Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 847170, 19 pages
doi:10.1155/2011/847170

Research Article

Common Fixed Point Theorems for a Finite Family of Discontinuous and Noncommutative Maps

Lai-Jiu Lin and Sung-Yu Wang

Department of Mathematics, National Changhua University of Education, Changhua 50058, Taiwan

Correspondence should be addressed to Lai-Jiu Lin, maljlin@cc.ncue.edu.tw

Received 30 December 2010; Accepted 20 February 2011

Academic Editor: Jong Kim

We study common fixed point theorems for a finite family of discontinuous and noncommutative single-valued functions defined in complete metric spaces. We also study a common fixed point theorem for two multivalued self-mappings and a stationary point theorem in complete metric spaces. Throughout this paper, we establish common fixed point theorems without commuting and continuity assumptions. In contrast, commuting or continuity assumptions are often assumed in common fixed point theorems. We also give examples to show our results. Results in this paper except those that generalized Banach contraction principle and those improve and generalize recent results in fixed point theorem are original and different from any existence result in the literature. The results in this paper will have some applications in nonlinear analysis and fixed point theory.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \multimap X$ be a multivalued map. We say that $x \in X$ is a stationary point of T if $T(x) = \{x\}$. The existence theorem of stationary point was first considered by Dancs et al. [1]. If S is a self-mapping (multivalued or single valued) defined on X, we denote $F(S)$ the collection of all the fixed points of S. In this paper,we need the following definitions.

Definition 1.1. A function $f : X \to X$ is called

(i) *contraction* if there exists $r \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \mathrm{rd}(x, y), \quad \forall x, y \in X, \tag{1.1}$$

(ii) *kannan* if there exists $\alpha \in [0,1/2)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, f(x)) + \alpha d(y, f(y)), \quad \forall x, y \in X, \tag{1.2}$$

(iii) *quasicontractive* if there is a constant $r \in (0,1)$ such that

$$d(f(x), f(y)) \leq rM(x,y), \quad \forall x, y \in X, \tag{1.3}$$

where $M(x,y) = \max\{d(x,y), d(x,f(x)), d(y,f(y)), d(x,f(y)), d(y,f(x))\}$.

(iv) *weakly contractive* if there exists a lower semicontinuous and nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) = 0$ if and only if $t = 0$ such that

$$d(f(x), f(y)) \leq d(x,y) - \varphi(d(x,y)), \quad \forall x, y \in X. \tag{1.4}$$

It is known that every contraction and every Kannan mapping has a unique fixed point in complete metric spaces Banach [2], Kannan [3] and every quasicontractive mapping has a unique fixed point in Banach spaces Ćirić [4], Rhoades [5]. In 2001, Rhoades [6] proved that every weakly contractive mapping has a unique fixed point in a complete metric space. Let f and g be self-maps defined on X; the following inequality was considered in the study of common fixed points theorems Rhoades [5], Chang [7]:

$$d(f(x), g(y)) \leq rM(x,y) \tag{1.5}$$

for some constant $r \in (0,1)$ and function $M : X \times X \to [0,+\infty)$.

If f and g satisfy the inequality (1.5) with

$$M(x,y) = \max\{d(x,y), d(x,f(x)), d(y,g(y)), d(x,g(y)), d(y,f(x))\}, \tag{1.6}$$

then f and g are said to be a couple of quasicontractive mappings which is studied by Rhoades [5]. Chang [7] prove that every couple of quasicontractive mappings has a unique common fixed point in Banach spaces. Recently, Zhang and Song [8] proved a common fixed point theorem in complete metric spaces under the following assumption:

$$d(f(x), g(y)) \leq M(x,y) - \varphi(M(x,y)), \tag{1.7}$$

where $M(x,y) = \max\{d(x,y), d(x,f(x)), d(y,g(y)), (1/2)[d(y,f(x)) + d(x,g(y))]\}$.

The result of Zhang and Song [8] generalized the results in [2, 3, 5, 6]. Motivated by Chang [7], Zhang and Song [8], it is natural to ask whether there is a common fixed point of f and g in X satisfy inequality (1.5) with $M(x,y) = d(x, f(y))$. In this paper, we give a positive answer to this question in complete metric spaces.

Let $\{T_i\}_{i=1}^m$ be a finite family of self-mappings on X. If there is a nondecreasing, lower semicontinuous function $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(x) = 0$ if and only if $x = 0$ such that for every $x, y \in X$,

$$d(T_i x, T_{i+1} y) \leq M_i(x,y), \quad \forall i \in \{1,2,\ldots,m\}, \tag{1.8}$$

where $M_i(x,y) = \max\{d(T_{i-1}x, T_i y), d(T_{i-1}x, T_i x), d(T_{i-1}y, T_i y)\}$, for all $i \in \{1,2,\ldots,m-1\}$, and

$$M_m(x,y) = \max\{d(T_{m-1}x, y) - \varphi(d(T_{m-1}x, y)), d(T_{m-1}x, T_m x) - \varphi(d(T_{m-1}x, T_m x))\}. \tag{1.9}$$

Here, T_0 is the identity map defined on X and $T_{m+1} = T_1$. We show that $\{T_i\}_{i=1}^m$ have a unique common fixed point if X is complete. As a special case of this result, we give a common fixed point theorem in complete metric spaces under the assumption that inequality (1.5) holds with $M(x,y) = d(x, f(y))$. One of our results generalized Banach contraction principle, an example is given (Example 2.12) to show that the maps T_i $(i = 1,2,\ldots,m)$ above need not to be continuous. The assumption of continuity is often used in the existence theorems of fixed points [6, 9–14]. We also give an example to show that the family $\{T_i\}_{i=1}^m$ above is not necessary to be commuting, and in contrast that the commutativity assumption is often used in the existence theorems of common fixed points [9, 10, 13, 15, 16]. Finally, we generalize some of our results to the case of multivalued maps.

Let $T, S : X \multimap X$ be multivalued maps satisfy

$$H(Tx, Sy) \leq \mathrm{rd}(x, Ty), \quad \forall x, y \in X. \tag{1.10}$$

for some $r \in [0,1)$ (where H denotes the Hausdorff metric). In fact, under the hypothesis that inequality (1.10) holds, we can show that $F(T) = F(S) \neq \emptyset$ and $Tx = Sx = F(T)$ for all $x \in F(T)$ if T and S have nonempty closed bounded values. Further we give a new stationary point theorem in complete metric spaces and illustrate with examples (Examples 3.4 and 3.8).

