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Research Article

Solving the Set Equilibrium Problems

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We study the weak solutions and strong solutions of set equilibrium problems in real Hausdorff topological vector space settings. Several new results of existence for the weak solutions and strong solutions of set equilibrium problems are derived. The new results extend and modify various existence theorems for similar problems.

1. Introduction and Preliminaries

Let X, Y, Z be arbitrary real Hausdorff topological vector spaces, let K be a nonempty closed convex set of X, and let $C \subset Y$ be a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} C \neq \emptyset$, that is, C is proper closed with $\operatorname{int} C \neq \emptyset$ and satisfies the following conditions:

(1) $\lambda C \subseteq C$, for all $\lambda > 0$;

(2) $C + C \subseteq C$;

(3) $C \cap (-C) = \{0\}$.

Letting A, B be two sets of Y, we can define relations "$\leq_C$" and "$\not\leq_C$" as follows:

(1) $A \leq_C B \Leftrightarrow B - A \subset C$;

(2) $A \not\leq_C B \Leftrightarrow B - A \not\subset C$.

Similarly, we can define the relations "$\leq_{\operatorname{int} C}$" and "$\not\leq_{\operatorname{int} C}$" if we replace the set C by $\operatorname{int} C$.

The trimapping $f : Z \times K \times K \to 2^Y$ and mapping $T : K \to 2^Z$ are given. The set equilibrium problem $(\mathrm{SEP})_I$ is to find an $\overline{x} \in K$ such that

$$f(\overline{s}, \overline{x}, y) \not\leq_{\operatorname{int} C} \{0\} \tag{1.1}$$

for all $y \in K$ and for some $\overline{s} \in T(\overline{x})$. Such solution is called a weak solution for $(\mathrm{SEP})_I$. We note that (1.1) is equivalent to the following one:

$$f(\overline{s}, \overline{x}, y) \not\subset -\operatorname{int} C \tag{1.2}$$

for all $y \in K$ and for some $\overline{s} \in T(\overline{x})$.

For the case when $\overline{s}$ does not depend on y, that is, to find an $\overline{x} \in K$ with some $\overline{s} \in T(\overline{x})$ such that

$$f(\overline{s}, \overline{x}, y) \not\leq_{\operatorname{int} C} \{0\} \tag{1.3}$$

for all $y \in K$, we will call this solution a strong solution of $(\mathrm{SEP})_I$. We also note that (1.3) is equivalent to the following one:

$$f(\overline{s}, \overline{x}, y) \not\subset -\operatorname{int} C \tag{1.2$'$}$$

for all $y \in K$.

We note that if f is a vector-valued function and the mapping $s \to f(s, x, y)$ is constant for each $x, y \in K$, then $(\mathrm{SEP})_I$ reduces to the vector equilibrium problem (VEP), which is to find $\overline{x} \in K$ such that

$$f(\overline{x}, y) \not\leq_{\operatorname{int} C} 0 \tag{1.4}$$

for all $y \in K$. Existence of a solution of this problem is investigated by Ansari et al. [1, 2].

If f is a vector-valued function and $Z = L(X, Y)$ which is denoted the space of all continuous linear mappings from X to Y and $f(s, x, y) = (s, y - x)$, where (s, y) denotes the evaluation of the linear mapping s at y, then $(\mathrm{SEP})_I$ reduces to (GVVIP): to find $\overline{x} \in K$ and $\overline{s} \in T(\overline{x})$ such that

$$(\overline{s}, y - \overline{x}) \not\leq_{\operatorname{int} C} 0 \tag{1.5}$$

for all $y \in K$. It has been studied by Chen and Craven [3].

If we consider $F : K \to K$, $Z = L(X, Y)$, $A : L(X, Y) \to L(X, Y)$, and $f(s, x, y) = (As, y - x) + F(y) - F(x)$, where (s, y) denotes the evaluation of the linear mapping s at y, then $(\mathrm{SEP})_I$ reduces to the (GVVIP) which is discussed by Huang and Fang [4] and Zeng and Yao [5]: to find a vector $\overline{x} \in K$ and $\overline{s} \in T(\overline{x})$ such that

$$(A\overline{s}, y - \overline{x}) + F(y) - F(\overline{x}) \not\leq_{\operatorname{int} C} 0, \quad \forall y \in K. \tag{1.6}$$

If $Z = L(X, Y)$, $T : K \to L(X, Y)$ is a single-valued mapping, $f(s, x, y) = (T(x), y - x)$, then $(\mathrm{SEP})_I$ reduces to the (weak) vector variational inequalities problem which is considered by Fang and Huang [6], Chiang and Yao [7], and Chiang [8] as follows: to find a vector $\overline{x} \in K$ such that

$$(T(\overline{x}), y - \overline{x}) \not\leq_{\operatorname{int} C} 0 \tag{1.7}$$

for all $y \in K$. The vector variational inequalities problem was first introduced by Giannessi [9] in finite-dimensional Euclidean space.

