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Research Article

Second-Order Contingent Derivative of the Perturbation Map in Multiobjective Optimization

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Some relationships between the second-order contingent derivative of a set-valued map and its profile map are obtained. By virtue of the second-order contingent derivatives of set-valued maps, some results concerning sensitivity analysis are obtained in multiobjective optimization. Several examples are provided to show the results obtained.

1. Introduction

In this paper, we consider a family of parametrized multiobjective optimization problems

$$\text{(PVOP)}\begin{cases}\min & f(u,x) = \left(f_1(u,x), f_2(u,x), \ldots, f_m(u,x)\right),\\ \text{s.t.} & u \in X(x) \subseteq R^p.\end{cases} \tag{1.1}$$

Here, u is a p-dimensional decision variable, x is an n-dimensional parameter vector, X is a nonempty set-valued map from R^n to R^p, which specifies a feasible decision set, and f is an objective map from $R^p \times R^n$ to R^m, where m, n, p are positive integers. The norms of all finite dimensional spaces are denoted by $\|\cdot\|$. C is a closed convex pointed cone with nonempty interior in R^m. The cone C induces a partial order $\leq_C$ on R^m, that is, the relation $\leq_C$ is defined by

$$y \leq_C y' \longleftrightarrow y' - y \in C, \quad \forall y, y' \in R^m. \tag{1.2}$$

We use the following notion. For any $y, y' \in R^m$,

$$y <_C y' \longleftrightarrow y' - y \in \text{int } C. \tag{1.3}$$

Based on these notations, we can define the following two sets for a set M in R^m:

(i) $y_0 \in M$ is a C-minimal point of M with respect to C if there exists no $y \in M$, such that $y \leq_C y_0$, $y \neq y_0$,

(ii) $y_0 \in M$ is a weakly C-minimal point of M with respect to C if there exists no $y \in M$, such that $y <_C y_0$.

The sets of C-minimal point and weakly C-minimal point of M are denoted by $\text{Min}_C M$ and $W\text{Min}_C M$, respectively.

Let G be a set-valued map from R^n to R^m defined by

$$G(x) = \{y \in R^m \mid y = f(u, x), \text{ for some } u \in X(x)\}. \tag{1.4}$$

$G(x)$ is considered as the feasible set map. In the vector optimization problem corresponding to each parameter valued x, our aim is to find the set of C-minimal point of the feasible set map $G(x)$. The set-valued map W from R^n to R^m is defined by

$$W(x) = \text{Min}_C G(x), \tag{1.5}$$

for any $x \in R^n$, and call it the perturbation map for (PVOP).

Sensitivity and stability analysis is not only theoretically interesting but also practically important in optimization theory. Usually, by sensitivity we mean the quantitative analysis, that is, the study of derivatives of the perturbation function. On the other hand, by stability we mean the qualitative analysis, that is, the study of various continuity properties of the perturbation (or marginal) function (or map) of a family of parametrized vector optimization problems.

Some interesting results have been proved for sensitivity and stability in optimization (see [1–16]). Tanino [5] obtained some results concerning sensitivity analysis in vector optimization by using the concept of contingent derivatives of set-valued maps introduced in [17], and Shi [8] and Kuk et al. [7, 11] extended some of Tanino's results. As for vector optimization with convexity assumptions, Tanino [6] studied some quantitative and qualitative results concerning the behavior of the perturbation map, and Shi [9] studied some quantitative results concerning the behavior of the perturbation map. Li [10] discussed the continuity of contingent derivatives for set-valued maps and also discussed the sensitivity, continuity, and closeness of the contingent derivative of the marginal map. By virtue of lower Studniarski derivatives, Sun and Li [14] obtained some quantitative results concerning the behavior of the weak perturbation map in parametrized vector optimization.

Higher order derivatives introduced by the higher order tangent sets are very important concepts in set-valued analysis. Since higher order tangent sets, in general, are not cones and convex sets, there are some difficulties in studying set-valued optimization problems by virtue of the higher order derivatives or epiderivatives introduced by the higher order tangent sets. To the best of our knowledge, second-order contingent derivatives of perturbation map in multiobjective optimization have not been studied until now. Motivated by the work reported in [5–11, 14], we discuss some second-order quantitative results concerning the behavior of the perturbation map for (PVOP).

The rest of the paper is organized as follows. In Section 2, we collect some important concepts in this paper. In Section 3, we discuss some relationships between the second-order contingent derivative of a set-valued map and its profile map. In Section 4, by the second-order contingent derivative, we discuss the quantitative information on the behavior of the perturbation map for (PVOP).