2. Fixed Point Theorems

Throughout this paper, let (X,d) be a complete metric space and let $\mathbb{N}$ be the set of all positive integers. In this section, all the self-maps on X are single valued. The following theorem is the main result in this section.

Theorem 2.1. *Let $\{T_i\}_{i=1}^m$ be a finite family of self-mappings on X. If there is a nondecreasing, lower semicontinuous function $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(x) = 0$ if and only if $x = 0$ such that for every $x, y \in X$,*

$$d(T_i x, T_{i+1} y) \leq M_i(x,y), \quad \forall i \in \{1,2,\ldots,m\}, \tag{2.1}$$

where

$$M_i(x,y) = \max\{d(T_{i-1}x, T_i y), d(T_{i-1}x, T_i x), d(T_{i-1}y, T_i y)\} \tag{2.2}$$

for all $i \in \{1, 2, \dots, m-1\}$, *and*

$$M_m(x, y) = \max\{d(T_{m-1}x, y) - \varphi(d(T_{m-1}x, y)), d(T_{m-1}x, T_m x) - \varphi(d(T_{m-1}x, T_m x))\}, \tag{2.3}$$

T_0 *is the identity map defined on* X *and* $T_{m+1} = T_1$.

Then, $\{T_i\}_{i=1}^m$ have a unique common fixed point.

Proof. For any fixed $x \in X$, take

$$\begin{aligned} &x_0 = x, x_1 = T_1 x_0, x_2 = T_2 x_0, \dots, x_m = T_m x_0, \\ &x_{m+1} = T_1 x_m, x_{m+2} = T_2 x_m, \dots, x_{2m} = T_m x_m, \\ &\vdots \end{aligned} \tag{2.4}$$

Continuing in this way, we obtain by induction a sequence $\{x_n\}$ in X such that $x_n = T_k x_{sm}$, whenever $n = k + sm$ with $1 \le k \le m$ and $s \ge 0$. Then, if $n = km$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T_m x_{(k-1)m}, T_1 x_{km}) \\ &\le M_m(x_{(k-1)m}, x_{km}) \\ &= d(x_{n-1}, x_n) - \varphi(d(x_{n-1}, x_n)) \\ &\le d(x_{n-1}, x_n). \end{aligned} \tag{2.5}$$

If $n = k + sm$ for some $1 \le k \le m-1$, then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T_k x_{sm}, T_{k+1} x_{sm}) \\ &\le M_k(x_{sm}, x_{sm}) \\ &= d(T_{k-1} x_{sm}, T_k x_{sm}) \\ &= d(x_{n-1}, x_n). \end{aligned} \tag{2.6}$$

Therefore $\{d(x_n, x_{n+1})\}$ is a decreasing and bounded below sequence,and there exists $r \ge 0$ such that $d(x_n, x_{n+1}) \to r$. Since φ is lower semicontinuous, $\varphi(r) \le \liminf_{n\to\infty} \varphi(d(x_n, x_{n+1}))$. Taking upper limits as $k \to \infty$ on two sides of the following inequality

$$d(x_{km}, x_{km+1}) \le d(x_{km-1}, x_{km}) - \varphi(d(x_{km-1}, x_{km})), \tag{2.7}$$

we have

$$r \leq r - \liminf_{n\to\infty} \varphi(d(x_n, x_{n+1})) \leq r - \varphi(r). \tag{2.8}$$

Then, $\varphi(r) = 0$ and, hence, $r = 0$.

$\{x_n\}$ is a Cauchy sequence in (X, d). Indeed, let $C_n = \sup\{d(x_j, x_k) : j, k \geq n\}$. Then $\{C_n\}$ is a decreasing sequence. If $\lim_{n\to\infty} C_n = 0$, we are done. Suppose that $\lim_{n\to\infty} C_n = C > 0$, choose $\varepsilon < C/(4m+6)$ small enough and select $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \varepsilon, \quad C_n < C + \varepsilon, \quad \forall n \geq N. \tag{2.9}$$

By the definition of C_{N+1}, there exist $k, l \geq N+1$ such that

$$d(x_k, x_l) > C_{N+1} - \varepsilon. \tag{2.10}$$

Since $d(x_n, x_{n+1}) < \varepsilon$, for all $n \geq N$. Replace k and l if necessary, we may assume that $k = m + sm$, $l = 1 + tm$ and

$$d(x_k, x_l) > C_{N+1} - \varepsilon - 2m\varepsilon = C_{N+1} - (2m+1)\varepsilon. \tag{2.11}$$

Hence,

$$d(x_{k-1}, x_{l-1}) > C - (2m+1)\varepsilon - 2\varepsilon = C - \frac{2m+3}{4m+6}C = \frac{C}{2}. \tag{2.12}$$

Then,

$$C_{N+1} - (2m+1)\varepsilon < d(x_k, x_l) = d(T_m x_{sm}, T_1 x_{tm}) \leq M_m(x_{sm}, x_{tm}). \tag{2.13}$$

We consider the following two cases:

(i) $M_m(x_{sm}, x_{tm}) = d(T_{m-1}x_{sm}, x_{tm}) - \varphi(d(T_{m-1}x_{sm}, x_{tm}))$

(ii) $M_m(x_{sm}, x_{tm}) = d(T_{m-1}x_{sm}, T_m x_{sm}) - \varphi(d(T_{m-1}x_{sm}, T_m x_{sm}))$.

If $M_m(x_{sm}, x_{tm}) = d(T_{m-1}x_{sm}, x_{tm}) - \varphi(d(T_{m-1}x_{sm}, x_{tm}))$, we have

$$\begin{aligned} C_{N+1} - (2m+1)\varepsilon &< d(T_{m-1}x_{sm}, x_{tm}) - \varphi(d(T_{m-1}x_{sm}, x_{tm})) \\ &= d(x_{k-1}, x_{l-1}) - \varphi(d(x_{k-1}, x_{l-1})) \\ &\leq d(x_{k-1}, x_{l-1}) - \varphi\left(\frac{C}{2}\right) \leq C_N - \varphi\left(\frac{C}{2}\right) \\ &< C + \varepsilon - \varphi\left(\frac{C}{2}\right). \end{aligned} \tag{2.14}$$

Then, $C_{N+1} + \varphi(C/2) < C + (2m+2)\varepsilon$. Since ε is arbitrary small positive number, if we take $\varepsilon < \varphi(C/2)/(2m+2)$. Then,

$$C + \varphi\left(\frac{C}{2}\right) \leq C_{N+1} + \varphi\left(\frac{C}{2}\right) < C + (2m+2)\varepsilon < C + \varphi\left(\frac{C}{2}\right). \tag{2.15}$$

This yields a contradiction.

