_n - z^* - \left[g_3(z_n) - g_3(z^*)\right]\right\| \le p_3\|z_n - z^*\|, \tag{3.20}$$

respectively. Substituting (3.18) and (3.19) into (3.17), we have

$$\|g_3(z_n) - g_3(z^*)\| \le \left(\Phi_{T_3}(r_3) + p_3\right)\|x_n - x^*\|. \tag{3.21}$$

Combining (3.16), (3.20), and (3.21) yields that

$$\|z_n - z^*\| \le p_3\|z_n - z^*\| + \left(\Phi_{T_3}(r_3) + p_3\right)\|x_n - x^*\|. \tag{3.22}$$

This implies that

$$\|z_n - z^*\| \le \frac{(\Phi_{T_3}(r_3) + p_3)}{1 - p_3}\|x_n - x^*\|. \tag{3.23}$$

Substituting (3.23) into (3.15), we have

$$\|y_n - y^*\| \le p_2\|y_n - y^*\| + (\Phi_{T_2}(r_2) + p_2)\frac{(\Phi_{T_3}(r_3) + p_3)}{1 - p_3}\|x_n - x^*\|, \tag{3.24}$$

that is,

$$\|y_n - y^*\| \le \frac{(\Phi_{T_2}(r_2) + p_2)(\Phi_{T_3}(r_3) + p_3)}{(1 - p_2)(1 - p_3)}\|x_n - x^*\|. \tag{3.25}$$

By (3.8) and (3.25), we obtain

$$\left\|y_n - y^* - r_1[T_1 y_n - T_1 y^*]\right\| \le \frac{\Phi_{T_1}(r_1)(\Phi_{T_2}(r_2) + p_2)(\Phi_{T_3}(r_3) + p_3)}{(1 - p_2)(1 - p_3)}\|x_n - x^*\|. \tag{3.26}$$

On the other hand, since g_1 is λ_1-strongly monotone and δ_1-Lipschitz mapping, we can show that

$$\|x_n - x^* - [g_1(x_n) - g_1(x^*)]\| \le p_1\|x_n - x^*\|, \tag{3.27}$$

$$\|y_n - y^* - [g_1(y_n) - g_1(y^*)]\| \le p_1\|y_n - y^*\|. \tag{3.28}$$

Substituting (3.25) into (3.28) yields that

$$\left\|y_n - y^* - [g_1(y_n) - g_1(y^*)]\right\| \le \frac{p_1(\Phi_{T_2}(r_2) + p_2)(\Phi_{T_3}(r_3) + p_3)}{(1 - p_2)(1 - p_3)}\|x_n - x^*\|. \tag{3.29}$$

Writing

$$\Diamond = \frac{(\Phi_{T_2}(r_2) + p_2)(\Phi_{T_3}(r_3) + p_3)}{(1 - p_2)(1 - p_3)} \tag{3.30}$$

and substituting (3.26), (3.27), and (3.29) into (3.7), we will get

$$\|x_{n+1} - x^*\| \le (1 - \alpha_n(1 - \tau(p_1 + p_1\Diamond + \Phi_{T_1}(r_1)\Diamond)))\|x_n - x^*\|. \tag{3.31}$$

Table 1

	μ_i	ν_i	$\left[0, \frac{\mu_i - \sqrt{\mu_i^2 - \nu_i^2}}{2\mu_i}\right) \cup \left[\frac{\mu_i + \sqrt{\mu_i^2 - \nu_i^2}}{2\mu_i}, 1\right)$	$\left(\frac{\nu_i - \sqrt{\nu_i^2 - \mu_i^2(4p_i)(1-p_i)}}{\mu_i^2}, \frac{\nu_i + \sqrt{\nu_i^2 - \mu_i^2(4p_i)(1-p_i)}}{\mu_i^2}\right)$
T_1	$\frac{1}{2}$	$\frac{1}{2}$	$[0,1)$	$(0,4) =: R_1$
T_2	$\frac{1}{4}$	$\frac{1}{4}$	$[0,1)$	$(0,8) =: R_2$
T_3	$\frac{1}{2}$	$\frac{1}{4}$	$\left[0, \frac{2-\sqrt{3}}{4}\right) \cup \left[\frac{2+\sqrt{3}}{4}, 1\right)$	$\left(\frac{7-\sqrt{22}}{7}, \frac{7+\sqrt{22}}{7}\right) =: R_3$

Notice that, by conditions (i) and (ii), we have

$$\prod_{i=1}^{3}\left(\frac{\Phi_{T_i}(r_i) + p_i}{1 - p_i}\right) < 1. \tag{3.32}$$