Summing up the above arguments, they show that for a suitable choice of the mapping T and the spaces X, Y, and Z, we can obtain a number of known classes of vector equilibrium problems, vector variational inequalities, and implicit generalized variational inequalities. It is also well known that variational inequality and its variants enable us to study many important problems arising in mathematical, mechanics, operations research, engineering sciences, and so forth.

In this paper we aim to derive some solvabilities for the set equilibrium problems. We also study some results of existence for the weak solutions and strong solutions of set equilibrium problems. Let K be a nonempty subset of a topological vector space X. A set-valued function Φ from K into the family of subsets of X is a KKM mapping if for any nonempty finite set $A \subset K$, the convex hull of A is contained in $\bigcup_{x \in A} \Phi(x)$. Let us first recall the following results.

Fan's Lemma (see [10]). *Let K be a nonempty subset of Hausdorff topological vector space X. Let $G : K \to 2^X$ be a KKM mapping such that for any $y \in K$, $G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$.*

Definition 1.1 (see [11]). Let Ω be a vector space, let Σ be a topological vector space, let K be a nonempty convex subset of Ω, and let $C \subset \Sigma$ be a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} C \neq \emptyset$, and $\varphi : K \to 2^{\Sigma}$ is said to be

(1) *C-convex* if $t\varphi(x_1) + (1-t)\varphi(x_2) \subset \varphi(tx_1 + (1-t)x_2) + C$ for every $x_1, x_2 \in K$ and $t \in [0,1]$;

(2) *naturally quasi -C-convex* if $\varphi(tx_1 + (1-t)x_2) \subset \operatorname{co}\{\varphi(x_1) \cup \varphi(x_2)\} - C$ for every $x_1, x_2 \in K$ and $t \in [0,1]$.

The following definition can also be found in [11].

Definition 1.2. Let Y be a Hausdorff topological vector space, let $C \subset Y$ be a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} C \neq \emptyset$, and let A be a nonempty subset of Y. Then

(1) a point $z \in A$ is called a *minimal point* of A if $A \cap (z - C) = \{z\}$; $\operatorname{Min} A$ is the set of all minimal points of A;

(2) a point $z \in A$ is called a *maximal point* of A if $A \cap (z + C) = \{z\}$; $\operatorname{Max} A$ is the set of all maximal points of A;

(3) a point $z \in A$ is called a *weakly minimal point* of A if $A \cap (z - \operatorname{int} C) = \emptyset$; $\operatorname{Min}_w A$ is the set of all weakly minimal points of A;

(4) a point $z \in A$ is called a *weakly maximal point* of A if $A \cap (z + \operatorname{int} C) = \emptyset$; $\operatorname{Max}_w A$ is the set of all weakly maximal points of A.

Definition 1.3. Let X, Y be two topological spaces. A mapping $T : X \to 2^Y$ is said to be

(1) upper semicontinuous if for every $x \in X$ and every open set V in Y with $T(x) \subset V$, there exists a neighborhood $W(x)$ of x such that $T(W(x)) \subset V$;

(2) lower semicontinuous if for every $x \in X$ and every open neighborhood $V(y)$ of every $y \in T(x)$, there exists a neighborhood $W(x)$ of x such that $T(u) \cap V(y) \neq \emptyset$ for all $u \in W(x)$;

(3) continuous if it is both upper semicontinuous and lower semicontinuous.

We note that T is lower semicontinuous at x_0 if for any net $\{x_\nu\} \subset X$, $x_\nu \to x_0$, $y_0 \in T(x_0)$ implies that there exists net $y_\nu \in T(x_\nu)$ such that $y_\nu \to y_0$. For other net-terminology properties about these two mappings, one can refer to [12].

Lemma 1.4 (see [13]). *Let X, Y, and Z be real topological vector spaces, and let K and C be nonempty subsets of X and Y, respectively. Let $F : K \times C \to 2^Z$, $S : K \to 2^C$ be set-valued mappings. If both F and S are upper semicontinuous with nonempty compact values, then the set-valued mapping $G : K \to 2^Z$ defined by*

$$G(x) = \bigcup_{y \in S(x)} F(x, y) = F(x, S(x)), \quad \forall x \in K \tag{1.8}$$

is upper semicontinuous with nonempty compact values.

By using similar technique of [11, Proposition 2.1], we can deduce the following lemma that slight-generalized the original one.