If $M_m(x_{sm}, x_{tm}) = d(T_{m-1}x_{sm}, T_m x_{sm}) - \varphi(d(T_{m-1}x_{sm}, T_m x_{sm}))$, we have

$$\begin{aligned} C_{N+1} - (2m+1)\varepsilon &< d(T_{m-1}x_{sm}, T_m x_{sm}) - \varphi(d(T_{m-1}x_{sm}, T_m x_{sm})) \\ &= d(x_{k-1}, x_k) - \varphi(d(x_{k-1}, x_k)) \leq \varepsilon. \end{aligned} \tag{2.16}$$

Then $C_{N+1} < (2m+2)\varepsilon < C/2$. This also yields a contradiction.

Therefore, $C = 0$ and $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, $\{x_n\}$ converges to a point in X, say z.

In order to show that z is the unique common fixed point of $\{T_i\}_{i=1}^m$. We first claim that $T_i z = z$, for all $i = 1, 2, \ldots, m$.

Indeed, for each $i \in \{1, 2, \ldots, m\}$,

$$d(T_i z, z) = \lim_{k\to\infty} d(T_i z, x_{i-1+km}) = \lim_{k\to\infty} d(T_i z, T_{i-1}x_{km}) \leq \lim_{k\to\infty} M_{i-1}(x_{km}, z). \tag{2.17}$$

We consider the following three cases:

(i) $M_{i-1}(x_{km}, z) = d(T_{i-2}x_{km}, T_{i-1}x_{km})$,

(ii) $M_{i-1}(x_{km}, z) = d(T_{i-2}z, T_{i-1}z)$,

(iii) $M_{i-1}(x_{km}, z) = d(T_{i-2}x_{km}, T_{i-1}z)$.

If $M_{i-1}(x_{km}, z) = d(T_{i-2}x_{km}, T_{i-1}x_{km})$, then

$$d(T_i z, z) \leq \lim_{k\to\infty} d(T_{i-2}x_{km}, T_{i-1}x_{km}) = \lim_{k\to\infty} d(x_{km+i-2}, x_{km+i-1}) = 0. \tag{2.18}$$

If $M_{i-1}(x_{km}, z) = d(T_{i-2}z, T_{i-1}z)$, then

$$\begin{aligned} d(T_i z, z) \leq d(T_{i-2}z, T_{i-1}z) &\leq M_{i-2}(z, z) = d(T_{i-3}z, T_{i-2}z) \\ &\ \ \vdots \\ &\leq d(z, T_1 z) = \lim_{t\to\infty} d(T_1 z, x_{tm}) = \lim_{t\to\infty} d\big(T_1 z, T_m x_{(t-1)m}\big) \\ &\leq \lim_{t\to\infty} d\big(z, T_{m-1}x_{(t-1)m}\big) - \varphi\big(d\big(z, T_{m-1}x_{(t-1)m}\big)\big) \\ &= \lim_{t\to\infty} d(z, x_{tm-1}) - \varphi(d(z, x_{tm-1})) \leq d(z, z) = 0. \end{aligned} \tag{2.19}$$

If $M_{i-1}(x_{km}, z) = d(T_{i-2}x_{km}, T_{i-1}z)$, then

$$d(T_i z, z) \leq \lim_{k\to\infty} d(T_{i-2}x_{km}, T_{i-1}z) = \lim_{k\to\infty} d(T_{i-1}z, x_{km+i-2}). \tag{2.20}$$

Continuing in this process, we show that $d(T_i z, z) \leq \lim_{k\to\infty} d(T_1 z, x_{km})$. By the same argument as in the case above, we see that $d(T_i z, z) = 0$.

Then, we see that $T_i z = z$, for all $i = 1, 2, \ldots, m$. Next, we claim that z is the unique fixed point of T_1. Indeed, for any $x \in F(T_1)$, we have

$$\begin{aligned} d(x,z) &= d(T_1 x, z) = d(T_1 x, T_m z) \\ &\leq d(x, T_{m-1}z) - \varphi(d(x, T_{m-1}z)) = d(x,z) - \varphi(d(x,z)). \end{aligned} \tag{2.21}$$

Then, $d(x,z) = 0$ and $y = z$. Therefore, z is the unique fixed point of T_1 and we complete the proof. □

Remark 2.2. (a) The sequence $\{x_n\}$ approaching to the unique common fixed point in Theorem 2.1 is different from those in [8, 11, 12, 16–19].

(b) The finite family $\{T_i\}_{i=1}^m$ of self-mappings in Theorem 2.1 is neither commuting nor continuous, which are often assumed in common fixed point theorems, see [6, 9–16]. In fact, the commuting and continuity assumptions are not needed throughout this paper and we will give examples (Examples 2.12–2.15) to show this fact.

As special cases of Theorem 2.1, we have the following theorems and corollaries.

Theorem 2.3. *Let $S, T : X \to X$, be self-mappings on X. If there is a nondecreasing, lower semicontinuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(x) = 0$ if and only if $x = 0$ such that*

$$\begin{aligned} d(Tx, Sy) &\leq \max\{d(x, Ty) - \varphi(d(x, Ty)), d(Ty, Sy) - \varphi(d(Ty, Sy))\}, \\ d(Tx, Sy) &\leq \max\{d(x, Ty), d(x, Tx), d(y, Ty)\}, \end{aligned} \tag{2.22}$$

for all $x, y \in X$. Then S and T have a unique common fixed point.

Proof. Take $m = 2$, $T_1 = T$ and $T_2 = S$ in Theorem 2.1, then Theorem 2.3 follows from Theorem 2.1. □

Corollary 2.4. *Let S, T be self-mappings on X. If there is a nondecreasing, lower semicontinuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(x) = 0$ if and only if $x = 0$ such that*

$$d(Tx, Sy) \leq d(x, Ty) - \varphi(d(x, Ty)) \quad \forall x, y \in X. \tag{2.23}$$

Then S and T have a unique common fixed point.

Corollary 2.5. *Let T be a self-mapping on X. If there is a nondecreasing, lower semicontinuous function $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(x) = 0$ if and only if $x = 0$ such that*

$$d\left(Tx, T^2y\right) \leq d(x, Ty) - \varphi(d(x, Ty)) \quad \forall x, y \in X. \tag{2.24}$$

Then T has a unique fixed point.

Proof. Take $S = T^2$ in Theorem 2.3, then Corollary 2.4 follows from Corollary 2.4. □

Remark 2.6. (a) Since $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$ for all $x, y \in X$ implies

$$d\left(Tx, T^2y\right) \leq d(x, Ty) - \varphi(d(x, Ty)), \quad \forall x, y \in X. \tag{2.25}$$

Corollary 2.5 generalizes Theorem 1 in Rhoades [6].

