Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 812813, 11 pages
doi:10.1155/2011/812813

Research Article

Asymptotically Pseudocontractions, Banach Operator Pairs and Best Simultaneous Approximations

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Received 3 December 2010; Accepted 12 January 2011

Academic Editor: Mohamed Amine Khamsi

The existence of common fixed points is established for the mappings where T is asymptotically f-pseudo-contraction on a nonempty subset of a Banach space. As applications, the invariant best simultaneous approximation and strong convergence results are proved. Presented results are generalizations of very recent fixed point and approximation theorems of Khan and Akbar (2009), Chen and Li (2007), Pathak and Hussain (2008), and several others.

1. Introduction and Preliminaries

We first review needed definitions. Let M be a subset of a normed space $(X, \|\cdot\|)$. The set $P_M(u) = \{x \in M : \|x-u\| = \operatorname{dist}(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $\operatorname{dist}(u, M) = \inf\{\|y-u\| : y \in M\}$. Suppose that A and G are bounded subsets of X. Then, we write

$$\begin{gathered} r_G(A) = \inf_{g \in G} \sup_{a \in A} \|a-g\|, \\ \operatorname{cent}_G(A) = \left\{ g_0 \in G : \sup_{a \in A} \|a-g_0\| = r_G(A) \right\}. \end{gathered} \tag{1.1}$$

The number $r_G(A)$ is called the *Chebyshev radius* of A w.r.t. G, and an element $y_0 \in \operatorname{cent}_G(A)$ is called a *best simultaneous approximation* of A w.r.t. G. If $A = \{u\}$, then $r_G(A) = \operatorname{dist}(u, G)$ and $\operatorname{cent}_G(A)$ is the set of all best approximations, $P_G(u)$, of u from G. We also refer the reader to Milman [1], and Vijayraju [2] for further details. We denote by $\mathbb{N}$ and $\operatorname{cl}(M)$ $(w\operatorname{cl}(M))$, the set of positive integers and the closure (weak closure) of a set M in X, respectively. Let $f,\ T : M \to M$ be mappings. The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($x = fx = Tx$). The pair $\{f, T\}$ is called

(1) *commuting* [3] if $Tfx = fTx$ for all $x \in M$,

(2) *compatible* (see [3, 4]) if $\lim_n \|Tfx_n - fTx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M,

(3) *weakly compatible* if they commute at their coincidence points; that is, if $fTx = Tfx$ whenever $fx = Tx$,

(4) *Banach operator pair*, if the set $F(f)$ is T-invariant, namely $T(F(f)) \subseteq F(f)$. Obviously, commuting pair (T, f) is a Banach operator pair but converse is not true in general, see [5, 6]. If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair (see, e.g., [5, 7, 8]).

The set M is called *q-starshaped* with $q \in M$, if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in M for all $x \in M$. The map f defined on a q-starshaped set M is called *affine* if

$$f\big((1-k)q + kx\big) = (1-k)fq + kfx, \quad \forall x \in M. \tag{1.2}$$

Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then, T and f are called,

(5) *R-subweakly commuting* on M (see [9]) if for all $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \leq R\,\mathrm{dist}(fx, [q, Tx])$,

(6) *uniformly R-subweakly commuting* on $M \setminus \{q\}$ (see [10]) if there exists a real number $R > 0$ such that $\|fT^n x - T^n fx\| \leq R\,\mathrm{dist}(fx, [q, T^n x])$, for all $x \in M \setminus \{q\}$ and $n \in \mathbb{N}$. The map $T : M \to X$ is said to be *demiclosed at* 0 if, for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converges to $0 \in X$, then $0 = Tx$.

The classical Banach contraction principle has numerous generalizations, extensions and applications. While considering Lipschitzian mappings, a natural question arises whether it is possible to weaken contraction assumption a little bit in Banach contraction principle and still obtain the existence of a fixed point. In this direction the work of Edelstein [11], Jungck [3], Park [12–18] and Suzuki [19] is worth to mention.