Lemma 1.5. *Let $\overline{L}$, $\overline{K}$ be two Hausdorff topological vector spaces, and let L, K be nonempty compact convex subsets of $\overline{L}$ and $\overline{K}$, respectively. Let $G : L \times K \to 2^{\mathbb{R}}$ be continuous mapping with nonempty compact valued on $L \times K$; the mapping $s \to -G(s, x)$ is naturally quasi $\mathbb{R}_+$-convex on L for each $x \in K$, and the mapping $x \to G(s, x)$ is $\mathbb{R}_+$-convex on K for each $s \in L$. Assume that for each $x \in K$, there exists $s_x \in L$ such that*

$$\operatorname{Min} G(s_x, x) \geq_{\mathbb{R}_+} \operatorname{Min} \bigcup_{x \in K} \operatorname{Max}_w \bigcup_{s \in L} G(s, x). \tag{1.9}$$

Then, one has

$$\operatorname{Min} \bigcup_{x \in K} \operatorname{Max}_w \bigcup_{s \in L} G(s, x) = \operatorname{Max} \bigcup_{s \in L} \operatorname{Min}_w \bigcup_{x \in K} G(s, x). \tag{1.10}$$

2. Existence Theorems for Set Equilibrium Problems

Now, we state and show our main results of solvabilities for set equilibrium problems.

Theorem 2.1. *Let X, Y, Z be real Hausdorff topological vector spaces, let K be a nonempty closed convex subset of X, and let $C \subset Y$ be a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} C \neq \emptyset$. Given mappings $f : Z \times K \times K \to 2^Y$, $T : K \to 2^Z$, and $\nu : K \times K \to 2^Y$, suppose that*

(1) *$\{0\} \leq_C \nu(x, x)$ for all $x \in K$;*

(2) *for each $x \in K$, there is an $s \in T(x)$ such that for all $y \in K$,*

$$\nu(x, y) \leq_C f(s, x, y), \tag{2.1}$$

(3) *for each* $x \in K$*, the set* $\{y \in K : \{0\} \nleq_C \nu(x,y)\}$ *is convex;*

(4) *there is a nonempty compact convex subset* D *of* K*, such that for every* $x \in K \setminus D$*, there is a* $y \in D$ *such that for all* $s \in T(x)$*,*

$$f(s,x,y) \leq_{\operatorname{int} C} \{0\}, \tag{2.2}$$

(5) *for each* $y \in K$*, the set* $\{x \in K : f(s,x,y) \leq_{\operatorname{int} C} \{0\}$ *for all* $s \in T(x)\}$ *is open in* K*.*

Then there exists an $\overline{x} \in K$ *which is a weak solution of (SEP)*$_I$*. That is, there is an* $\overline{x} \in K$ *such that*

$$f(\overline{s},\overline{x},y) \nleq_{\operatorname{int} C} \{0\} \tag{2.3}$$

for all $y \in K$ *and for some* $\overline{s} \in T(\overline{x})$*.*

Proof. Define $\Omega : K \to 2^D$ by

$$\Omega(y) = \{x \in D : f(s,x,y) \nleq_{\operatorname{int} C} \{0\} \text{ for some } s \in T(x)\} \tag{2.4}$$

for all $y \in K$. From condition (5) we know that for each $y \in K$, the set $\Omega(y)$ is closed in K, and hence it is compact in D because of the compactness of D.

Next, we claim that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property, and then the whole intersection $\bigcap_{y \in K} \Omega(y)$ is nonempty and any element in the intersection $\bigcap_{y \in K} \Omega(y)$ is a solution of (SEP)$_I$, for any given nonempty finite subset N of K. Let $D_N = \operatorname{co}\{D \cup N\}$, the convex hull of $D \cup N$. Then D_N is a compact convex subset of K. Define the mappings $S, R : D_N \to 2^{D_N}$, respectively, by

$$\begin{gathered} S(y) = \{x \in D_N : f(s,x,y) \nleq_{\operatorname{int} C} \{0\} \text{ for some } s \in T(x)\}, \\ R(y) = \{x \in D_N : \{0\} \leq_C \nu(x,y)\}, \end{gathered} \tag{2.5}$$

for each $y \in D_N$. From conditions (1) and (2), we have

$$\{0\} \leq_C \nu(y,y) \quad \forall y \in D_N, \tag{2.6}$$

and for each $y \in K$, there is an $s \in T(y)$ such that

$$\nu(y,y) - f(s,y,y) \leq_C \{0\}. \tag{2.7}$$

Hence $\{0\} \leq_C f(s,y,y)$, and then $y \in S(y)$ for all $y \in D_N$.

We can easily see that S has closed values in D_N. Since, for each $y \in D_N$, $\Omega(y) = S(y) \cap D$, if we prove that the whole intersection of the family $\{S(y) : y \in D_N\}$ is nonempty, we can deduce that the family $\{\Omega(y) : y \in K\}$ has finite intersection property because $N \subset D_N$ and due to condition (4). In order to deduce the conclusion of our theorem, we can apply Fan's lemma if we claim that S is a KKM mapping. Indeed, if S is not a KKM mapping, neither is R since $R(y) \subset S(y)$ for each $y \in D_N$. Then there is a nonempty finite subset M of D_N such that

$$\operatorname{co} M \not\subset \bigcup_{u \in M} R(u). \tag{2.8}$$

Thus there is an element $\overline{u} \in \operatorname{co} M \subset D_N$ such that $\overline{u} \notin R(u)$ for all $u \in M$, that is, $\{0\} \not\leq_C v(\overline{u}, u)$ for all $u \in M$. By (3), we have

$$\overline{u} \in \operatorname{co} M \subset \{y \in K : \{0\} \not\leq_C v(\overline{u}, y)\}, \tag{2.9}$$

and hence $\{0\} \not\leq_C v(\overline{u}, \overline{u})$ which contradicts (2.6). Hence R is a KKM mapping, and so is S. Therefore, there exists an $\overline{x} \in K$ which is a solution of $(\mathrm{SEP})_I$. This completes the proof. □