2. Preliminaries

In this section, we state several important concepts.

Let $F : R^n \to 2^{R^m}$ be nonempty set-valued maps. The efficient domain and graph of F are defined by

$$\operatorname{dom}(F) = \{x \in R^n \mid F(x) \neq \emptyset\},$$
$$\operatorname{gph}(F) = \left\{(x, y) \in R^n \times R^m \mid y \in F(x), x \in R^n\right\}, \tag{2.1}$$

respectively. The profile map F_+ of F is defined by $F_+(x) = F(x) + C$, for every $x \in \operatorname{dom}(F)$, where C is the order cone of R^m.

Definition 2.1 (see [18]). A base for C is a nonempty convex subset Q of C with $0_{R^m} \notin clQ$, such that every $c \in C$, $c \neq 0_{R^m}$, has a unique representation of the form αb, where $b \in Q$ and $\alpha > 0$.

Definition 2.2 (see [19]). F is said to be locally Lipschitz at $x_0 \in R^n$ if there exist a real number $\gamma > 0$ and a neighborhood $U(x_0)$ of x_0, such that

$$F(x_1) \subseteq F(x_2) + \gamma\|x_1 - x_2\| B_{R^m}, \quad \forall x_1,\ x_2 \in U(x_0), \tag{2.2}$$

where B_{R^m} denotes the closed unit ball of the origin in R^m.

3. Second-Order Contingent Derivatives for Set-Valued Maps

In this section, let X be a normed space supplied with a distance d, and let A be a subset of X. We denote by $d(x, A) = \inf_{y \in A} d(x, y)$ the distance from x to A, where we set $d(x, \emptyset) = +\infty$. Let Y be a real normed space, where the space Y is partially ordered by nontrivial pointed closed convex cone $C \subset Y$. Now, we recall the definitions in [20].

Definition 3.1 (see [20]). Let A be a nonempty subset X, $x_0 \in cl(A)$, and $u \in X$, where $cl(A)$ denotes the closure of A.

(i) The second-order contingent set $T_A^{(2)}(x_0, u)$ of A at (x_0, u) is defined as

$$T_A^{(2)}(x_0, u) = \left\{ x \in X \mid \exists h_n \longrightarrow 0^+, x_n \longrightarrow x, \text{s.t. } x_0 + h_n u + h_n^2 x_n \in A \right\}. \tag{3.1}$$

(ii) The second-order adjacent set $T_A^{\flat(2)}(x_0, u)$ of A at (x_0, u) is defined as

$$T_A^{\flat(2)}(x_0, u) = \left\{ x \in X \mid \forall h_n \longrightarrow 0^+, \exists x_n \longrightarrow x, \text{s.t. } x_0 + h_n u + h_n^2 x_n \in A \right\}. \tag{3.2}$$

Definition 3.2 (see [20]). Let X, Y be normed spaces and $F : X \to 2^Y$ be a set-valued map, and let $(x_0, y_0) \in \text{gph}(F)$ and $(u, v) \in X \times Y$.

(i) The set-valued map $D^{(2)}F(x_0, y_0, u, v)$ from X to Y defined by

$$\text{gph}\Big(D^{(2)}F(x_0, y_0, u, v) = T^{(2)}_{\text{gph}(F)}(x_0, y_0, u, v), \tag{3.3}$$

is called second-order contingent derivative of F at (x_0, y_0, u, v).

(ii) The set-valued map $D^{\flat(2)}F(x_0, y_0, u, v)$ from X to Y defined by

$$\text{gph}\Big(D^{\flat(2)}F(x_0, y_0, u, v) = T^{\flat(2)}_{\text{gph}(F)}(x_0, y_0, u, v), \tag{3.4}$$

is called second-order adjacent derivative of F at (x_0, y_0, u, v).

Definition 3.3 (see [21]). The C-domination property is said to be held for a subset H of Y if $H \subset \text{Min}_C H + C$.

Proposition 3.4. *Let $(x_0, y_0) \in gph(F)$ and $(u, v) \in X \times Y$, then*

$$D^{(2)}F(x_0, y_0, u, v)(x) + C \subseteq D^{(2)}(F + C)(x_0, y_0, u, v)(x), \tag{3.5}$$

for any $x \in X$.

Proof. The conclusion can be directly obtained similarly as the proof of [5, Proposition 2.1]. □

It follows from Proposition 3.4 that

$$\text{dom}\Big[D^{(2)}F(x_0, y_0, u, v)\Big] \subseteq \text{dom}\Big[D^{(2)}F_+(x_0, y_0, u, v)\Big]. \tag{3.6}$$

Note that the inclusion of

$$D^{(2)}F_+(x_0,y_0,u,v)(x) \subseteq D^{(2)}F(x_0,y_0,u,v)(x)+C, \tag{3.7}$$

may not hold. The following example explains the case.