(b) Corollary 2.4 is equivalent to Corollary 2.5.

Proof. It suffices to show that $S = T^2$ in Corollary 2.4. Indeed, for each $y \in X$, there exists $x \in X$ such that $x = T(y)$. By the hypothesis in Corollary 2.4, $d(T^2y, Sy) = d(Tx, Sy) \leq d(x, Ty) = 0$ and we complete the proof.

(c) In Theorem 2.3, the map S is not necessary equal T^2, see Example 2.14. In fact, the maps S and T in Theorem 2.3 are not necessary to be commuting, see Example 2.15. □

Theorem 2.7. *Let $\{T_i\}_{i=1}^m$ be a finite family of self-mappings on X. If there exists $r \in [0,1)$ such that for every $x, y \in X$*

$$d(T_i x, T_{i+1} y) \leq M_i(x, y), \quad \forall i \in \{1, 2, \dots, m\}, \tag{2.26}$$

where

$$M_i(x, y) = \begin{cases} \max\{d(T_{i-1}x, T_i y), d(T_{i-1}x, T_i x), d(T_{i-1}y, T_i y)\}, & \text{if } i \in \{1, 2, \dots, m-1\}, \\ r \max\{d(T_{m-1}x, y), d(T_{m-1}x, T_m x)\}, & \text{if } i = m, \end{cases} \tag{2.27}$$

and T_0 is the identity map defined on X and $T_{m+1} = T_1$.

Then $\{T_i\}_{i=1}^m$ have a unique common fixed point.

Proof. Take $\varphi(t) = (1 - r)t$ for all $t \in [0, \infty)$, then Theorem 2.7 follows from Theorem 2.1. □

Theorem 2.8. *Let S, T be self-mappings on X. If there exists $r \in [0,1)$ such that*

$$\begin{gathered} d(Tx, Sy) \leq r \max\{d(x, Ty), d(Ty, Sy)\}, \\ d(Tx, Sy) \leq \max\{d(x, Ty), d(x, Tx), d(y, Ty)\}, \end{gathered} \tag{2.28}$$

for all $x, y \in X$. Then S and T have unique common fixed point.

Proof. Take, $m = 2$, $T_1 = T$ and $T_2 = S$ in Theorem 2.7, then Theorem 2.8 follows from Theorem 2.7. □

Corollary 2.9. *Let S, T be self-mappings on X and if there exists $r \in [0,1)$ such that*

$$d(Tx, Sy) \leq \mathrm{rd}(x, Ty), \quad \forall x, y \in \mathrm{X}. \tag{2.29}$$

Then, S and T have a unique common fixed point.

Corollary 2.10. *Let T be a self-map on X and if there exists $r \in [0,1)$ such that*

$$d\left(Tx, T^2 y\right) \leq \mathrm{rd}(x, Ty), \quad \forall x, y \in \mathrm{X}. \tag{2.30}$$

Then T has a unique fixed point.

Proof. Take $S = T^2$ in Corollary 2.9, then Corollary 2.10 follows from Corollary 2.9. □

Remark 2.11. (i) If T is contractive, then there exists $r \in [0,1)$ such that $d(Tx, T^2y) \leq \mathrm{rd}(x, Ty)$ for all $x, y \in \mathrm{X}$, but the converse is not true. It is obvious that Corollary 2.9 is a special case of Corollaries 2.4 and 2.10 is a generalization of Banach contraction principle. Further we see that Corollary 2.9 is equivalent to Corollary 2.10 by the same argument as in Remark 2.6.

(ii) A map T satisfies $d(Tx, T^2y) \leq \mathrm{rd}(x, Ty)$ for all $x, y \in \mathrm{X}$ and for some $r \in [0,1)$ is neither continuous nor nonexpansive. We give an example (Example 2.12) to show this fact.

Example 2.12. Let $T : [0, 1/2] \to [0, 1/2]$ be defined by

$$T(x) = \begin{cases} x^2, & \text{if } x \in \left[0, \dfrac{1}{4}\right), \\ 0, & \text{if } x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right]. \end{cases} \tag{2.31}$$

Then, $d(Tx, T^2y) \leq (1/2)d(x, Ty)$ for all $x, y \in [0, 1/2]$ and T is not continuous.

Proof. We consider the following three cases:

(i) $x \in [0, 1/4), y \in [0, 1/2]$,

(ii) $x \in [1/4, 1/2], y \in [1/4, 1/2]$,

(iii) $x \in [1/4, 1/2], y \in [0, 1/4)$.

If $x \in [0, 1/4)$, $y \in [0, 1/2]$, then

$$\begin{aligned} d\left(Tx, T^2y\right) &= d\left(x^2, (Ty)^2\right) = \left|x^2 - (Ty)^2\right| \\ &= |x - Ty||x + Ty| \leq \frac{1}{2}|x - Ty| = \frac{1}{2}d(x, Ty). \end{aligned} \tag{2.32}$$

If $x \in [1/4, 1/2]$, $y \in [1/4, 1/2]$, then

$$d\left(Tx, T^2 y\right) = d(0,0) = 0 \text{ and hence } d\left(Tx, T^2 y\right) \le \frac{1}{2} d(x, Ty). \tag{2.33}$$

If $x \in [1/4, 1/2]$, $y \in [0, 1/4)$, then

$$\begin{aligned} d\left(Tx, T^2 y\right) &= d\left(0, y^4\right) = y^4 < \frac{1}{256}, \\ d(x, Ty) &= d\left(x, y^2\right) = x - y^2 \ge \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{3}{16} \end{aligned} \tag{2.34}$$

and, hence $d(Tx, T^2 y) \le (1/2) d(x, Ty)$. It is obvious that T is not continuous at $x = 1/4$ but $d(Tx, T^2 y) \le (1/2) d(x, Ty)$ for all $x, y \in [0, 1/2]$. □

Example 2.13. Let T be the same as in Example 2.12. and take $T_i = T^i$ for all $i \in \mathbb{N}$. Since $F(T_1) \subseteq F(T_i)$ for all $i \in \mathbb{N}$. By Example 2.12 and Corollary 2.10, we see that $\{T_i\}_{i \in \mathbb{N}}$ has a unique common fixed point. But for each $i \in \mathbb{N}$, T_i is not continuous, the results in [13, 16] do not work in this example. Further it is obvious that the family $\{T_i\}_{i \in \mathbb{N}}$ have a unique common fixed point 0.