Schu [20] introduced the concept of asymptotically pseudocontraction and proved the existence and convergence of fixed points for this class of maps (see also [21]). Recently, Chen and Li [5] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Hussain [6], Ćirić et al. [7], Khan and Akbar [22, 23] and Pathak and Hussain [8]. More recently, Zhou [24] established a demiclosedness principle for a uniformly L-Lipschitzian asymptotically pseudocontraction map and as an application obtained a fixed point result for asymptotically pseudocontraction in the setup of a Hilbert space. In this paper, we are able to join the concepts of uniformly f-Lipschitzian, asymptotically f-pseudocontraction and Banach operator pair to get the result of Zhou [24] in the setting of a Banach space. As a consequence, the common fixed point and approximation results of Al-Thagafi [25], Beg et al. [10], Chidume et al. [26], Chen and Li [5], Cho et al. [27], Khan and Akbar [22, 23], Pathak and Hussain [8], Schu [28] and Vijayraju [2] are extended to the class of asymptotically f-pseudocontraction maps.

2. Main Results

Let X be a real Banach space and M be a subset of X. Let $f, g\ T : M \to M$ be mappings. Then T is called

(a) an (f, g)*-contraction* if there exists $0 \le k < 1$ such that $\|Tx - Ty\| \le k\|fx - gy\|$ for any $x, y \in M$; if $k = 1$, then T is called f*-nonexpansive*,

(b) *asymptotically* (f, g)*-nonexpansive* [2] if there exists a sequence $\{k_n\}$ of real numbers with $k_n \ge 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - T^n y\right\| \le k_n \left\|fx - gy\right\| \tag{2.1}$$

for all $x, y \in M$ and for each $n \in \mathbb{N}$; if $g = \mathrm{id}$, then T is called f*-asymptotically nonexpansive map*,

(c) *pseudocontraction* if and only if for each $x, y \in M$, there exists $j(x - y) \in J(x - y)$ such that

$$\left\langle Tx - Ty, j(x - y)\right\rangle \le \left\|x - y\right\|^2, \tag{2.2}$$

where $J : X \to 2^{X^*}$ is the *normalized duality mapping* defined by

$$J(u) = \left\{j \in X^* : \langle u, j\rangle = \|u\|^2,\ \left\|j\right\| = \|u\|\right\}; \tag{2.3}$$

(d) *strongly pseudocontraction* if and only if for each $x, y \in M$, there exists $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\left\langle Tx - Ty, j(x - y)\right\rangle \le k\left\|x - y\right\|^2; \tag{2.4}$$

(e) *asymptotically* (f, g)*-pseudocontractive* if and only if for each $n \in \mathbb{N}$ and $x, y \in M$, there exists $j(x - y) \in J(x - y)$ and a constant $k_n \ge 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\langle T^n x - T^n y, j(x - y)\right\rangle \le k_n \left\|fx - gy\right\|^2. \tag{2.5}$$

If $g = \mathrm{id}$ in (2.5), then T is called *asymptotically* f*-pseudocontractive* [20, 24, 27],

(f) *uniformly* (f, g)*-Lipschitzian* if there exists some $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \le L\left\|fx - gy\right\|, \tag{2.6}$$

for all $x, y \in M$ and for each $n \in \mathbb{N}$; if $g = \mathrm{id}$, then T is called *uniformly* f*-Lipschitzian* [20, 24, 29].

The map T is called *uniformly asymptotically regular* [2, 10] on M, if for each $\eta > 0$, there exists $N(\eta) = N$ such that $\|T^n x - T^{n+1} x\| < \eta$ for all $n \ge N$ and all $x \in M$.

The class of asymptotically pseudocontraction contains properly the class of asymptotically nonexpansive mappings and every asymptotically nonexpansive mapping is a uniformly L-Lipschitzian [2, 24]. For further details, we refer to [21, 24, 27, 29, 30].

In 1974, Deimling [30] proved the following fixed point theorem.

Theorem D. *Let T be self-map of a closed convex subset K of a real Banach space X. Assume that T is continuous strongly pseudocontractive mapping. Then, T has a unique fixed point.*

The following result extends and improves Theorem 3.4 of Beg et al. [10], Theorem 2.10 in [22], Theorems 2.2 of [25] and Theorem 4 in [31].