Example 3.5. Let $X = R$, $Y = R$, and $C = R_+$. Consider a set-valued map $F : X \to 2^Y$ defined by

$$F(x)\begin{cases}\{y \mid y \geq x^2\} & \text{if } x \leq 0,\\ \{x^2,-1\} & \text{if } x > 0.\end{cases} \tag{3.8}$$

Let $(x_0,y_0) = (0,0) \in \mathrm{gph}(F)$ and $(u,v) = (1,0)$, then, for any $x \in X$,

$$D^{(2)}F_+(x_0,y_0,u,v)(x) = R, \qquad D^{(2)}F(x_0,y_0,u,v)(x) = \{1\}. \tag{3.9}$$

Thus, one has

$$D^{(2)}F_+(x_0,y_0,u,v)(x) \not\subseteq D^{(2)}F(x_0,y_0,u,v)(x)+C, \quad x \in X, \tag{3.10}$$

which shows that the inclusion of (3.7) does not hold here.

Proposition 3.6. *Let $(x_0,y_0) \in gph(F)$ and $(u,v) \in X \times Y$. Suppose that C has a compact base Q, then for any $x \in X$,*

$$\mathrm{Min}_C D^{(2)}F_+(x_0,y_0,u,v)(x) \subseteq D^{(2)}F(x_0,y_0,u,v)(x). \tag{3.11}$$

Proof. Let $x \in X$. If $\mathrm{Min}_C D^{(2)}F_+(x_0,y_0,u,v)(x) = \emptyset$, then (3.11) holds trivially. So, we assume that $\mathrm{Min}_C D^{(2)}F_+(x_0,y_0,u,v)(x) \neq \emptyset$, and let

$$y \in \mathrm{Min}_C D^{(2)}F_+(x_0,y_0,u,v)(x). \tag{3.12}$$

Since $y \in D^{(2)}F_+(x_0,y_0,u,v)(x)$, there exist sequences $\{h_n\}$ with $h_n \to 0^+$, $\{(x_n,y_n)\}$ with $(x_n,y_n) \to (x,y)$, and $\{c_n\}$ with $c_n \in C$, such that

$$y_0 + h_n v + h_n^2(y_n - c_n) \in F\left(x_0 + h_n u + h_n^2 x_n\right), \quad \text{for any } n. \tag{3.13}$$

It follows from $c_n \in C$ and C has a compact base Q that there exist some $\alpha_n > 0$ and $b_n \in Q$, such that, for any n, one has $c_n = \alpha_n b_n$. Since Q is compact, we may assume without loss of generality that $b_n \to b \in Q$.

We now show $\alpha_n \to 0$. Suppose that $\alpha_n \nrightarrow 0$, then for some $\varepsilon > 0$, we may assume without loss of generality that $\alpha_n \geq \varepsilon$, for all n, by taking a subsequence if necessary. Let $\overline{c}_n = (\varepsilon/\alpha_n)c_n$, then, for any n, $c_n - \overline{c}_n \in C$ and

$$y_0 + h_n v + h_n^2(y_n - \overline{c}_n) \in F_+\left(x_0 + h_n u + h_n^2 x_n\right). \tag{3.14}$$

Since $\overline{c}_n = (\varepsilon/\alpha_n)c_n = \varepsilon b_n$, for all $n, \overline{c}_n \to \varepsilon b \neq 0_Y$. Thus, $y_n - \overline{c}_n \to y - \varepsilon b$. It follows from (3.14) that

$$y - \varepsilon b \in D^{(2)}F_+(x_0, y_0, u, v)(x), \tag{3.15}$$

which contradicts (3.12), since $\varepsilon b \in C$. Thus, $\alpha_n \to 0$ and $y_n - c_n \to y$. Then, it follows from (3.13) that $y \in \mathrm{D}^{(2)}F(x_0, y_0, u, v)(x)$. So,

$$\mathrm{Min}_C D^{(2)}F_+(x_0, y_0, u, v)(x) \subseteq D^{(2)}F(x_0, y_0, u, v)(x), \tag{3.16}$$

and the proof of the proposition is complete. □

Note that the inclusion of

$$W\mathrm{Min}_C D^{(2)}F_+(x_0, y_0, u, v)(x) \subseteq D^{(2)}F(x_0, y_0, u, v)(x), \tag{3.17}$$

may not hold under the assumptions of Proposition 3.6. The following example explains the case.