This implies that

$$\Diamond < \frac{1 - p_1}{\Phi_{T_1}(r_1) + p_1}, \tag{3.33}$$

that is,

$$\Delta =: p_1 + p_1\Diamond + \Phi_{T_1}(r_1)\Diamond < 1. \tag{3.34}$$

Put

$$\begin{aligned} a_n &= \|x_n - x^*\|, \\ \lambda_n &= \alpha_n(1 - \tau\Delta). \end{aligned} \tag{3.35}$$

By condition (iii), in view of (3.32) and (3.34), we see that $\tau\Delta \in (0,1)$; this implies $\lambda_n \in (0,1)$. Meanwhile, from condition (iv), we also have $\sum_{n=0}^{\infty} \lambda_n = \infty$. Hence, all conditions of Lemma 2.4 are satisfied, and we can conclude that $x_n \to x^*$ as $n \to \infty$. Consequently, from (3.23) and (3.25), we know that $z_n \to z^*$ and $y_n \to y^*$ as $n \to \infty$, respectively. This completes the proof.

□

Example 3.3. Let $H = [0,1]$ and $C = [0,1/2]$. For $i = 1,2,3$, let $T_i, g_i : H \to H$ be mappings which are defined by $T_1(x) = x/2$, $T_2(x) = x/4$, $T_3(x) = x^2/4$, $g_1(x) = x$, and $g_2(x) = g_3(x) = (27/28)x$. Then, one can show that $p_1 = 0$ and $p_2 = p_3 = 1/28$. Consequently, we have Table 1.

It follows that the condition (i) of Theorem 3.2 is satisfied. Moreover, if for each $i = 1,2,3$ the real number r_i belongs to R_i, then we can check that $\prod_{i=1}^{3}((\Phi_{T_i}(r_i) + p_i)/(1 - p_i)) < 1$.

Now let $\gamma \in (1,\infty)$ be a fixed positive real number and $\alpha \in (0, 1/\gamma\prod_{i=1}^{3}((\Phi_{T_i}(r_i)+p_i)/(1-p_i)))$. If $S : H \to H$ is a mapping which is defined by

$$S(x) = \alpha x^{\gamma}, \quad \forall x \in H. \tag{3.36}$$

Then we know that the conditions (ii) and (iii) of Theorem 3.2 are satisfied. In fact, we have $(0,0,0) \in \mathrm{SGVID}(\Xi, \Lambda, C)$ and $0 \in F(S)$.

Applying our Theorem 3.2, the following results are obtained immediately.

Corollary 3.4. *Let C be a closed convex subset of a real Hilbert space H. Let $T_i : H \to H$ be ν_i-strongly monotone and μ_i-Lipschitz mapping, and let $g : H \to H$ be λ-strongly monotone and δ-Lipschitz mapping for $i = 1,2,3$. Let $S : H \to H$ be a τ-Lipschitz mapping such that $(\mathrm{SGVID}(\Xi, g, C))_1 \cap F(S) \neq \emptyset$. Let r_1, r_2, and r_3 be positive real numbers that generate the problem* (2.7). *For arbitrary chosen initial $x_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ such that*

$$\begin{aligned} g(z_n) &= P_C[g(x_n) - r_3T_3x_n], \\ g(y_n) &= P_C[g(z_n) - r_2T_2z_n], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S(x_n - g(x_n) + P_C[g(y_n) - r_1T_1y_n]). \end{aligned} \tag{3.37}$$

Put $p = \sqrt{1+\delta^2-2\lambda}$. If the following control conditions are satisfied:

(i) $p \in [0, (\mu_i - \sqrt{\mu_i^2 - \nu_i^2})/2\mu_i) \cup [(\mu_i + \sqrt{\mu_i^2 - \nu_i^2})/2\mu_i, 1)$, *for each $i = 1,2,3$,*

(ii) $|r_i - \nu_i/\mu_i^2| < \sqrt{\nu_i^2 - \mu_i^2(4p)(1-p)}/\mu_i^2$, *for each $i = 1,2,3$,*

(iii) $\tau\prod_{i=1}^{3}((\Phi_{T_i}(r_i)+p)/(1-p)) < 1$,

(iv) $\sum_{n=0}^{\infty}\alpha_n = \infty$,

then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (3.37) *converge strongly to x^*, y^*, and z^*, respectively, such that $(x^*, y^*, z^*) \in \mathrm{SGVID}(\Xi, g, C)$ and $x^* \in F(S)$.*