Theorem 2.1. *Let f, T be self-maps of a subset M of a real Banach space X. Assume that $F(f)$ is closed (resp., weakly closed) and convex, T is uniformly f-Lipschitzian and asymptotically f-pseudocontractive which is also uniformly asymptotically regular on M. If $\mathrm{cl}(T(M))$ is compact (resp., $w\,\mathrm{cl}(T(M))$ is weakly compact and $id - T$ is demiclosed at 0) and $T(F(f)) \subseteq F(f)$, then $F(T) \cap F(f) \neq \emptyset$.*

Proof. For each $n \geq 1$, define a self-map T_n on $F(f)$ by

$$T_n x = (1 - \mu_n) q + \mu_n T^n x, \tag{2.7}$$

where $\mu_n = \lambda_n / k_n$ and $\{\lambda_n\}$ is a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty}\lambda_n = 1$ and $q \in F(f)$. Since $T^n(F(f)) \subset F(f)$ and $F(f)$ is convex with $q \in F(f)$, it follows that T_n maps $F(f)$ into $F(f)$. As $F(f)$ is convex and $\mathrm{cl}\,T(F(f)) \subseteq F(f)$ (resp. $w\,\mathrm{cl}\,T(F(f)) \subseteq F(f)$), so $\mathrm{cl}\,T_n(F(f)) \subseteq F(f))$ (resp. $w\,\mathrm{cl}\,T_n(F(f)) \subseteq F(f))$ for each $n \geq 1$. Since T_n is a strongly pseudocontractive on $F(f)$, by Theorem D, for each $n \geq 1$, there exists $x_n \in F(f)$ such that $x_n = f x_n = T_n x_n$. As $T(F(f))$ is bounded, so $\|x_n - T^n x_n\| = (1-\mu_n)\|T^n x_n - q\| \to 0$ as $n \to \infty$. Now,

$$\begin{aligned} \|x_n - Tx_n\| &= \|x_n - T^n x_n\| + \left\|T^n x_n - T^{n+1} x_n\right\| + \left\|T^{n+1} x_n - T x_n\right\| \\ &\leq \|x_n - T^n x_n\| + \left\|T^n x_n - T^{n+1} x_n\right\| + L\left\|f T^n x_n - f x_n\right\|. \end{aligned} \tag{2.8}$$

Since for each $n \geq 1$, $T^n(F(f)) \subseteq F(f)$ and $x_n \in F(f)$, therefore $T^n x_n \in F(f)$. Thus $fT^n x_n = T^n x_n$. Also T is uniformly asymptotically regular, we have from (2.8)

$$\|x_n - Tx_n\| \leq \|x_n - T^n x_n\| + \left\|T^n x_n - T^{n+1} x_n\right\| + L\|T^n x_n - x_n\| \longrightarrow 0, \tag{2.9}$$

as $n \to \infty$. Thus $x_n - Tx_n \to 0$ as $n \to \infty$. As $\mathrm{cl}\,T(M)$ is compact, so there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z \in \mathrm{cl}(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f))$ and $\mathrm{cl}\,T(F(f)) \subseteq F(f)$, therefore $z \in F(f)$. Moreover,

$$\|Tx_m - Tz\| \leq L\|f x_m - f z\| = L\|x_m - z\| \leq L\|x_m - Tx_m\| + L\|Tx_m - z\|. \tag{2.10}$$

Taking the limit as $m \to \infty$, we get $z = Tz$. Thus, $M \cap F(T) \cap F(f) \neq \emptyset$ proves the first case.

Since a weakly closed set is closed, by Theorem D, for each $n \geq 1$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. The weak compactness of $w\,\mathrm{cl}(T(M))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ converging weakly to $y \in w\,\mathrm{cl}(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f))$ and $w\,\mathrm{cl}\,T(F(f)) \subseteq F(f)$, so $y \in F(f)$. Moreover, we have, $x_m - Tx_m \to 0$ as $m \to \infty$. If $\mathrm{id} - T$ is demiclosed at 0, then $y = Ty$. Thus, $M \cap F(T) \cap F(f) \neq \emptyset$. □

Remark 2.2. By comparing Theorem 3.4 of Beg et al. [10] with the first case of Theorem 2.1, their assumptions "M is closed and q-starshaped, $fM = M$, $T(M \setminus \{q\}) \subset f(M) \setminus \{q\}$, f, T are continuous, f is linear, $q \in F(f)$, $\mathrm{cl}\,T(M \setminus \{q\})$ is compact, T is asymptotically f-nonexpansive and T and f are uniformly R-subweakly commuting on M" are replaced with "M is nonempty set, $F(f)$ is closed, convex, $T(F(f)) \subseteq F(f)$, $\mathrm{cl}\,T(M)$ is compact, T is uniformly f-Lipschitzian and asymptotically f-pseudocontractive".