Example 3.7. Let $X = R$, $Y = R^2$, and $C = R_+^2$. Obviously, C has a compact base. Consider a set-valued map $F : X \to 2^Y$ defined by

$$F(x) = \left\{(y_1, y_2) \mid y_1 \geq x,\ y_2 = x^2\right\}. \tag{3.18}$$

Let $(x_0, y_0) = (0, (0, 0)) \in \mathrm{gph}(F)$ and $(u, v) = (1, (1, 0))$. For any $x \in X$,

$$\begin{gathered} D^{(2)}F_+(x_0, y_0, u, v)(x) = \{(y_1, y_2) \mid y_1 \geq x,\ y_2 \geq 1\}, \\ D^{(2)}F(x_0, y_0, u, v)(x) = \{(y_1, 1) \mid y_1 \geq x\}. \end{gathered} \tag{3.19}$$

Then, for any $x \in X$, $W\mathrm{Min}_C D^{(2)}F_+(x_0, y_0, u, v)(x) = \{(y_1, 1) \mid y_1 \geq x\} \cup \{(x, y_2) \mid y_2 \geq 1\}$. So, the inclusion of (3.17) does not hold here.

Proposition 3.8. *Let $(x_0, y_0) \in gph(F)$ and $(u, v) \in X \times Y$. Suppose that C has a compact base Q and $P(x) := D^{(2)}F_+(x_0, y_0, u, v)(x)$ satisfies the C-domination property for all $x \in K :=$ $\mathrm{dom}[D^{(2)}F(x_0, y_0, u, v)]$, then for any $x \in K$,*

$$\mathrm{Min}_C D^{(2)}F_+(x_0, y_0, u, v)(x) = \mathrm{Min}_C D^{(2)}F(x_0, y_0, u, v)(x). \tag{3.20}$$

Proof. From Proposition 3.4, one has

$$D^{(2)}F(x_0, y_0, u, v)(x) + C \subseteq D^{(2)}F_+(x_0, y_0, u, v)(x), \quad \text{for any } x \in K. \tag{3.21}$$

It follows from the C-domination property of $D^{(2)}F_+(x_0, y_0, u, v)(x)$ and Proposition 3.6 that

$$\begin{aligned} \mathrm{D}^{(2)}F_+(x_0, y_0, u, v)(x) &\subseteq \mathrm{Min}_C D^{(2)}F_+(x_0, y_0, u, v)(x) + C \\ &\subseteq D^{(2)}F(x_0, y_0, u, v)(x) + C, \quad \text{for any } x \in K, \end{aligned} \tag{3.22}$$

and then

$$D^{(2)}F(x_0, y_0, u, v)(x) + C = D^{(2)}F_+(x_0, y_0, u, v)(x), \quad \text{for any } x \in K. \tag{3.23}$$

Thus, for any $x \in K$,

$$\mathrm{Min}_C D^{(2)}F_+(x_0, y_0, u, v)(x) = \mathrm{Min}_C D^{(2)}F(x_0, y_0, u, v)(x), \tag{3.24}$$

and the proof of the proposition is complete. □

The following example shows that the C-domination property of $P(x)$ in Proposition 3.8 is essential.

Example 3.9 ($P(x)$ does not satisfy the C-domination property). Let $X = R$, $Y = R^2$, and $C = R^2_+$, and let $F : X \to 2^Y$ be defined by

$$F(x) = \begin{cases} \{(0,0)\} & \text{if } x \le 0, \\ \{(0,0), (-x, -\sqrt{x})\} & \text{if } x > 0, \end{cases} \tag{3.25}$$

then

$$F_+(x) = \begin{cases} R^2_+ & \text{if } x \le 0, \\ \{(y_1, y_2) \mid y_1 \ge -x, y_2 \ge -\sqrt{x}\} & \text{if } x > 0. \end{cases} \tag{3.26}$$

Let $(x_0, y_0) = (0, (0,0)) \in \mathrm{gph}(F)$, $(u, v) = (1, (0,0))$, then, for any $x \in X$,

$$D^{(2)}F(x_0, y_0, u, v)(x) = \{(0,0)\}, \qquad P(x) = D^{(2)}F_+(x_0, y_0, u, v)(x) = R^2. \tag{3.27}$$

Obviously, $P(x)$ does not satisfy the C-domination property and

$$\mathrm{Min}_C D^{(2)}F_+(x_0, y_0, u, v)(x) \ne \mathrm{Min}_C D^{\flat(2)}F(x_0, y_0, u, v)(x). \tag{3.28}$$

4. Second-Order Contingent Derivative of the Perturbation Maps

The purpose of this section is to investigate the quantitative information on the behavior of the perturbation map for (PVOP) by using second-order contingent derivative. Hereafter in this paper, let $x_0 \in E$, $y_0 \in W(x_0)$, and $(u, v) \in R^n \times R^m$, and let C be the order cone of R^m.