Example 2.14. Let $X = \{\pm 1/8^k\}_{k=0}^{\infty} \cup \{0\}$ and define maps $T, S : X \to X$ by $T(x) = x/8$ and $S(x) = -x/8$. Then $S \neq T^2$ and we see that T and S have a unique common fixed point.

Proof. It suffices to show that there exists $r \in [0, 1)$ such that

$$\begin{aligned} d(Tx, Sy) &\le r \max\{d(x, Ty), d(Ty, Sy)\}, \\ d(Tx, Sy) &\le \max\{d(x, Ty), d(x, Tx), d(y, Ty)\}, \end{aligned} \tag{2.35}$$

for all $x, y \in X$.

We have to consider the following two cases:

(i) $|x| \ge |y|$,

(ii) $|x| < |y|$.

If $|x| \ge |y|$, then

$$\begin{aligned} d(Tx, Sy) &= d\left(\frac{x}{8}, \frac{-y}{8}\right) = \left|\frac{x}{8} + \frac{y}{8}\right| \le \left|\frac{x}{8}\right| + \left|\frac{y}{8}\right| \le \left|\frac{x}{4}\right|, \\ d(x, Ty) &= d\left(x, \frac{-y}{8}\right) = \left|x + \frac{y}{8}\right| \ge |x| - \left|\frac{y}{8}\right| \ge |x| - \left|\frac{x}{8}\right| = \left|\frac{7x}{8}\right|. \end{aligned} \tag{2.36}$$

If $|x| < |y|$, then

$$\begin{aligned} d(Tx, Sy) &= d\left(\frac{x}{8}, \frac{-y}{8}\right) = \left|\frac{x}{8} + \frac{y}{8}\right| \le \left|\frac{x}{8}\right| + \left|\frac{y}{8}\right| \le \left|\frac{y}{64}\right| + \left|\frac{y}{8}\right| = \left|\frac{9y}{64}\right|, \\ d(Ty, Sy) &= d\left(\frac{y}{8}, \frac{-y}{8}\right) = \left|\frac{y}{8} + \frac{y}{8}\right| = \left|\frac{y}{4}\right|, \\ d(y, Ty) &= d\left(y, \frac{y}{8}\right) = \left|y - \frac{y}{8}\right| = \left|\frac{7y}{8}\right|. \end{aligned} \tag{2.37}$$

If we take $r = 9/16$, then by Theorem 2.8, we see that S and T have a unique common fixed point. In fact, 0 is the unique common fixed point of S and T. □

By the same argument as in Example 2.14, we give the following example to show that the maps T and S in Theorem 2.8 are not necessary to be commuting.

Example 2.15. Let $X = \{\pm 1/8^{2k}\}_{k=0}^{\infty} \cup \{0\}$ and define maps $T, S : X \to X$ by $T(x) = x^2$ and $S(x) = -x^2$. Then S and T are not commuting and we see that T and S have a unique common fixed point.

Proof. It suffices to show that there exists $r \in [0, 1)$ such that

$$\begin{aligned} d(Tx, Sy) &\le r \max\{d(x, Ty), d(Ty, Sy)\}, \\ d(Tx, Sy) &\le \max\{d(x, Ty), d(x, Tx), d(y, Ty)\}, \end{aligned} \tag{2.38}$$

for all $x, y \in X$.

We have to consider the following two cases:

(i) $|x| \ge |y|$,

(ii) $|x| < |y|$.

If $|x| \ge |y|$, then

$$\begin{aligned} d(Tx, Sy) &= d\left(x^2, -y^2\right) = \left|x^2 + y^2\right| = x^2 + y^2 \le 2x^2, \\ d(x, Ty) &= d\left(x, y^2\right) = \left|x - y^2\right| \ge |x| - y^2 \ge 8x^2 - y^2 \ge 7x^2. \end{aligned} \tag{2.39}$$

If $|x| < |y|$, then

$$\begin{aligned} d(Tx, Sy) &= d\left(x^2, -y^2\right) = \left|x^2 + y^2\right| \le \frac{y^2}{64} + y^2 = \frac{65}{64}y^2, \\ d(Ty, Sy) &= d\left(y^2, -y^2\right) = 2y^2, \\ d(y, Ty) &= d\left(y, -y^2\right) = \left|y + y^2\right| \ge |y| - y^2 \ge 8y^2 - y^2 = 7y^2. \end{aligned} \tag{2.40}$$

If we take $r = 65/128$, then by Theorem 2.8, we see S and T have a unique common fixed point. In fact, 0 is the unique common fixed point of S and T. □

3. A Common Fixed Point Theorem of Set-Valued Maps and a Stationary Point Theorem

In this section, we study a fixed point theorem and a stationary point theorem which generalize a fixed point theorem in Section 2.

In this section, let $\mathrm{CB}(X)$ be the class of all nonempty bounded closed subsets of X and for $A, B \in \mathrm{CB}(X)$, let $H(A, B)$ be the Hausdorff metric of A and B and let $d(x, B) = \inf_{b\in B} d(x, b)$ for all $x \in X$.

Lemma 3.1 (see [20]). *For all $A, B \in CB(X)$, $\varepsilon > 0$ and $a \in A$, there exists $b \in B$ such that $d(a, b) \le H(A, B) + \varepsilon$.*

Theorem 3.2. *Let $S, T : X \to CB(X)$ be multivalued maps. If there exists $r \in [0, 1)$ such that*

$$H(Tx, Sy) \le \mathrm{r}d(x, Ty), \quad \forall x, y \in X. \tag{3.1}$$

Then $F(T) = F(S) \neq \emptyset$ and $Tx = Sx = F(T)$ for all $x \in F(T)$.