If M is weakly closed and f is weakly continuous, then $F(f)$ is weakly closed and hence closed, thus we obtain the following.

Corollary 2.3. *Let f, T be self-maps of a weakly closed subset M of a Banach space X. Assume that f is weakly continuous, $F(f)$ is nonempty and convex, T is uniformly f-Lipschitzian and asymptotically f-pseudocontractive which is also uniformly asymptotically regular on M. If $\mathrm{cl}(T(M))$ is compact (resp. $w\,\mathrm{cl}(T(M))$ is weakly compact and id $-$ T is demiclosed at 0) and (T, f) is a Banach operator pair, then $F(T) \cap F(f) \neq \emptyset$.*

A mapping f on M is called *pointwise asymptotically nonexpansive* [32, 33] if there exists a sequence $\{\alpha_n\}$ of functions such that

$$\|f^n x - f^n y\| \leq \alpha_n(x) \|x - y\| \tag{2.11}$$

for all $x, y \in M$ and for each $n \in \mathbb{N}$ where $\alpha_n \to 1$ pointwise on M.

An asymptotically nonexpansive mapping is pointwise asymptotically nonexpansive. A pointwise asymptotically nonexpansive map f defined on a closed bounded convex subset of a uniformly convex Banach space has a fixed point and $F(f)$ is closed and convex [32, 33]. Thus we obtain the following.

Corollary 2.4. *Let f be a pointwise asymptotically nonexpansive self-map of a closed bounded convex subset M of a uniformly convex Banach space X. Assume that T is a self-map of M which is uniformly f-Lipschitzian, asymptotically f-pseudocontractive and uniformly asymptotically regular. If $\mathrm{cl}(T(M))$ is compact (resp. $w\,\mathrm{cl}(T(M))$ is weakly compact and id $-$ T is demiclosed at 0) and $T(F(f)) \subseteq F(f)$, then $F(T) \cap F(f) \neq \emptyset$.*

Corollary 2.5 (see [24, Theorem 3.3]). *Let T be self-map of a closed bounded and convex subset M of a real Hilbert space X. Assume that T is uniformly Lipschitzian and asymptotically pseudocontractive which is also uniformly asymptotically regular on M. Then, $F(T) \neq \emptyset$.*

Corollary 2.6. *Let X be a Banach space and T and f be self-maps of X. If $u \in X$, $D \subseteq P_M(u)$, $D_0 := D \cap F(f)$ is closed (resp. weakly closed) and convex, $\mathrm{cl}(T(D))$ is compact (resp. $w\,\mathrm{cl}(T(D))$ is weakly compact and id $-$ T is demiclosed at 0), T is uniformly f-Lipschitzian and asymptotically f-pseudocontractive which is also uniformly asymptotically regular on D, and $T(D_0) \subseteq D_0$, then$P_M(u) \cap F(T) \cap F(f) \neq \emptyset$.*

Remark 2.7. Corollary 2.6 extends Theorems 4.1 and 4.2 of Chen and Li [5] to a more general class of asymptotically f-pseudocontractions.

Theorem 2.1 can be extended to uniformly (f, g)-Lipschitzian and asymptotically (f, g)-pseudocontractive map which extends Theorem 2.10 of [22] to asymptotically (f, g)-pseudocontractions.