Definition 4.1. We say that G is C-minicomplete by W near x_0 if

$$G(x) \subseteq W(x) + C, \quad \forall x \in V(x_0), \tag{4.1}$$

where $V(x_0)$ is some neighborhood of x_0.

Remark 4.2. Let C be a convex cone. Since $W(x) \subseteq G(x)$, the C-minicompleteness of G by W near x_0 implies that

$$W(x) + C = G(x) + C, \quad \forall x \in V(x_0). \tag{4.2}$$

Hence, if G is C-minicomplete by W near x_0, then

$$D^{(2)}(W + C)(x_0, y, u, v) = D^{(2)}(G + C)(x_0, y, u, v), \quad \forall y \in W(x_0). \tag{4.3}$$

Theorem 4.3. *Suppose that the following conditions are satisfied:*

(i) *G is locally Lipschitz at x_0;*

(ii) *$D^{(2)}G(x_0, y_0, u, v) = D^{\flat(2)}G(x_0, y_0, u, v)$;*

(iii) *G is C-minicomplete by W near x_0;*

(iv) *there exists a neighborhood $U(x_0)$ of x_0, such that for any $x \in U(x_0)$, $W(x)$ is a single point set,*

then, for all $x \in R^n$,

$$D^{(2)}W(x_0, y_0, u, v)(x) \subseteq \mathrm{Min}_C D^{(2)}G(x_0, y_0, u, v)(x). \tag{4.4}$$

Proof. Let $x \in R^n$. If $D^{(2)}W(x_0, y_0, u, v)(x) = \emptyset$, then (4.4) holds trivially. Thus, we assume that $D^{(2)}W(x_0, y_0, u, v)(x) \neq \emptyset$. Let $y \in D^{(2)}W(x_0, y_0, u, v)(x)$, then there exist sequences $\{h_n\}$ with $h_n \to 0^+$ and $\{(x_n, y_n)\}$ with $(x_n, y_n) \to (x, y)$, such that

$$\begin{aligned} y_0 + h_n v + h_n^2 y_n &\in W\left(x_0 + h_n u + h_n^2 x_n\right) \\ &\subseteq G\left(x_0 + h_n u + h_n^2 x_n\right), \quad \forall n. \end{aligned} \tag{4.5}$$

So, $y \in D^{(2)}G(x_0, y_0, u, v)(x)$.

Suppose that $y \notin \mathrm{Min}_C D^{(2)}G(x_0, y_0, u, v)(x)$, then there exists $\overline{y} \in D^{(2)}G(x_0, y_0, u, v)(x)$, such that

$$y - \overline{y} \in C \setminus \{0_Y\}. \tag{4.6}$$

Since $D^{(2)}G(x_0, y_0, u, v) = D^{\flat(2)}G(x_0, y_0, u, v)$, for the preceding sequence $\{h_n\}$, there exists a sequence $\{(\overline{x}_n, \overline{y}_n)\}$ with $(\overline{x}_n, \overline{y}_n) \to (x, \overline{y})$, such that

$$y_0 + h_n v + h_n^2 \overline{y}_n \in G\left(x_0 + h_n u + h_n^2 \overline{x}_n\right), \quad \forall n. \tag{4.7}$$

It follows from the locally Lipschitz continuity of G that there exist $\gamma > 0$ and a neighborhood $V(x_0)$ of x_0, such that

$$G(x_1) \subseteq G(x_2) + \gamma \|x_1 - x_2\| B_{R^m}, \quad \forall x_1,\ x_2 \in V(x_0), \tag{4.8}$$

where B_{R^m} is the closed ball of R^m.