Theorem 2.2. *Let X, Y, Z be real Hausdorff topological vector spaces, let K be a nonempty closed convex subset of X, and let $C \subset Y$ be a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} C \neq \emptyset$. Let the mapping $f : Z \times K \times K \to 2^Y$ be such that for each $y \in K$, the mappings $(s, x) \to f(s, x, y)$ and $T : K \to 2^Z$ are upper semicontinuous with nonempty compact values and $v : K \times K \to 2^Y$. Suppose that conditions (1)–(4) of Theorem 2.1 hold. Then there exists an $\overline{x} \in K$ which is a solution of $(\mathrm{SEP})_I$. That is, there is an $\overline{x} \in K$ such that*

$$f(\overline{s}, \overline{x}, y) \not\leq_{\operatorname{int} C} \{0\} \tag{2.10}$$

for all $y \in K$ and for some $\overline{s} \in T(\overline{x})$.

Proof. For any fixed $y \in K$, we define the mapping $G : K \to 2^Y$ by

$$G(x) = \bigcup_{s \in Tx} f(s, x, y) \tag{2.11}$$

for all $s \in Z$ and $x \in K$. Since the mappings $(s, x) \to f(s, x, y)$ and $T : K \to 2^Z$ are upper semicontinuous with nonempty compact values, by Lemma 1.4, we know that G is upper semicontinuous on K with nonempty compact values. Hence, for each $y \in K$, the set

$$\{x \in K : f(s, x, y) \leq_{\operatorname{int} C} \{0\}\ \forall s \in T(x)\} = \{x \in K : G(x) \subset (-\operatorname{int} C)\} \tag{2.12}$$

is open in K. Then all conditions of Theorem 2.1 hold. From Theorem 2.1, $(\mathrm{SEP})_I$ has a solution. □

In order to discuss the results of existence for the strong solution of $(\mathrm{SEP})_I$, we introduce the condition $(\mathfrak{Y})$. It is obviously fulfilled that if $Y = \mathbb{R}$, f is single-valued function.

Theorem 2.3. *Under the framework of Theorem 2.2, one has a weak solution $\overline{x}$ of $(\mathrm{SEP})_I$ with $\overline{s} \in T(\overline{x})$. In addition, if $Y = \mathbb{R}$, $C = \mathbb{R}_+$, and K is compact, $T(\overline{x})$ is convex, the mapping $(s, x) \to f(s, \overline{x}, x)$ is continuous with nonempty compact valued on $T(\overline{x}) \times K$, the mapping $s \to -f(s, \overline{x}, x)$*

is naturally quasi $\mathbb{R}_+$-convex on $T(\overline{x})$ for each $x \in K$, and the mapping $x \to f(s,\overline{x},x)$ is $\mathbb{R}_+$-convex on K for each $s \in T(\overline{x})$. Assuming that for each $x \in K$, there exists $t_x \in T(\overline{x})$ such that

$$\operatorname{Min} f(t_x,\overline{x},x) \geq_C \operatorname{Min} \bigcup_{x\in K} \operatorname{Max}_w \bigcup_{s\in T(\overline{x})} f(s,\overline{x},x), \tag{$\mathfrak{Y}$}$$

then $\overline{x}$ is a strong solution of (SEP)$_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that

$$f(\overline{s},\overline{x},x) \nleq_{\operatorname{int} C} \{0\} \tag{2.13}$$

for all $x \in K$. Furthermore, the set of all strong solutions of (SEP)$_I$ is compact.

Proof. From Theorem 2.2, we know that $\overline{x} \in K$ such that (1.1) holds for all $x \in K$ and for some $\overline{s} \in T(\overline{x})$. Then we have $\operatorname{Min} \bigcup_{x\in K} \operatorname{Max} \bigcup_{s\in T(\overline{x})} f(s,\overline{x},x) \geq_C 0$.

From condition ($\mathfrak{Y}$) and the convexity of $T(\overline{x})$, Lemma 1.5 tells us that $\operatorname{Max} \bigcup_{s\in T(\overline{x})} \operatorname{Min}_w \bigcup_{x\in K} f(s,\overline{x},x) \geq_C 0$. Then there is an $\overline{s} \in T(\overline{x})$ such that $\operatorname{Min}_w \bigcup_{x\in K} f(\overline{s},\overline{x},x) \geq_C 0$. Thus for all $\rho \in \bigcup_{x\in K} f(\overline{s},\overline{x},x)$, we have $\rho \geq_C 0$. Hence there exists $\overline{s} \in T(\overline{x})$ such that

$$f(\overline{s},\overline{x},x) \nleq_{\operatorname{int} C} \{0\} \tag{2.14}$$

for all $x \in K$. Such an $\overline{x}$ is a strong solution of (SEP)$_I$.