Theorem 2.8. *Let f, g, T be self-maps of a subset M of a Banach space X. Assume that $F(f) \cap F(g)$ is closed (resp. weakly closed) and convex, T is uniformly (f, g)-Lipschitzian and asymptotically (f, g)-pseudocontractive which is also uniformly asymptotically regular on M. If $\mathrm{cl}(T(M))$ is compact (resp. $w\,\mathrm{cl}(T(M))$ is weakly compact and $id - T$ is demiclosed at 0) and $T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Proof. For each $n \geq 1$, define a self-map T_n on $F(f) \cap F(g)$ by

$$T_n x = (1 - \mu_n) q + \mu_n T^n x, \tag{2.12}$$

where $\mu_n = \lambda_n / k_n$ and $\{\lambda_n\}$ is a sequence of numbers in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $q \in F(f) \cap F(g)$. Since $T^n(F(f) \cap F(g)) \subset F(f) \cap F(g)$ and $F(f) \cap F(g)$ is convex with $q \in F(f) \cap F(g)$, it follows that T_n maps $F(f) \cap F(g)$ into $F(\mathrm{f}) \cap F(g)$. As $F(f) \cap F(g)$ is convex and $\mathrm{cl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ (resp. $w\,\mathrm{cl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$), so $\mathrm{cl}\, T_n(F(f) \cap F(g)) \subseteq F(f) \cap F(g))$ (resp. $w\,\mathrm{cl}\, T_n(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$) for each $n \geq 1$. Further, since T_n is a strongly pseudocontractive on $F(f) \cap F(g)$, by Theorem D, for each $n \geq 1$, there exists $x_n \in F(f) \cap F(g)$ such that $x_n = f x_n = g x_n = T_n x_n$. Rest of the proof is similar to that of Theorem 2.1. □

Corollary 2.9. *Let f, g, T be self-maps of a subset M of a Banach space X. Assume that $F(f) \cap F(g)$ is closed (resp. weakly closed) and convex, T is uniformly (f, g)-Lipschitzian and asymptotically (f, g)-pseudocontractive which is also uniformly asymptotically regular on M. If $\mathrm{cl}(T(M))$ is compact (resp. $w\,\mathrm{cl}(T(M))$ is weakly compact and $id - T$ is demiclosed at 0) and (T, f) and (T, g) are Banach operator pairs, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Corollary 2.10. *Let X be a Banach space and T, f, and g be self-maps of X. If $y_1, y_2 \in X$, $D \subseteq \mathrm{cent}_K(\{y_1, y_2\})$, where $\mathrm{cent}_K(A)$ is the set of best simultaneous approximations of A w.r.t K. Assume that $D_0 := D \cap F(f) \cap F(g)$ is closed (resp. weakly closed) and convex, $\mathrm{cl}(T(D))$ is compact (resp. $w\,\mathrm{cl}(T(D))$ is weakly compact and $id - T$ is demiclosed at 0), T is uniformly (f, g)-Lipschitzian and asymptotically (f, g)-pseudocontractive which is also uniformly asymptotically regular on D, and $T(D_0) \subseteq D_0$, then $\mathrm{cent}_K(\{y_1, y_2\} \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Remark 2.11. (1) Theorem 2.2 and 2.7 of Khan and Akbar [23] are particular cases of Corollary 2.10.

(2) By comparing Theorem 2.2 of Khan and Akbar [23] with the first case of Corollary 2.10, their assumptions "$\mathrm{cent}_K(\{y_1, y_2\})$ is nonempty, compact, starshaped with respect to an element $q \in F(f) \cap F(g)$, $\mathrm{cent}_K(\{y_1, y_2\})$ is invariant under T, f and g, (T, f) and (T, g) are Banach operator pairs on $\mathrm{cent}_K(\{y_1, y_2\})$, $F(f)$ and $F(g)$ are q-starshaped with $q \in F(f) \cap F(g)$, f and g are continuous and T is asymptotically (f, g)-nonexpansive on D," are replaced with "$D \subseteq \mathrm{cent}_K(\{y_1, y_2\})$, $D_0 := D \cap F(f) \cap F(g)$ is closed and convex, $T(D_0) \subseteq D_0$, $\mathrm{cl}(T(D))$ is compact and T is uniformly (f, g)-Lipschitzian and asymptotically (f, g)-pseudocontractive on D."