From assumption (iii), there exists a neighborhood $V_1(x_0)$ of x_0, such that

$$G(x) \subseteq W(x) + C, \quad \forall x \in V_1(x_0). \tag{4.9}$$

Naturally, there exists $N > 0$, such that

$$x_0 + h_n u + h_n^2 x_n, x_0 + h_n u + h_n^2 \overline{x}_n \in U(x_0) \cap V(x_0) \cap V_1(x_0), \quad \forall n > N. \tag{4.10}$$

Therefore, it follows from (4.7) and (4.8) that for any $n > N$, there exists $b_n \in B_{R^m}$, such that

$$y_0 + h_n v + h_n^2 (\overline{y}_n - \gamma \|\overline{x}_n - x_n\| b_n) \in G\left(x_0 + h_n u + h_n^2 x_n\right). \tag{4.11}$$

Thus, from (4.5), (4.9), and assumption (iv), one has

$$\begin{aligned} &y_0 + h_n v + h_n^2 (\overline{y}_n - \gamma \|\overline{x}_n - x_n\| b_n) - \left(y_0 + h_n v + h_n^2 y_n\right) \\ &\quad = h_n^2 (\overline{y}_n - \gamma \|\overline{x}_n - x_n\| b_n - y_n) \in C, \quad \forall n > N, \end{aligned} \tag{4.12}$$

and then it follows from $\overline{y}_n - \gamma \|\overline{x}_n - x_n\| b_n - y_n \to \overline{y} - y$ and C is a closed convex cone that

$$\overline{y} - y \in C, \tag{4.13}$$

which contradicts (4.6). Thus, $y \in \mathrm{Min}_C D^{(2)}G(x_0, y_0, u, v)(x)$ and the proof of the theorem is complete. □

The following two examples show that the assumption (iv) in Theorem 4.3 is essential.

Example 4.4 ($W(x)$ is not a single-point set near x_0). Let $C = \{(y_1, y_2) \in R_+^2 \mid y_1 \geq y_2\}$ and $G : R_+ \to 2^{R^2}$ be defined by

$$G(x) = C \cup \left\{(y_1, y_2) \mid y_1 \geq x^2 + x,\ y_2 \geq x^2\right\}, \tag{4.14}$$

then

$$W(x) = \{(0,0)\} \cup \left\{(y_1, y_2) \mid y_1 = x^2 + x,\ y_2 > x^2 + x\right\}. \tag{4.15}$$

Let $x_0 = 0$, $y_0 = (0,0)$, and $(u,v) = (1,(1,1))$, then $W(x)$ is not a single-point set near x_0, and it is easy to check that other assumptions of Theorem 4.3 are satisfied.

For any $x \in R$, one has

$$\begin{gathered} D^{(2)}G(x_0, y_0, u, v)(x) = \{(y_1, y_2) \mid y_1 \in R,\ y_1 \geq y_2\} \cup \{(y_1, y_2) \mid y_1 \geq 1 + x,\ y_2 \in R\}, \\ D^{(2)}W(x_0, y_0, u, v)(x) = \{(1 + x, y_2) \mid y_2 \geq 1 + x\}, \end{gathered} \tag{4.16}$$

and then

$$\mathrm{Min}_C D^{(2)}G(x_0, y_0, u, v)(x) = \{(1 + x, y_2) \mid y_2 > 1 + x\}. \tag{4.17}$$

Thus, for any $x \in R$, the inclusion of (4.4) does not hold here.

Example 4.5 ($W(x)$ is not a single-point set near x_0). Let $C = \{(y_1, y_2) \in R^2_+ \mid y_1 = 0\}$ and $G : R \to 2^{R^2}$ be defined by

$$G(x) = \begin{cases} C & \text{if } x = 0, \\ C \cup \left\{(y_1, y_2) \mid y_1 = x,\ y_2 \geq -\sqrt{1 + |x|}\right\} & \text{if } x \neq 0, \end{cases} \tag{4.18}$$

then

$$W(x) = \begin{cases} \{(0,0)\} & \text{if } x = 0, \\ \left\{(0,0), \left(x, -\sqrt{1 + |x|}\right)\right\} & \text{if } x \neq 0. \end{cases} \tag{4.19}$$

Let $x_0 = 0$, $y_0 = (0,0)$, and $(u,v) = (0,(0,0))$, then $W(x)$ is not a single-point set near x_0, and it is easy to check that other assumptions of Theorem 4.3 are satisfied.

For any $x \in R$, one has

$$\begin{gathered} D^{(2)}G(x_0, y_0, u, v)(x) = D^{\flat(2)}G(x_0, y_0, u, v)(x) = C \cup \{(y_1, y_2) \mid y_1 = x,\ y_2 \in R\}, \\ D^{(2)}W(x_0, y_0, u, v)(x) = \{(0,0)\}, \end{gathered} \tag{4.20}$$

and then

$$\mathrm{Min}_C D^{(2)}G(x_0, y_0, u, v)(0) = \emptyset. \tag{4.21}$$

Thus, for $x = 0$, the inclusion of (4.4) does not hold here.

Now, we give an example to illustrate Theorem 4.3.