Proof. For any fixed $x \in X$ and $0 < \varepsilon < 1$. Take $x_0 = x$, and let $x_1 \in Tx_0$. By Lemma 3.1, we may choose $x_2 \in Sx_0$ such that $d(x_1, x_2) \le H(Tx_0, Sx_0) + \varepsilon$, $x_3 \in Tx_2$ such that $d(x_2, x_3) \le H(Sx_0, Tx_2) + \varepsilon^2$, $x_4 \in Sx_2$ such that $d(x_3, x_4) \le H(Tx_2, Sx_2) + \varepsilon^3, \ldots$. Continuing in this process, we obtain by induction a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\begin{gathered} x_{2n} \in Sx_{2n-2}, x_{2n+1} \in Tx_{2n}, \\ d(x_{2n+1}, x_{2n+2}) \le H(Tx_{2n}, Sx_{2n}) + \varepsilon^{2n+1}, \\ d(x_{2n}, x_{2n+1}) \le H(Sx_{2n-2}, Tx_{2n}) + \varepsilon^{2n}. \end{gathered} \tag{3.2}$$

Therefore,

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\le H(Sx_{2n-2}, Tx_{2n}) + \varepsilon^{2n} \\ &\le \mathrm{r}d(x_{2n}, Tx_{2n-2}) + \varepsilon^{2n} \le \mathrm{r}d(x_{2n}, x_{2n-1}) + \varepsilon^{2n}, \\ d(x_{2n+1}, x_{2n+2}) &\le H(Tx_{2n}, Sx_{2n}) + \varepsilon^{2n+1} \\ &\le \mathrm{r}d(x_{2n}, Tx_{2n}) + \varepsilon^{2n+1} \le \mathrm{r}d(x_{2n}, x_{2n+1}) + \varepsilon^{2n+1}. \end{aligned} \tag{3.3}$$

Therefore, $d(x_n, x_{n+1}) \leq \mathrm{rd}(x_{n-1}, x_n) + \varepsilon^n$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \mathrm{rd}(x_{n-1}, x_n) + \varepsilon^n \\
&\leq r\left(\mathrm{rd}(x_{n-2}, x_{n-1}) + \varepsilon^{n-1}\right) + \varepsilon^n \\
&= r^2 d(x_{n-2}, x_{n-1}) + r\varepsilon^{n-1} + \varepsilon^n \\
&\leq r^2\left(\mathrm{rd}(x_{n-3}, x_{n-2}) + \varepsilon^{n-2}\right) + r\varepsilon^{n-1} + \varepsilon^n \\
&= r^3 d(x_{n-3}, x_{n-2}) + r^2\varepsilon^{n-2} + r\varepsilon^{n-1} + \varepsilon^n \\
&\vdots \\
&\leq r^n d(x_0, x_1) + \sum_{k=1}^{n} r^{n-k}\varepsilon^k.
\end{aligned} \tag{3.4}$$

This shows that

$$\begin{aligned}
\sum_{n=1}^{\infty} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{\infty} r^n d(x_0, x_1) + \sum_{n=1}^{\infty}\left[\varepsilon^n\left(\sum_{k=0}^{\infty} r^k\right)\right] \\
&= \frac{1}{1-r}\left[\mathrm{rd}(x_0, x_1) + \frac{\varepsilon}{1-\varepsilon}\right] < \infty
\end{aligned} \tag{3.5}$$

and $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. Since

$$\begin{aligned}
d(Tz, z) = \lim_{n\to\infty} d(Tz, x_{2n+2}) &\leq \lim_{n\to\infty} H(Tz, Sx_{2n}) \\
&\leq r\lim_{n\to\infty} d(z, Tx_{2n}) \leq r\lim_{n\to\infty} d(z, x_{2n+1}) = \mathrm{rd}(z, z) = 0
\end{aligned} \tag{3.6}$$

and $H(Tz, Sz) \leq \mathrm{rd}(z, Tz) = 0$, $z \in Tz = Sz$. Therefore $z \in F(T) \neq \emptyset$ and $z \in F(S) \neq \emptyset$.

To complete the proof, it suffices to show the following four cases:

(i) $F(T) \subseteq Tz$ and $Sx = Tx$ for all $x \in F(T)$,

(ii) $Tz \subseteq F(T)$,

(iii) $Tx = Tz$ for all $x \in F(T)$,

(iv) $F(S) \subseteq Tz$.

For any $x \in F(T)$,

$$d(x, Tz) \leq H(Tx, Sz) \leq \mathrm{rd}(x, Tz). \tag{3.7}$$

This shows that $d(x, Tz) = 0$ and $x \in Tz$. Further

$$H(Sx, Tx) \leq \mathrm{rd}(x, Tx) = 0 \tag{3.8}$$

and $x \in Sx = Tx$.

For any $x \in Tz$,

$$d(x, Tx) \leq H(Sz, Tx) \leq \mathrm{rd}(Tz, x) = 0. \tag{3.9}$$

This shows that $x \in Tx$. Till now, we see that $Tz = F(T) \subseteq F(S)$ and $Sx = Tx$ for all $x \in F(T)$.

For any $x \in F(T)$,

$$H(Tx, Sz) \leq \mathrm{rd}(x, Tz) = \mathrm{rd}(x, F(T)) = 0. \tag{3.10}$$

Hence $Tx = Sz = Tz$.

It remains to show that $F(S) \subseteq Tz = F(T)$.

Indeed, for any $x \in F(S)$,

$$\begin{aligned} d(x, Tz) &\leq H(Sx, Tz) \\ &\leq \mathrm{rd}(Tx, z) \leq rH(Tx, Sz) \leq r^2 d(x, Tz) \end{aligned} \tag{3.11}$$

and $d(x, Tz) = 0$. Then $x \in Tz$ and $F(S) \subseteq Tz$. □

Remark 3.3. (a) If one of S and T in Theorem 3.2 is single valued, then the set $F(T) = F(S)$ is singleton and the maps S and T have a unique common fixed point in X. Therefore, Theorem 3.2 is a generalization of Corollary 2.9, but Theorem 3.2 is not a generalization of Theorem 5 Nadler [20].

(b) The sequence $\{x_n\}$ approaches to the common fixed point z of S and T in Theorem 3.2 is different from those in [20–25].

(c) By Example 2.12, we see that both T and S in Theorem 3.2 are neither to be upper semicontinuous nor to be lower semicontinuous (multivalued maps). Further the maps T and S are not necessary to be commuting. We give an example below.

Example 3.4. Let $X = \{\pm 1/8^k\}_{k=1}^{\infty} \cup \{0\}$ and let maps $T, S : X \to X$ be defined by $T(x) = \{\pm x/8^k\}_{k=1}^{\infty} \cup \{0\}$ and $S(x) = \{|x|/8^{k+1}\}_{k=1}^{\infty} \cup \{0\}$. Then we see that T and S have a unique common fixed point.