Corollary 3.5. *Let C be a closed convex subset of a real Hilbert space H. Let $T : H \to H$ be ν-strongly monotone and μ-Lipschitz continuous mapping, and let $g : H \to H$ be λ-strongly monotone and δ-Lipschitz mapping. Let $S : H \to H$ be a τ-Lipschitz mapping such that $(\mathrm{SGVI}(T, g, C))_1 \cap F(S) \neq \emptyset$. Let r_1, r_2, and r_3 be positive real numbers that generate the problem* (2.8). *For arbitrary chosen initial $x_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ such that*

$$\begin{aligned} g(z_n) &= P_C[g(x_n) - r_3Tx_n], \\ g(y_n) &= P_C[g(z_n) - r_2Tz_n], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S(x_n - g(x_n) + P_C[g(y_n) - r_1Ty_n]). \end{aligned} \tag{3.38}$$

If the following control conditions are satisfied:

(i) $p \in [0, (\mu - \sqrt{\mu^2 - \nu^2})/2\mu) \cup [(\mu + \sqrt{\mu^2 - \nu^2})/2\mu, 1)$, *where $p = \sqrt{1+\delta^2-2\lambda}$,*

(ii) $|r - \nu/\mu^2| < \sqrt{\nu^2 - \mu^2(4p)(1-p)}/\mu^2$, *where* $r = \max\{r_1, r_2, r_3\}$,

(iii) $\tau\prod_{i=1}^{3}((\Phi_T(r_i) + p)/(1-p)) < 1$,

(iv) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

then the sequences $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *generated by* (3.38) *converge strongly to* x^*, y^*, *and* z^*, *respectively, such that* $(x^*, y^*, z^*) \in \mathrm{SGVI}(T, g, C)$ *and* $x^* \in F(S)$.

Corollary 3.6. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T_i : H \to H$ *be* ν_i*-strongly monotone and* μ_i*-Lipschitz continuous mapping for* $i = 1, 2, 3$. *Let* $S : C \to C$ *be a* τ*-Lipschitz mapping such that* $(\mathrm{SVID}(\Xi, C))_1 \cap F(S) \neq \emptyset$. *Let* r_1, r_2, *and* r_3 *be positive real numbers that generate the problem* (2.9). *For arbitrary chosen initial* $x_0 \in H$, *compute the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *such that*

$$\begin{aligned} z_n &= P_C[x_n - r_3 T_3 x_n], \\ y_n &= P_C[z_n - r_2 T_2 z_n], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S P_C[y_n - r_1 T_1 y_n]. \end{aligned} \tag{3.39}$$

If the following control conditions are satisfied:

(i) $r_i \in (0, 2\nu_i/\mu_i^2)$, *for each* $i = 1, 2, 3$,

(ii) $\tau\prod_{i=1}^{3}\Phi_T(r_i) < 1$,

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

then the sequences $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *generated by* (3.39) *converge strongly to* x^*, y^*, *and* z^*, *respectively, such that* $(x^*, y^*, z^*) \in \mathrm{SVID}(\Xi, C)$ *and* $x^* \in F(S)$.

Proof. Since the identity mapping is 1-strongly monotone and 1-Lipschitz mapping, it follows that the number p, defined in Corollary 3.4, is identically zero. Hence, the required result can be obtained immediately. □

Corollary 3.7. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : H \to H$ *be* ν*-strongly monotone and* μ*-Lipschitz mapping. Let* $S : C \to C$ *be a* τ*-Lipschitz mapping such that* $(\mathrm{SVI}(T, C))_1 \cap F(S) \neq \emptyset$. *Let* r_1, r_2, *and* r_3 *be positive real numbers that generate the problem* (2.10). *For arbitrary chosen initial* $x_0 \in H$, *compute the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *such that*

$$\begin{aligned} z_n &= P_C[x_n - r_3 T x_n], \\ y_n &= P_C[z_n - r_2 T z_n], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S P_C[y_n - r_1 T y_n]. \end{aligned} \tag{3.40}$$

If the following control conditions are satisfied:

(i) $r_i \in (0, 2\nu/\mu^2)$, *for each* $i = 1, 2, 3$,

(ii) $\tau\prod_{i=1}^{3}\Phi_T(r_i) < 1$,

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *generated by* (3.40) *converge strongly to* x^*, y^*, *and* z^*, *respectively, such that* $(x^*, y^*, z^*) \in \mathrm{SVI}(T, C)$ *and* $x^* \in F(S)$.