Finally, to see that the solution set of (SEP)$_I$ is compact, it is sufficient to show that the solution set is closed due to the coercivity condition (4) of Theorem 2.2. To this end, let Γ denote the solution set of (SEP)$_I$. Suppose that net $\{x_\alpha\} \subset \Gamma$ which converges to some p. Fix any $y \in K$. For each α, there is an $s_\alpha \in T(x_\alpha)$ such that

$$f(s_\alpha,x_\alpha,y) \nleq_{\operatorname{int} C} \{0\}. \tag{2.15}$$

Since T is upper semicontinuous with compact values and the set $\{x_\alpha\} \cup \{p\}$ is compact, it follows that $T(\{x_\alpha\} \cup \{p\})$ is compact. Therefore without loss of generality, we may assume that the sequence $\{s_\alpha\}$ converges to some s. Then $s \in T(p)$ and $f(s_\alpha,x_\alpha,y) \not\subset -\operatorname{int} C$. Let $\Omega = \{(s,x) \in (\bigcup_{z\in K} T(z)) \times K : f(s,x,y) \subset -\operatorname{int} C\}$. Since the mapping $(s,x) \to f(s,x,y)$ is upper semicontinuous with nonempty compact values, the set Ω is open in $(\bigcup_{z\in K} T(z)) \times K$. Hence $(\bigcup_{z\in K} T(z)) \times K \setminus \Omega$ is closed in $(\bigcup_{z\in K} T(z)) \times K$. By the facts $(s_\alpha,x_\alpha) \in (\bigcup_{z\in K} T(z)) \times K \setminus \Omega$ and $(s_\alpha,x_\alpha) \to_\alpha (s,p)$, we have $(s,p) \in (\bigcup_{z\in K} T(z)) \times K \setminus \Omega$. This implies that $f(s,p,y) \not\subset -\operatorname{int} C$. We then obtain

$$f(s,p,y) \nleq_{\operatorname{int} C} \{0\}. \tag{2.16}$$

Hence $p \in \Gamma$ and Γ is closed. □

We would like to point out that condition ($\mathfrak{Y}$) is fulfilled if we take $Y = \mathbb{R}$ and f is a single-valued function. The following is a concrete example for both Theorems 2.1 and 2.3.

Example 2.4. Let $X = Y = \mathbb{R}$, $Z = L(X,Y)$, $K = [1,2]$, $C = \mathbb{R}_+$, and $D = [1,2]$. Choose $T : K \to 2^{L(X,Y)}$ to be defined by $T(x) = \{ax : a \in [1,2]\} \in 2^{L(X,Y)}$ for every $x \in K$ and f :

$T(K) \times K \times K \to 2^Y$ is defined by $f(s,x,y) = \{(a+\delta)x(y-x) : \delta \in [0,1]\}$, where $x \in K$, $s \in T(x)$ with $s = ax$, for some $a \in [1,2]$, $y \in K$, and $\nu : K \times K \to 2^Y$ is defined by

$$\nu(x,y) = \begin{cases} \{x(y-x)\}, & y \geq x, \\ \{3x(y-x)\}, & y \leq x. \end{cases} \tag{2.17}$$

Then all conditions of Theorems 2.1 and 2.3 are satisfied. By Theorems 2.1 and 2.3, respectively, the $(\text{SEP})_I$ not only has a weak solution, but also has a strong solution. A simple geometric discussion tells us that $\overline{x} = 1$ is a strong solution for $(\text{SEP})_I$.

Corollary 2.5. *Under the framework of Theorem 2.1, one has a weak solution $\overline{x}$ of $(SEP)_I$ with $\overline{s} \in T(\overline{x})$. In addition, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, K is compact, $T(\overline{x})$ is convex, $\mathbb{R}_+$-convex on $T(\overline{x})$ for each $x \in K$ and the mapping $x \to f(s,\overline{x},x)$ is $\mathbb{R}_+$-convex on K for each $s \in T(\overline{x})$, $f : Z \times K \times K \to 2^Y$ such that $(s,x) \to f(s,x,y)$ is continuous with nonempty compact values for each $y \in K$, and $T : K \to 2^Z$ is upper semicontinuous with nonempty compact values. Assume that condition $(\mathfrak{Y})$ holds, then $\overline{x}$ is a strong solution of $(SEP)_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that*

$$f(\overline{s},\overline{x},x) \nleq_{\operatorname{int} C} \{0\} \tag{2.18}$$

for all $x \in K$. Furthermore, the set of all strong solutions of $(SEP)_I$ is compact.