Example 4.6. Let $C = R^2_+$ and $G : R \to 2^{R^2}$ be defined by

$$G(x) = \left\{(y_1, y_2) \in R^2 \mid x \le y_1 \le x + x^2, x - x^2 \le y_2 \le x\right\}, \quad \forall x \in R, \tag{4.22}$$

then

$$W(x) = \left\{\left(x, x - x^2\right)\right\}, \quad \forall x \in R. \tag{4.23}$$

Let $(x_0, y_0) = (0, (0,0)) \in \text{gph}(G)$, $(u, v) = (1, (1,1))$. By directly calculating, for all $x \in R$, one has

$$\begin{aligned} D^{(2)}G(x_0, y_0, u, v)(x) &= D^{\flat(2)}G(x_0, y_0, u, v)(x) \\ &= \{(y_1, y_2) \mid x \le y_1 \le x + 1, x - 1 \le y_1 \le x\}, \\ D^{(2)}W(x_0, y_0, u, v)(x) &= \{(x, x-1)\}. \end{aligned} \tag{4.24}$$

Then, it is easy to check that assumptions of Theorem 4.3 are satisfied, and the inclusion of (4.4) holds.

Theorem 4.7. *If $P(x) := D^{(2)}G_+(x_0, y_0, u, v)(x)$ fulfills the C-domination property for all $x \in \Omega :=$* $\text{dom}[D^{(2)}G(x_0, y_0, u, v)]$ *and G is C-minicomplete by W near x_0, then*

$$\text{Min}_C D^{(2)}G(x_0, y_0, u, v)(x) \subseteq D^{(2)}W(x_0, y_0, u, v)(x), \quad \textit{for any } x \in \Omega. \tag{4.25}$$

Proof. Since $C \subset R^n$, C has a compact base. Then, it follows from Propositions 3.6 and 3.8 and Remark 4.2 that for any $x \in \Omega$, one has

$$\begin{aligned} \text{Min}_C D^{(2)}G(x_0, y_0, u, v)(x) &= \text{Min}_C D^{(2)}G_+(x_0, y_0, u, v)(x) \\ &= \text{Min}_C D^{(2)}W_+(x_0, y_0, u, v)(x) \\ &\subseteq D^{(2)}W(x_0, y_0, u, v)(x). \end{aligned} \tag{4.26}$$

Then, the conclusion is obtained and the proof is complete. □

Remark 4.8. If the C-domination property of $P(x)$ is not satisfied in Theorem 4.7, then Theorem 4.7 may not hold. The following example explains the case.

Example 4.9 ($P(x)$ does not satisfy the C-domination property for $x \in \Omega$). Let $C = R^2_+$ and $G : R \to R^2$ be defined by

$$G(x) = \begin{cases} \{(0,0)\} & \text{if } x \le 0, \\ \{(0,0), (-x, -\sqrt{x})\} & \text{if } x > 0, \end{cases} \tag{4.27}$$

then,

$$G_+(x) = \begin{cases} R_+^2 & \text{if } x \leq 0, \\ \{(y_1, y_2) \mid y_1 \geq -x,\ y_2 \geq -\sqrt{x}\} & \text{if } x > 0. \end{cases} \tag{4.28}$$

Let $(x_0, y_0) = (0, (0, 0)) \in \operatorname{gph}(F)$, $(u, v) = (1, (0, 0))$, then, for any $x \in \Omega = R$,

$$W(x) = \begin{cases} \{(0, 0)\} & \text{if } x \leq 0, \\ \{(y_1, y_2) \mid y_1 = -x,\ y_2 = -\sqrt{x}\} & \text{if } x > 0, \end{cases} \tag{4.29}$$

for any $x \in \Omega$,

$$\begin{gathered} D^{(2)}G(x_0, y_0, u, v)(x) = \{(0, 0)\}, \qquad P(x) = D^{(2)}G_+(x_0, y_0, u, v)(x) = R^2, \\ D^{(2)}W(x_0, y_0, u, v)(x) = \emptyset. \end{gathered} \tag{4.30}$$

Hence, $P(x)$ does not satisfy the C-domination property, and $\operatorname{Min}_C D^{(2)}G(x_0, y_0, u, v)(x) = \{(0, 0)\}$. Then,

$$\operatorname{Min}_C D^{(2)}G(x_0, y_0, u, v)(x) \not\subseteq D^{\flat(2)}W(x_0, y_0, u, v)(x). \tag{4.31}$$

Theorem 4.10. *Suppose that the following conditions are satisfied:*

(i) *G is locally Lipschitz at x_0;*

(ii) *$D^{(2)}G(x_0, y_0, u, v) = D^{\flat(2)}G(x_0, y_0, u, v)$;*

(iii) *G is C-minicomplete by W near x_0;*

(iv) *there exists a neighborhood $U(x_0)$ of x_0, such that for any $x \in U(x_0)$, $W(x)$ is a single-point set;*