Remark 3.8. Corollary 3.9 mainly improves and extends the results of Verma [20].

Corollary 3.9. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : H \to H$ *be* ν*-strongly monotone and* μ*-Lipschitz mapping, and let* $g : H \to H$ *be* δ*-strongly monotone and* λ*-Lipschitz mapping. Let* $S : H \to H$ *be a* τ*-Lipschitz mapping such that* $\mathrm{GVI}(T, g, C) \cap F(S) \neq \emptyset$. *Put* $r = \nu/\mu^2$ *be a fixed positive real number. For arbitrary chosen initial* $x_0 \in H$, *compute the sequence* $\{x_n\}$ *such that*

$$\begin{aligned} g(z_n) &= P_C[g(x_n) - rTx_n], \\ g(y_n) &= P_C[g(z_n) - rTz_n], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S\big(g(x_n) - x_n + P_C[g(y_n) - rTy_n]\big). \end{aligned} \tag{3.41}$$

If the following control conditions are satisfied:

(i) $p \in [0, (\mu - \sqrt{\mu^2 - \nu^2})/2\mu) \cup [(\mu + \sqrt{\mu^2 - \nu^2})/2\mu, 1)$, *where* $p = \sqrt{1 + \delta^2 - 2\lambda}$,

(ii) $\tau \in (0, \mu(1-p)/(\mu p + \sqrt{\mu^2 - \nu^2}))$,

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

then the sequences $\{x_n\}$ *generated by* (3.41) *converges strongly to* x^*, *such that* $x^* \in \mathrm{GVI}(T, C) \bigcap F(S)$.

Proof. Notice that $\Phi_T(0) = 1$ and $\Phi_T(\nu/\mu^2) = \sqrt{\mu^2 - \nu^2}/\mu$. Consequently, condition (ii) implies that

$$\tau\left(\frac{p + \Phi_T(\nu/\mu^2)}{1-p}\right) < 1. \tag{3.42}$$

Moreover, by setting $r_2 = r_3 = 0$, we see that the problem $\mathrm{SGVI}(T, g, C)$ is reduced to the problem $\mathrm{GVI}(T, g, C)$. Using these observations, one can easily see that the required conclusion is followed immediately from the Corollary 3.5. □

Remark 3.10. Corollary 3.9 extends the results in [24] in some extent.

In light of Corollaries 3.6 and 3.9, we obtain the following result immediately.

Corollary 3.11. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : H \to H$ *be* ν*-strongly monotone and* μ*-Lipschitz mapping. Let* $S : C \to C$ *be a* τ*-Lipschitz mapping such that*

$VI(T,C)\cap F(S)\neq\emptyset$. Let $r=\nu/\mu^2$ be a fixed positive real number. For arbitrary chosen initial $x_0\in H$, compute the sequence $\{x_n\}$ such that

$$
\begin{aligned}
z_n &= P_C[x_n - rTx_n],\\
y_n &= P_C[z_n - rTz_n],\\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n SP_C[y_n - rTy_n].
\end{aligned}
\tag{3.43}
$$

If the following control conditions are satisfied:

(i) $\tau\in(0,\mu/\sqrt{\mu^2-\nu^2})$,

(ii) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

then the sequences $\{x_n\}$ generated by (3.43) converges strongly to x^*, such that $x^*\in VI(T,C)\cap F(S)$.

Remark 3.12. Corollary 3.11 extends and improves the main result announced by Noor and Huang [26], from a class nonexpansive mappings to a class of any Lipschitzian mappings.

Remark 3.13. The choice $r=\nu/\mu^2$ is a possible sharp for applying Corollaries 3.9 and 3.11 to a wide class of Lipschitz mappings. Indeed, notice that

$$\Phi_T\left(\frac{\nu}{\mu^2}\right)=\frac{\sqrt{\mu^2-\nu^2}}{\mu}=\inf_{r\in[0,\infty)}\{\Phi_T(r)\}. \tag{3.44}$$

Since both Corollaries 3.9 and 3.11 are special cases of Corollary 3.5, thus, based on condition (iii) of Corollary 3.5, our remark is asserted.

Now we show an application of Theorem 3.2. Recall that a mapping $Q:H\to H$ is said to be *asymptotically strict pseudocontraction* if there exists a constant $\lambda\in[0,1)$ satisfying

$$\|Q^n x - Q^n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + \lambda\|(I-Q^n)x-(I-Q^n)y\|^2 \tag{3.45}$$

for all $x,y\in H$ and all integer $n\ge 1$, where $\gamma_n\ge 0$ for all $n\ge 1$ such that $\gamma_n\to 0$ as $n\to\infty$. In this case, we also say Q is an asymptotically λ-strict pseudocontraction.