Proof. It suffices to show that there exists $r \in [0, 1)$ such that

$$H(Tx, Sy) \leq \mathrm{rd}(x, Ty), \quad \forall x, y \in X. \tag{3.12}$$

We have to consider the following two cases:

(i) $|x| \geq |y|$

(ii) $|x| < |y|$

If $|x| \geq |y|$, we have

$$H(Tx, Sy) = \frac{|x|}{8}, \qquad H(x, Ty) \geq |x| - \frac{|x|}{8} = \frac{7}{8}|x|. \tag{3.13}$$

If $|x| < |y|$, we have

$$H(Tx, Sy) \leq \frac{|y|}{64}, \qquad H(x, Ty) = |x| + \frac{|y|}{8} \geq \frac{1}{8}|y|. \tag{3.14}$$

If we take $r = 1/7$, then by Theorem 3.2, we see S and T have a unique common fixed point. In fact, 0 is the unique common fixed point of S and T. □

Corollary 3.5. *Let $T : X \multimap X$ be a multivalued map with nonempty compact values and $r \in [0, 1)$ such that*

$$H\left(Tx, T^2 y\right) \leq \mathrm{rd}(x, Ty), \quad \forall x, y \in X. \tag{3.15}$$

Then $F(T) \neq \emptyset$ and $Tx = F(T)$ for all $x \in F(T)$.

Similarly, we have the following existence theorem of stationary points.

Theorem 3.6. *Let $T : X \to CB(X)$ be a multivalued map, $S : X \to X$ be a single valued function. If $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing, lower semicontinuous function with $\varphi(a) - \varphi(b) \leq a - b$ for all $a \geq b \geq 0$ and $\varphi(x) = 0$ if and only if $x = 0$. Suppose that*

$$H(Tx, Sy) \leq d(x, Ty) - \varphi(d(x, Ty)), \quad \forall x, y \in X. \tag{3.16}$$

Then T has a unique stationary point, say $z \in X$. In fact, $F(T) = F(S) = \{z\}$ and $Tz = \{Sz\} = \{z\}$.

Proof. For any fixed $x_0 \in X$, let $x_1 \in Tx_0$, $x_2 = Sx_0$, $x_3 \in Tx_2$, $x_4 = Sx_2, \ldots$. Continuing in this process, we obtain by induction a sequence $\{x_n\}$ such that $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} = Sx_{2n}$. Since

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq H(x_{2n}, Tx_{2n}) \\ &= H(Sx_{2n-2}, Tx_{2n}) \leq d(x_{2n}, Tx_{2n-2}) - \varphi(d(x_{2n}, Tx_{2n-2})) \\ &\leq d(x_{2n}, x_{2n-1}) - \varphi(d(x_{2n}, x_{2n-1})) \leq d(x_{2n-1}, x_{2n}), \\ d(x_{2n+1}, x_{2n+2}) &\leq H(Tx_{2n}, x_{2n+2}) \\ &= H(Tx_{2n}, Sx_{2n}) \leq d(x_{2n}, Tx_{2n}) - \varphi(d(x_{2n}, Tx_{2n})) \\ &\leq d(x_{2n}, x_{2n+1}) - \varphi(d(x_{2n}, x_{2n+1})) \leq d(x_{2n}, x_{2n+1}). \end{aligned} \tag{3.17}$$

Then, $d(x_n, x_{n+1})$ is a decreasing and bounded below sequence, and hence there exist $r \geq 0$ such that $d(x_n, x_{n+1}) \to r$. Since φ is lower semicontinuous, $\varphi(r) \leq \liminf_{n\to\infty}\varphi(d(x_n, x_{n+1}))$. Taking upper limits as $n \to \infty$ on two sides of the following inequality

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) - \varphi(d(x_{n-1}, x_n)), \tag{3.18}$$

we have

$$r \leq r - \liminf_{n\to\infty}\varphi(d(x_n, x_{n+1})) \leq r - \varphi(r). \tag{3.19}$$

Then, $\varphi(r) = 0$ and hence $r = 0$.

$\{x_n\}$ is a Cauchy sequence in (X, d). Indeed, let $C_n = \sup\{d(x_j, x_k) : j, k \geq n\}$. Then $\{C_n\}$ is a decreasing sequence. If $\lim_{n\to\infty} C_n = 0$, we are done. Suppose that $\lim_{n\to\infty} C_n = C > 0$, choose $\varepsilon < C/10$ small enough and select $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \varepsilon, \quad C_n < C + \varepsilon, \quad \forall n \geq N. \tag{3.20}$$

By the definition of C_{N+1}, there exists $k, l \geq N + 1$ such that

$$d(x_k, x_l) > C_{N+1} - \varepsilon. \tag{3.21}$$

Since $d(x_n, x_{n+1}) < \varepsilon$, for all $n \geq N$. Replace k and l if necessary, we may assume that $k = 2s$, $l = 1 + 2t$ and

$$d(x_k, x_l) > C_{N+1} - \varepsilon - 2\varepsilon = C_{N+1} - 3\varepsilon. \tag{3.22}$$

Hence,

$$d(x_{k-1}, x_{l-1}) > C - 3\varepsilon - 2\varepsilon > C - \frac{5}{10}C = \frac{C}{2}. \tag{3.23}$$

Then,

$$\begin{aligned}
C_{N+1} - 3\varepsilon &< d(x_k, x_l) \\
&= d(Sx_{2s-2}, x_{2t+1}) \leq H(Sx_{2s-2}, Tx_{2t}) \\
&\leq d(x_{2t}, Tx_{2s-2}) - \varphi(d(x_{2t}, Tx_{2s-2})) \\
&\leq d(x_{2t}, x_{2s-1}) - \varphi(d(x_{2t}, x_{2s-1})) \\
&= d(x_{l-1}, x_{k-1}) - \varphi(d(x_{l-1}, x_{k-1})) \\
&\leq C_N - \varphi\left(\frac{C}{2}\right) < C + \varepsilon - \varphi\left(\frac{C}{2}\right),
\end{aligned} \tag{3.24}$$

and $C_{N+1}+\varphi(C/2) < C+4\varepsilon$. Since ε is arbitrary small positive number, if we take $\varepsilon < \varphi(C/2)/4$. Then

$$C + \varphi\left(\frac{C}{2}\right) \leq C_{N+1} + \varphi\left(\frac{C}{2}\right) < C + 4\varepsilon < C + \varphi\left(\frac{C}{2}\right). \tag{3.25}$$

This yields a contradiction. Therefore $C = 0$ and $\{x_n\}$ is a Cauchy sequence in (X, d). By the completeness of (X, d), $\{x_n\}$ converges to a point in X, say z.

Since,

$$\begin{aligned} d(Tz, z) &= \lim_{k\to\infty} d(Tz, x_{2k+2}) \leq \lim_{k\to\infty} H(Tz, Sx_{2k}) \\ &\leq \lim_{k\to\infty} \left[d(z, Tx_{2k}) - \varphi(d(z, Tx_{2k}))\right] \\ &\leq \lim_{k\to\infty} d(z, Tx_{2k}) \leq \lim_{k\to\infty} d(z, x_{2k+1}) = 0 \\ H(Sz, Tz) &\leq d(z, Tz) - \varphi(d(z, Tz)) \leq 0. \end{aligned} \tag{3.26}$$

It follows that $Tz = \{Sz\} = \{z\}$.