(3) By comparing Theorem 2.7 of Khan and Akbar [23] with the second case of Corollary 2.10, their assumptions "$\mathrm{cent}_K(\{y_1, y_2\})$ is nonempty, weakly compact, starshaped with respect to an element $q \in F(f) \cap F(g)$, $\mathrm{cent}_K(\{y_1, y_2\})$ is invariant under T, f and g, (T, f) and (T, g) are Banach operator pairs on $\mathrm{cent}_K(\{y_1, y_2\})$, $F(f)$ and $F(g)$ are q-starshaped with $q \in F(f) \cap F(g)$, f and g are continuous under weak and strong topologies, $f - T$ is demiclosed at 0 and T is asymptotically (f, g)-nonexpansive on D," are replaced with "$D \subseteq \mathrm{cent}_K(\{y_1, y_2\})$, $D_0 := D \cap F(f) \cap F(g)$ is weakly closed and convex, $T(D_0) \subseteq D_0$, $w\,\mathrm{cl}(T(D))$ is weakly compact and $\mathrm{id} - T$ is demiclosed at 0 and T is uniformly (f, g)-Lipschitzian and asymptotically (f, g)-pseudocontractive on D."

We denote by $\Im_0$ the class of closed convex subsets of X containing 0. For $M \in \Im_0$, we define $M_u = \{x \in M : \|x\| \leq 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$ (see [9, 25]).

Theorem 2.12. *Let f, g, T be self-maps of a Banach space X. If $u \in X$ and $M \in \Im_0$ such that $T(M_u) \subseteq M$, $\mathrm{cl}(T(M_u))$ is compact (resp. $w\,\mathrm{cl}(T(M_u))$ is weakly compact) and $\|Tx - u\| \leq \|x - u\|$ for all $x \in M_u$, then $P_M(u)$ is nonempty, closed and convex with $T(P_M(u)) \subseteq P_M(u)$. If, in addition, $D \subseteq P_M(u)$, $D_0 := D \cap F(f) \cap F(g)$ is closed (resp. weakly closed) and convex, $\mathrm{cl}(T(D))$ is compact (resp. $w\,\mathrm{cl}(T(D))$ is weakly compact and $\mathrm{id} - T$ is demiclosed at 0), T is uniformly (f, g)-Lipschitzian and asymptotically (f, g)-pseudocontractive which is also uniformly asymptotically regular on D, and $T(D_0) \subseteq D_0$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Proof. We may assume that $u \notin M$. If $x \in M \setminus M_u$, then $\|x\| > 2\|u\|$. Note that

$$\|x - u\| \geq \|x\| - \|u\| > \|u\| \geq \mathrm{dist}(u, M). \tag{2.13}$$

Thus, $\mathrm{dist}(u, M_u) = \mathrm{dist}(u, M) \leq \|u\|$. If $\mathrm{cl}(T(M_u))$ is compact, then by the continuity of norm, we get $\|z - u\| = \mathrm{dist}(u, \mathrm{cl}(T(M_u)))$ for some $z \in \mathrm{cl}(T(M_u))$.

If we assume that $w\,\mathrm{cl}(T(M_u))$ is weakly compact, using Lemma 5.5 in [34, page 192], we can show the existence of a $z \in w\,\mathrm{cl}(T(M_u))$ such that $\mathrm{dist}(u, w\,\mathrm{cl}(T(M_u))) = \|z - u\|$.

Thus, in both cases, we have

$$\mathrm{dist}(u, M_u) \leq \mathrm{dist}(u, \mathrm{cl}\,T(M_u)) \leq \mathrm{dist}(u, T(M_u)) \leq \|Tx - u\| \leq \|x - u\|, \tag{2.14}$$

for all $x \in M_u$. Hence $\|z - u\| = \mathrm{dist}(u, M)$ and so $P_M(u)$ is nonempty, closed and convex with $T(P_M(u)) \subseteq P_M(u)$. The compactness of $\mathrm{cl}(T(M_u))$ (resp. weak compactness of $w\,\mathrm{cl}(T(M_u))$) implies that $\mathrm{cl}(T(D))$ is compact (resp. $w\,\mathrm{cl}(T(D))$ is weakly compact). The result now follows from Theorem 2.8. □

Remark 2.13. Theorem 2.12 extends Theorems 4.1 and 4.2 in [25], Theorem 8 in [31], and Theorem 2.15 in [22].

Definition 2.14. Let M be a nonempty closed subset of a Banach space X, $I, T : M \to M$ be mappings and $C = \{x \in M : h(x) = \min_{z \in M} h(z)\}$. Then I and T are said to satisfy property (S) [10, 27] if the following holds: for any bounded sequence $\{x_n\}$ in M, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ implies $C \cap F(I) \cap F(T) \neq \emptyset$.