Lemma 3.14 (see [29]). *Let $Q:H\to H$ be an asymptotically λ-strict pseudocontraction. Then, for each $n\ge 1$, Q^n satisfies the Lipschitz condition*

$$\|Q^n x - Q^n y\| \le L_n\|x-y\|,\quad \forall x,y\in H, \tag{3.46}$$

where $L_n=(\lambda+\sqrt{1+\gamma_n(1-\lambda)})/(1-\lambda)$.

For each $i=1,2,3$, let $T_i:H\to H$ be a ν_i-strongly monotone and μ_i-Lipschitz mapping, and let $g_i:H\to H$ be a δ_i-strongly monotone and λ_i-Lipschitz mapping. Put

$$\xi=\prod_{i=1}^{3}\left(\frac{\Phi_{T_i}(r_i)+p_i}{1-p_i}\right), \tag{3.47}$$

where p_i is defined as in Theorem 3.2, for each $i = 1,2,3$, and r_1, r_2, r_3 are positive real numbers that generate the problem (2.5). Notice that, if $\xi \in (0,(1-\lambda)/(1+\lambda))$, then there exists a natural number j such that $L_j < 1/\xi$, since $L_n \downarrow ((1+\lambda)/(1-\lambda))$ as $n \to \infty$. Using this observation, we can apply Theorem 3.2 to obtain the following result.

Example 3.15. Let H be a real Hilbert space. For each $i = 1,2,3$, let $T_i : H \to H$ be a ν_i-strongly monotone and μ_i-Lipschitz mapping, and let $g_i : H \to H$ be a δ_i-strongly monotone and λ_i-Lipschitz mapping. Assume that the problem (2.5) is generated by the positive real numbers r_1, r_2, and r_3 such that the conditions (i) and (ii) in Theorem 3.2 are satisfied. Let $Q : H \to H$ be an asymptotically λ-strict pseudocontraction satisfying $\xi \in (0,(1-\lambda)/(1+\lambda))$, and let $j \in \mathbb{N}$ be a natural number such that $L_j < 1/\xi$, where L_j is defined as in Lemma 3.14. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences generated by Algorithm 1 with $S =: Q^j$.

If $(\mathrm{SQVID}(\Xi,\Lambda,C))_1 \cap F(Q) \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to x^*, y^*, and z^*, respectively, such that $(x^*, y^*, z^*) \in \mathrm{SGVID}(\Xi,\Lambda,C)$ and $x^* \in F(Q)$. Indeed, let $(x^*, y^*, z^*) \in \mathrm{SQVID}(\Xi,\Lambda,C)$ be such that $x^* \in F(Q)$. It follows that $x^* \in F(Q^n)$ for all $n \in \mathbb{N}$. Using this one together with the fact that $\xi L_j < 1$, as an application of Theorem 3.2, we know that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to x^*, y^*, and z^*, respectively.

Remark 3.16. If $\lambda = 0$, then Q is fallen to a class of mappings as asymptotically nonexpansive mapping. Hence, Example 3.15 can be viewed as an extension of the main result announced by Cho and Qin [25] in some aspects.

Remark 3.17. Recall that a mapping $T : H \to H$ is said to be

(i) *μ-cocoercive* if there exists a constant $\mu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \mu \|Tx - Ty\|^2, \quad \forall x, y \in H, \tag{3.48}$$

(ii) *relaxed μ-cocoercive* if there exists a constant $\mu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\mu) \|Tx - Ty\|^2, \quad \forall x, y \in H, \tag{3.49}$$

(iii) *relaxed (μ,ν)-cocoercive* if there exist constants $\mu, \nu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\mu) \|Tx - Ty\|^2 + \nu\|x-y\|^2, \quad \forall x, y \in H. \tag{3.50}$$

Obviously, the class of the relaxed (μ,ν)-cocoercive mappings is the most general one, of course, larger than the class of strongly monotone mappings. However, it is worth noting that, if the mapping T is relaxed (μ,ν)-cocoercive and τ-Lipschitz mapping such that $\nu - \mu\tau^2 > 0$, T must be a $(\nu - \mu\tau^2)$-strongly monotone. Hence, the results that appeared in this paper can be also applied to a class of the relaxed cocoercive mappings. In conclusion, for a suitable and appropriate choice of the mappings T, g and parameters r, our results include many important known results given by many authors as special cases.

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