Theorem 2.6. *Let X, Y, Z, K, C, T, f be as in Theorem 2.1. Assume that the mapping $y \to f(s,x,y)$ is C-convex on K for each $x \in K$ and $s \in T(x)$ such that*

(1) *for each $x \in K$, there is an $s \in T(x)$ such that $f(s,x,x) \nleq_{\operatorname{int} C} \{0\}$;*

(2) *there is a nonempty compact convex subset D of K, such that for every $x \in K \setminus D$, there is a $y \in D$ such that for all $s \in T(x)$,*

$$f(s,x,y) \leq_{\operatorname{int} C} \{0\}, \tag{2.19}$$

(3) *for each $y \in K$, the set $\{x \in K : f(s,x,y) \leq_{\operatorname{int} C} \{0\}$ for all $s \in T(x)\}$ is open in K.*

Then there is an $\overline{x} \in K$ which is a weak solution of $(SEP)_I$.

Proof. For any given nonempty finite subset N of K. Letting $D_N = \operatorname{co}(D \cup N)$, then D_N is a nonempty compact convex subset of K. Define $S : D_N \to 2^{D_N}$ as in the proof of Theorem 2.1, and for each $y \in K$, let

$$\Omega(y) = \{x \in D : f(s,x,y) \nleq_{\operatorname{int} C} \{0\} \text{ for some } s \in T(x)\}. \tag{2.20}$$

We note that for each $x \in D_N$, $S(x)$ is nonempty and closed since $x \in S(x)$ by conditions (1) and (3). For each $y \in K$, $\Omega(y)$ is compact in D. Next, we claim that the mapping S is a KKM mapping. Indeed, if not, there is a nonempty finite subset M of D_N, such that $\operatorname{co} M \not\subset \bigcup_{x \in M} S(x)$. Then there is an $x^* \in \operatorname{co} M \subset D_N$ such that

$$f(s,x^*,x) \leq_{\operatorname{int} C} \{0\} \tag{2.21}$$

for all $x \in M$ and $s \in T(x^*)$. Since the mapping

$$x \longrightarrow f(s, x^*, x) \tag{2.22}$$

is C-convex on D_N, we can deduce that

$$f(s, x^*, x^*) \leq_{\operatorname{int} C} \{0\} \tag{2.23}$$

for all $s \in T(x^*)$. This contradicts condition (1). Therefore, S is a KKM mapping, and by Fan's lemma, we have $\bigcap_{x \in D_N} S(x) \neq \emptyset$. Note that for any $u \in \bigcap_{x \in D_N} S(x)$, we have $u \in D$ by condition (2). Hence, we have

$$\bigcap_{y \in N} \Omega(y) = \bigcap_{y \in N} S(y) \cap D \neq \emptyset, \tag{2.24}$$

for each nonempty finite subset N of K. Therefore, the whole intersection $\bigcap_{y \in K} \Omega(y)$ is nonempty. Let $\overline{x} \in \bigcap_{y \in K} \Omega(y)$. Then $\overline{x}$ is a solution of $(\mathrm{SEP})_I$. □

Corollary 2.7. *Let X, Y, Z, K, C, T, f be as in Theorem 2.1. Assume that the mapping $y \to f(s, x, y)$ is C-convex on K for each $x \in K$ and $s \in T(x)$, $f : Z \times K \times K \to 2^Y$ such that $(s, x) \to f(s, x, y)$ is continuous with nonempty compact values for each $y \in K$, and $T : K \to 2^Z$ is upper semicontinuous with nonempty compact values. Suppose that*

(1) *for each $x \in K$, there is an $s \in T(x)$ such that $f(s, x, x) \nleq_{\operatorname{int} C} \{0\}$;*

(2) *there is a nonempty compact convex subset D of K, such that for every $x \in K \setminus D$, there is a $y \in D$ such that for all $s \in T(x)$,*

$$f(s, x, y) \leq_{\operatorname{int} C} \{0\}. \tag{2.25}$$

Then there is an $\overline{x} \in K$ which is a weak solution of $(\mathrm{SEP})_I$.

Proof. Using the technique of the proof in Theorem 2.2 and applying Theorem 2.6, we have the conclusion. □

The following result is another existence theorem for the strong solutions of $(\mathrm{SEP})_I$. We need to combine Theorem 2.6 and use the technique of the proof in Theorem 2.3.

Theorem 2.8. *Under the framework of Theorem 2.6, on has a weak solution $\overline{x}$ of $(\mathrm{SEP})_I$ with $\overline{s} \in T\overline{x}$. In addition, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, K is compact, $T(\overline{x})$ is convex and the mapping $s \to -f(s, \overline{x}, x)$ is naturally quasi C-convex on $T(\overline{x})$ for each $x \in K$, $f : Z \times K \times K \to 2^Y$ such that $(s, x) \to f(s, x, y)$ is continuous with nonempty compact values for each $y \in K$, and $T : K \to 2^Z$ is upper semicontinuous with nonempty compact values. Assuming that condition $(\mathfrak{Y})$ holds, then $\overline{x}$ is a strong solution of $(\mathrm{SEP})_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that*

$$f(\overline{s}, \overline{x}, x) \nleq_{\operatorname{int} C} \{0\} \tag{2.26}$$

for all $x \in K$. Furthermore, the set of all strong solutions of (SEP)$_I$ is compact.