The normal structure coefficient $N(X)$ of a Banach space X is defined [10, 26] by $N(X) = \inf\{\mathrm{diam}(M)/r_C(M) : M$ is nonempty bounded convex subset of X with

$\operatorname{diam}(M) > 0\}$, where $r_C(M) = \inf_{x\in M}\{\sup_{y\in M}\|x-y\|\}$ is the Chebyshev radius of M relative to itself and $\operatorname{diam}(M) = \sup_{x,y\in M}\|x-y\|$ is diameter of M. The space X is said to have the uniform normal structure if $\tilde{N}(X) > 1$. A Banach limit LIM is a bounded linear functional on l^∞ such that $\lim\inf_{n\to\infty} t_n \le \mathrm{LIM} t_n \le \lim\sup_{n\to\infty} t_n$ and $\mathrm{LIM} t_n = \mathrm{LIM} t_{n+1}$ for all bounded sequences $\{t_n\}$ in l^∞. Let $\{x_n\}$ be bounded sequence in X. Then we can define the real-valued continuous convex function f on X by $f(z) = \mathrm{LIM}\|x_n - z\|^2$ for all $z \in X$.

The following lemmas are well known.

Lemma 2.15 (see [10, 27]). *Let X be a Banach space with uniformly Gâteaux differentiable norm and $u \in X$. Let $\{x_n\}$ be bounded sequence in X. Then $f(u) = \inf_{z\in X} f(z)$ if and only if $\mathrm{LIM}\langle z, J(x_n - u)\rangle = 0$ for all $z \in X$, where $J : X \to X^*$ is the normalized duality mapping and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.*

Lemma 2.16 (see [10, 26]). *Let M be a convex subset of a smooth Banach space X, D be a nonempty subset of M and P be a retraction from M onto D. Then P is sunny and nonexpansive if and only if $\langle x - Px, J(z - Px)\rangle \le 0$ for all $x \in M$ and $z \in D$.*

Now, we are ready to prove strong convergence to nearest common fixed points of asymptotically f-pseudocontraction mappings.

Theorem 2.17. *Let M be a subset of a reflexive real Banach space X with uniformly Gâteaux differentiable norm. Let f and T be self-maps on M such that $F(f)$ is closed and convex, T is continuous, uniformly asymptotically regular, uniformly f-Lipschitzian and asymptotically f-pseudocontractive with a sequence $\{k_n\}$. Let $\{\lambda_n\}$ be sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty}\lambda_n = 1$ and $\lim_{n\to\infty}(k_n - 1)/(k_n - \lambda_n) = 0$. If $T(F(f)) \subset F(f)$, then we have the following.*

(A) *For each $n \ge 1$, there is exactly one x_n in M such that*

$$f x_n = x_n = (1-\mu_n)q + \mu_n T^n x_n \tag{2.15}$$

(B) *If $\{x_n\}$ is bounded and f and T satisfy property (S), then $\{x_n\}$ converges strongly to $Pq \in F(T) \cap F(f)$, where P is the sunny nonexpansive retraction from M onto $F(T)$.*

Proof. Part (A) follows from the proof of Theorem 2.1.

(B) As in Theorem 2.1, we get $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded, we can define a function $h : M \to R^+$ by $h(z) = \mathrm{LIM}\|x_n - z\|^2$ for all $z \in M$. Since h is continuous and convex, $h(z) \to \infty$ as $\|z\| \to \infty$ and X is reflexive, $h(z_0) = \min_{z\in M} h(z)$ for some $z_0 \in M$. Clearly, the set $C = \{x \in M : h(x) = \min_{z\in M} h(z)\}$ is nonempty. Since $\{x_n\}$ is bounded and f and T satisfy property (S), it follows that $C \cap F(f) \cap F(T) \ne \emptyset$. Suppose that $v \in C \cap F(f) \cap F(T)$, then by Lemma 2.15, we have

$$\mathrm{LIM}\langle x - v, J(x_n - v)\rangle \le 0 \quad \forall x \in M. \tag{2.16}$$