Further for all $x \in F(S)$,

$$\begin{aligned} d(x, z) &\leq H(Sx, Tz) \leq d(Tx, z) - \varphi(d(Tx, z)) \\ &\leq H(Tx, z) = H(Tx, Sz) \leq d(x, Tz) - \varphi(d(x, Tz)) \\ &= d(x, z) - \varphi(d(x, z)). \end{aligned} \tag{3.27}$$

Then, $d(x, z) = 0$ and $x = z$.

For all $x \in F(T)$,

$$\begin{aligned} d(x, z) &\leq H(Tx, z) = H(Tx, Sz) \\ &\leq d(x, Tz) - \varphi(d(x, Tz)) = d(x, z) - \varphi(d(x, z)). \end{aligned} \tag{3.28}$$

Then, $d(x, z) = 0$ and hence $x = z$. □

Remark 3.7. (a) The single valued map S in Theorem 3.6 is not necessary to be continuous (see Example 2.12), but the continuity assumption is used in Theorem 3.2 [21, 22, 25] and Theorem 2.1 Ćirić and Ume [23]. We give an example to show that T and S in Theorem 3.6 are not necessary to be commuting.

(b) Theorems 3.2 and 3.6 are different and Theorem 3.6 is also a generalization of Corollary 2.9.

Example 3.8. Let $X = \{\pm 1/8^k\}_{k=1}^{\infty} \cup \{0\}$ and let maps $T : X \multimap X$ and $S : X \to X$ be defined by $T(x) = \{\pm x/8^k\}_{k=1}^{\infty} \cup \{0\}$ and $S(x) = |x|/64$. Then we see that T has a unique stationary point.

Proof. It suffices to show that there exists $r \in [0,1)$ such that

$$H(Tx, Sy) \le \mathrm{rd}(x, Ty), \quad \forall x, y \in X. \tag{3.29}$$

We have to consider the following two cases:

(i) $|x| \ge |y|$,

(ii) $|x| < |y|$.

If $|x| \ge |y|$, we have

$$H(Tx, Sy) = \frac{|x|}{8} + \frac{|y|}{64} \le \frac{9}{64}|x|, \qquad H(x, Ty) = |x| + \frac{|y|}{8} \ge |x|. \tag{3.30}$$

If $|x| < |y|$, we have

$$H(Tx, Sy) = \frac{|x|}{8} + \frac{|y|}{64} \le \frac{|y|}{32}, \qquad H(x, Ty) = |x| + \frac{|y|}{8} \ge \frac{1}{8}|y|. \tag{3.31}$$

If we take $r = 1/4$. Then by Theorem 3.6, we see T has a unique stationary point. In fact, 0 is the unique stationary point of T. □

References

[1] S. Dancs, M. Hegedüs, and P. Medvegyev, "A general ordering and fixed-point principle in complete metric space," *Acta Scientiarum Mathematicarum*, vol. 46, no. 1–4, pp. 381–388, 1983.

[2] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrales," *Fundamenta Mathematican*, vol. 3, pp. 133–181, 1922.

[3] R. Kannan, "Some results on fixed points—II," *The American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.

[4] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, pp. 267–273, 1974.

[5] B. E. Rhoades, "A comparison of various definitions of contractive mappings," *Transactions of the American Mathematical Society*, vol. 226, pp. 257–290, 1977.

[6] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 47, pp. 2683–2693, 2001.

[7] S. S. Chang, "Random fixed point theorem in probabilistic analysis," *Nonlinear Analysis*, vol. 5, no. 2, pp. 113–122, 1981.

[8] Q. Zhang and Y. Song, "Fixed point theory for generalized φ-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 75–78, 2009.

[9] M. A. Al-Thagafi, "Common fixed points and best approximation," *Journal of Approximation Theory*, vol. 85, no. 3, pp. 318–323, 1996.

[10] D. Ilić and V. Rakočević, "Common fixed points for maps on cone metric space," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 876–882, 2008.

[11] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.

[12] S. Ishikawa, "Fixed points and iteration of a nonexpansive mapping in a Banach space," *Proceedings of the American Mathematical Society*, vol. 59, no. 1, pp. 65–71, 1976.

[13] A. Kaewcharoen and W. A. Kirk, "Nonexpansive mappings defined on unbounded domains," *Fixed Point Theory and Applications*, vol. 2006, Article ID 82080, 13 pages, 2006.

[14] P. Oliveira, "Two results on fixed points," *Nonlinear Analysis. Theory, Methods & Applicationsand Methods*, vol. 47, pp. 2703–2717, 2001.

[15] T. Kuczumow, S. Reich, and D. Shoikhet, "The existence and non-existence of common fixed points for commuting families of holomorphic mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 43, no. 1, pp. 45–59, 2001.

[16] J.-Z. Xiao and X.-H. Zhu, "Common fixed point theorems on weakly contractive and nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 469357, 8 pages, 2008.

[17] W. A. Kirk, "On successive approximations for nonexpansive mappings in Banach spaces," *Glasgow Mathematical Journal*, vol. 12, pp. 6–9, 1971.

[18] R. P. Pant, "Common fixed point theorems for contractive maps," *Journal of Mathematical Analysis and Applications*, vol. 226, no. 1, pp. 251–258, 1998.

[19] T. Shimizu and W. Takahashi, "Strong convergence to common fixed points of families of nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 211, no. 1, pp. 71–83, 1997.

[20] S. B. Nadler, Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.

[21] A. Ahmad and M. Imdad, "Some common fixed point theorems for mappings and multi-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 218, no. 2, pp. 546–560, 1998.

[22] A. Ahmed and A. R. Khan, "Some common fixed point theorems for non-self-hybrid contractions," *Journal of Mathematical Analysis and Applications*, vol. 213, no. 1, pp. 275–286, 1997.

[23] L. B. Ćirić and J. S. Ume, "Some common fixed point theorems for weakly compatible mappings," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 488–499, 2006.

[24] P. Z. Daffer and H. Kaneko, "Fixed points of generalized contractive multi-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, pp. 655–666, 1995.

[25] M. Imdad and L. Khan, "Fixed point theorems for a family of hybrid pairs of mappings in metrically convex spaces," *Fixed Point Theory and Applications*, no. 3, pp. 281–294, 2005.