Using the technique of the proof in Theorem 2.3, we have the following result.

Corollary 2.9. *Under the framework of Corollary 2.7, one has a weak solution $\overline{x}$ of (SEP)$_I$ with $\overline{s} \in T\overline{x}$. In addition, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, K is compact, $T(\overline{x})$ is convex, and the mapping $s \to -f(s, \overline{x}, x)$ is naturally quasi C-convex on $T(\overline{x})$ for each $x \in K$. Assuming that condition ($\mathfrak{Y}$) holds, then $\overline{x}$ is a strong solution of (SEP)$_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that*

$$f(\overline{s}, \overline{x}, x) \nleq_{\operatorname{int} C} \{0\} \tag{2.27}$$

for all $x \in K$. Furthermore, the set of all strong solutions of (SEP)$_I$ is compact.

Next, we discuss the existence results of the strong solutions for (SEP)$_I$ with the set K without compactness setting from Theorems 2.10 to 2.14 below.

Theorem 2.10. *Letting X be a finite-dimensional real Banach space, under the framework of Theorem 2.1, one has a weak solution $\overline{x}$ of (SEP)$_I$ with $\overline{s} \in T(\overline{x})$. In addition, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, $T(\overline{x})$ is convex, $f(s, x, x) = \{0\}$ for all $s \in T(x)$ and for all $x \in K$, the mapping $y \to f(s, x, y)$ is C-convex on K for each $x \in K$ and $s \in T(x)$ and the mapping $s \to -f(s, \overline{x}, x)$ is naturally quasi C-convex on $T(\overline{x})$ for each $x \in K$, $f : Z \times K \times K \to 2^Y$ such that $(s, x) \to f(s, x, y)$ is continuous for each $y \in K$, and $T : K \to 2^Z$ is upper semicontinuous with nonempty compact values. Assume that for some $r > \|\overline{x}\|$, such that for each $x \in K_r$, there is a $t_x \in T(\overline{x})$ such that the condition*

$$\operatorname{Min} f(t_x, \overline{x}, x) \geq_C \operatorname{Min} \bigcup_{x \in K_r} \operatorname{Max}_w \bigcup_{s \in T(\overline{x})} f(s, \overline{x}, x) \tag{$\mathfrak{Y}'$}$$

is satisfied, where $K_r \doteq \overline{B}(0, r) \cap K$. Then $\overline{x}$ is a strong solution of (SEP)$_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that

$$f(s, \overline{x}, x) \nleq_{\operatorname{int} C} \{0\} \tag{2.28}$$

for all $x \in K$. Furthermore, the set of all strong solutions of (SEP)$_I$ is compact.

Proof. Let us choose $r > \|\overline{x}\|$ such that condition ($\mathfrak{Y}'$) holds. Letting $\overline{B}(0, r) = \{x \in X : \|x\| \leq r\}$, then the set K_r is nonempty and compact in X. We replace K by K_r in Theorem 2.3; all conditions of Theorem 2.3 hold. Hence by Theorem 2.3, we have $\overline{s} \in T(\overline{x})$ such that

$$f(\overline{s}, \overline{x}, z) \nleq_{\operatorname{int} C} \{0\} \tag{2.29}$$

for all $z \in K_r$. For any $x \in K$, choose $t \in (0, 1)$ small enough such that $(1 - t)\overline{x} + tx \in K_r$. Putting $z = (1 - t)\overline{x} + tx$ in (2.29), we have

$$f(\overline{s}, \overline{x}, (1 - t)\overline{x} + tx) \nleq_{\operatorname{int} C} \{0\}. \tag{2.30}$$

We note that

$$f(\overline{s},\overline{x},(1-t)\overline{x}+tx) \leq_C (1-t)f(\overline{s},\overline{x},\overline{x}) + tf(\overline{s},\overline{x},x) = tf(\overline{s},\overline{x},x) \tag{2.31}$$

which implies that

$$f(\overline{s},\overline{x},x) \not\leq_{\text{int}\, C} \{0\} \tag{2.32}$$

for all $x \in K$. This completely proves the theorem. □

Corollary 2.11. *Letting X be a finite-dimensional real Banach space, under the framework of Theorem 2.2, one has a weak solution $\overline{x}$ of (SEP)$_I$ with $\overline{s} \in T(\overline{x})$. In addition, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, $T(\overline{x})$ is convex, $f(s,x,x) = \{0\}$ for all $s \in T(x)$ and for all $x \in K$, the mapping $y \to f(s,x,y)$ is C-convex on K for each $x \in K$ and $s \in T(x)$, and the mapping $s \to -f(s,\overline{x},x)$ is naturally quasi C-convex on $T(\overline{x})$ for each $x \in K$. Assume that for some $r > \|\overline{x}\|$, condition $(\mathfrak{Y}')$ holds. Then $\overline{x}$ is a strong solution of (SEP)$_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that*

$$f(s,\overline{x},x) \not\leq_{\text{int}\, C} \{0\} \tag{2.33}$$

for all $x \in K$. Furthermore, the set of all strong solutions of (SEP)$_I$ is compact.