In particular, we have

$$\mathrm{LIM}\langle q - v, J(x_n - v)\rangle \le 0. \tag{2.17}$$

From (2.8), we have

$$x_n - T^n x_n = (1-\mu_n)q - T^n x_n = \frac{1-\mu_n}{\mu_n}(q - x_n). \tag{2.18}$$

Now, for any $v \in C \cap F(f) \cap F(T)$, we have

$$\begin{aligned} \langle x_n - T^n x_n, J(x_n - v)\rangle &= \langle x_n - v + T^n v - T^n x_n, J(x_n - v)\rangle \\ &\geq -(k_n - 1)\|x_n - v\|^2 \\ &\geq -(k_n - 1)K^2 \end{aligned} \tag{2.19}$$

for some $K > 0$. It follows from (2.18) that

$$\langle x_n - q, J(x_n - v)\rangle \leq \frac{k_n - 1}{k_n - \lambda_n} K^2. \tag{2.20}$$

Hence we have

$$\mathrm{LIM}\langle x_n - q, J(x_n - v)\rangle \leq 0. \tag{2.21}$$

This together with (2.17) implies that $\mathrm{LIM}\langle x_n - v, J(x_n - v)\rangle = \mathrm{LIM}\|x_n - v\|^2 = 0$.

Thus there is a subsequence $\{x_m\}$ of $\{x_n\}$ which converges strongly to v. Suppose that there is another subsequence $\{x_j\}$ of $\{x_n\}$ which converges strongly to y (say). Since T is continuous and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, y is a fixed point of T. It follows from (2.21) that

$$\langle v - q, J(v - y)\rangle \leq 0, \qquad \langle y - q, J(y - v)\rangle \leq 0. \tag{2.22}$$

Adding these two inequalities, we get

$$\langle v - y, J(v - y)\rangle = \|v - y\|^2 \leq 0 \quad \text{and thus } v = y. \tag{2.23}$$

Consequently, $\{x_n\}$ converges strongly to $v \in F(f) \cap F(T)$. We can define now a mapping P from M onto $F(T)$ by $\lim_{n\to\infty} x_n = Pq$. From (2.21), we have $\langle q - Pq, J(v - Pq)\rangle \leq 0$ for all $q \in M$ and $v \in F(T)$. Thus by Lemma 2.16, P is the sunny nonexpansive retraction on M. Notice that $x_n = fx_n$ and $\lim_{n\to\infty} x_n = Pq$, so by the same argument as in the proof of Theorem 2.1 we obtain, $Pq \in F(f)$. □

Remark 2.18. Theorem 2.17 extends Theorem 1 in [27]. Notice that the conditions of the continuity and linearity of f are not needed in Theorem 3.6 of Beg et al. [10]; moreover, we have obtained the conclusion for more general class of uniformly f-Lipschitzian and asymptotically f-pseudocontractive map T without any type of commutativity of f and T.

Corollary 2.19 (see [26, Theorem 3.1]). *Let M be a closed convex bounded subset of a real Banach space X with uniformly Gâteaux differentiable norm possessing uniform normal structure. Let $T : M \to M$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$. Let $u \in M$ be fixed, $\{\lambda_n\}$ be sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty}\lambda_n = 1$ and $\lim_{n\to\infty}(k_n - 1)/(k_n - \lambda_n) = 0$. Then,*

(A) *for each $n \geq 1$, there is unique x_n in M such that*

$$x_n = (1-\mu_n)u + \mu_n T^n x_n, \tag{2.24}$$

(B) *if $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 2.20. (1) Theorem 2.17 improves and extends the results of Beg et al. [10], Cho et al. [27], and Schu [20, 28] to more general class of Banach operators.

(2) It would be interesting to prove similar results in Modular Function Spaces (cf. [29]).

(3) Let $X = \mathbb{R}$ with the usual norm and $M = [0,1]$. A mapping T is defined by $Tx = x$, for $x \in [0,1/2]$ and $Tx = 0$, for $x \in (1/2,1]$ and $f(x) = x$ on M. Clearly, T is not f-nonexpansive [21] (e.g., $\|T(3/4) - T(1/2)\| = 1/2$ and $\|f(3/4) - f(1/2)\| = 1/4$). But, T is a f-pseudocontractive mapping.

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