(v) *for any $x \in \Omega := \operatorname{dom}[D^{(2)}G(x_0, y_0, u, v)]$, $D^{(2)}G_+(x_0, y_0, u, v)(x)$ fulfills the C-domination property;*

then

$$D^{(2)}W(x_0, y_0, u, v)(x) = \operatorname{Min}_C D^{(2)}G(x_0, y_0, u, v)(x), \quad \forall x \in \Omega. \tag{4.32}$$

Proof. It follows from Theorems 4.3 and 4.7 that (4.32) holds. The proof of the theorem is complete. □

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References

[1] A. V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, vol. 165 of *Mathematics in Science and Engineering*, Academic Press, Orlando, Fla, USA, 1983.

[2] W. Alt, "Local stability of solutions to differentiable optimization problems in Banach spaces," *Journal of Optimization Theory and Applications*, vol. 70, no. 3, pp. 443–466, 1991.

[3] S. W. Xiang and W. S. Yin, "Stability results for efficient solutions of vector optimization problems," *Journal of Optimization Theory and Applications*, vol. 134, no. 3, pp. 385–398, 2007.

[4] J. Zhao, "The lower semicontinuity of optimal solution sets," *Journal of Mathematical Analysis and Applications*, vol. 207, no. 1, pp. 240–254, 1997.

[5] T. Tanino, "Sensitivity analysis in multiobjective optimization," *Journal of Optimization Theory and Applications*, vol. 56, no. 3, pp. 479–499, 1988.

[6] T. Tanino, "Stability and sensitivity analysis in convex vector optimization," *SIAM Journal on Control and Optimization*, vol. 26, no. 3, pp. 521–536, 1988.

[7] H. Kuk, T. Tanino, and M. Tanaka, "Sensitivity analysis in vector optimization," *Journal of Optimization Theory and Applications*, vol. 89, no. 3, pp. 713–730, 1996.

[8] D. S. Shi, "Contingent derivative of the perturbation map in multiobjective optimization," *Journal of Optimization Theory and Applications*, vol. 70, no. 2, pp. 385–396, 1991.

[9] D. S. Shi, "Sensitivity analysis in convex vector optimization," *Journal of Optimization Theory and Applications*, vol. 77, no. 1, pp. 145–159, 1993.

[10] S. J. Li, "Sensitivity and stability for contingent derivative in multiobjective optimization," *Mathematica Applicata*, vol. 11, no. 2, pp. 49–53, 1998.

[11] H. Kuk, T. Tanino, and M. Tanaka, "Sensitivity analysis in parametrized convex vector optimization," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 2, pp. 511–522, 1996.

[12] F. Ferro, "An optimization result for set-valued mappings and a stability property in vector problems with constraints," *Journal of Optimization Theory and Applications*, vol. 90, no. 1, pp. 63–77, 1996.

[13] M. A. Goberna, M. A. Lópes, and M. I. Todorov, "The stability of closed-convex-valued mappings and the associated boundaries," *Journal of Mathematical Analysis and Applications*, vol. 306, no. 2, pp. 502–515, 2005.

[14] X. K. Sun and S. J. Li, "Lower Studniarski derivative of the perturbation map in parametrized vector optimization," *Optimization Letters*. In press.

[15] T. D. Chuong and J.-C. Yao, "Coderivatives of efficient point multifunctions in parametric vector optimization," *Taiwanese Journal of Mathematics*, vol. 13, no. 6A, pp. 1671–1693, 2009.

[16] K. W. Meng and S. J. Li, "Differential and sensitivity properties of gap functions for Minty vector variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 386–398, 2008.

[17] J.-P. Aubin, "Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions," in *Mathematical Analysis and Applications, Part A*, L. Nachbin, Ed., vol. 7 of *Adv. in Math. Suppl. Stud.*, pp. 159–229, Academic Press, New York, NY, USA, 1981.

[18] R. B. Holmes, *Geometric Functional Analysis and Its Applications*, Graduate Texts in Mathematics, no. 2, Springer, New York, NY, USA, 1975.

[19] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Pure and Applied Mathematics (New York), John Wiley & Sons, New York, NY, USA, 1984.

[20] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, vol. 2 of *Systems & Control: Foundations & Applications*, Biekhäuser, Boston, Mass, USA, 1990.

[21] D. T. Luc, *Theory of Vector Optimization*, vol. 319 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 1989.