Using a similar argument to that of the proof in Theorem 2.10 and combining Theorem 2.6 and Corollary 2.7, respectively, we have the following two results of existence for the strong solution of (SEP)$_I$.

Theorem 2.12. *Let X be a finite-dimensional real Banach space, under the framework of Theorem 2.6, one has a weak solution $\overline{x}$ of (SEP)$_I$ with $\overline{s} \in T(\overline{x})$. In addition, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, $T(\overline{x})$ is convex, $f(s,x,x) = \{0\}$ for all $s \in T(x)$ and for all $x \in K$, the mapping $s \to -f(s,\overline{x},x)$ is naturally quasi C-convex on $T(\overline{x})$ for each $x \in K$, $f : Z \times K \times K \to 2^Y$ such that $(s,x) \to f(s,x,y)$ is continuous for each $y \in K$, and $T : K \to 2^Z$ is upper semicontinuous with nonempty compact values. Assume that for some $r > \|\overline{x}\|$, condition $(\mathfrak{Y}')$ holds. Then $\overline{x}$ is a strong solution of (SEP)$_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that*

$$f(\overline{s},\overline{x},x) \not\leq_{\text{int}\, C} \{0\} \tag{2.34}$$

for all $x \in K$. Furthermore, the set of all strong solutions of (SEP)$_I$ is compact.

In order to illustrate Theorems 2.10 and 2.12 more precisely, we provide the following concrete example.

Example 2.13. Let $X = Y = \mathbb{R}$, $Z = L(X,Y)$, $K = [1,2]$, $C = \mathbb{R}_+$, and $D = [1,2]$. Choose $T : K \to 2^{L(X,Y)}$ to be defined by $T(x) = [x/2, x] \in 2^{L(X,Y)}$ for every $x \in K$ and $f : T(K) \times K \times K \to 2^Y$

is defined by $f(s,x,y) = \{sx(y-x) : s \in T(x)\}$, where $x \in K$, $y \in K$, and $\nu : K \times K \to 2^Y$ is defined by

$$\nu(x,y) = \begin{cases} \left\{\dfrac{x^2(y-x)}{2}\right\}, & y \geq x, \\ \{x^2(y-x)\}, & y \leq x. \end{cases} \tag{2.35}$$

We claim that condition $(\mathfrak{Y}')$ holds. Indeed, We know that the weak solution $\overline{x} = 1$. For each $x \in K_r = [1,r] \cap [1,2]$, if we choose any $t_x \in T(1)$, then $\operatorname{Min} f(t_x, \overline{x}, x) = \operatorname{Min}\{t_x(x-1)\} = \{t_x(x-1)\}$ and $\operatorname{Min}\bigcup_{x\in K_r} \operatorname{Max}_w \bigcup_{s\in T(\overline{x})} f(s,\overline{x},x) = \operatorname{Min}\bigcup_{x\in[1,r]\cap[1,2]} \operatorname{Max}_w\{s(x-1) : s \in [1/2,1]\} = \operatorname{Min}\bigcup_{x\in[1,r]\cap[1,2]}\{x-1\} = \operatorname{Min}[0,r-1] \cap [0,1] = \{0\}$. Hence condition $(\mathfrak{Y}')$ and all other conditions of Theorems 2.10 and 2.12 are satisfied. By Theorems 2.10 and 2.12, respectively, the $(\text{SEP})_I$ not only has a weak solution, but also has a strong solution. We can see that $\overline{x} = 1$ is a strong solution for $(\text{SEP})_I$.

Theorem 2.14. *Letting X be a finite-dimensional real Banach space, under the framework of Corollary 2.7, one has a weak solution $\overline{x}$ of $(\text{SEP})_I$ with $\overline{s} \in T(\overline{x})$. In addition, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, $T(\overline{x})$ is convex, $f(s,x,x) = \{0\}$ for all $s \in T(x)$ and for all $x \in K$, and the mapping $s \to -f(s,\overline{x},x)$ is naturally quasi C-convex on $T(\overline{x})$ for each $x \in K$. Assume that for some $r > \|\overline{x}\|$, condition $(\mathfrak{Y}')$ holds. Then $\overline{x}$ is a strong solution of $(\text{SEP})_I$; that is, there exists $\overline{s} \in T(\overline{x})$ such that*

$$f(\overline{s},\overline{x},x) \nleq_{\operatorname{int} C} \{0\} \tag{2.36}$$

for all $x \in K$. Furthermore, the set of all strong solutions of $(\text{SEP})_I$ is compact.

We would like to point out an open question naturally arising from Theorem 2.3: is Theorem 2.3 extendable to the case of $Y = \mathbb{R}^p$ or more general spaces, such as Hausdorff topological vector spaces?

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