Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 207016, 18 pages
doi:10.1155/2008/207016

Research Article

The Characterizations of Extreme Amenability of Locally Compact Semigroups

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Received 15 May 2008; Revised 22 August 2008; Accepted 10 November 2008

Recommended by Michael Tom

We demonstrate that the characterizations of topological extreme amenability. In particular, we prove that for every locally compact semigroup S with a right identity, the condition $\mu \odot (F \times G) = (\mu \odot F) \times (\mu \odot G)$, for F, G in $M(S)^*$, and $0 < \mu \in M(S)$, implies that $\mu = \varepsilon_a$, for some $a \in S$ (ε_a is a Dirac measure). We also obtain the conditions for which $M(S)^*$ is topologically extremely left amenable.

1. Introduction

Let S be a locally compact (Hausdorff) semigroup such that its multiplication is separately continuous. We denote by $m(S)$ the Banach algebra under the supremum norm of all bounded real-valued functions on S. For a topological semigroup S, let $\mathrm{BM}(S)$ and $\mathrm{CB}(S)$ be the closed subalgebras of $m(S)$ consisting of all Borel measurable functions and all continuous functions on S, respectively. Let $C_0(S)$ be the subalgebra of $\mathrm{CB}(S)$ consisting of the functions which vanish at infinity. Let $M(S)$ be the Banach space of all bounded regular Borel (signed) measures on S with a total variation norm. Let $M_0(S) = \{\mu \in M(S) : \mu \geq 0 \text{ and } \|\mu\| = 1\}$ be the set of all probability measures in $M(S)$.

It is known that $M(S) \simeq C_0(S)^*$ via the correspondence $\mu \to \overline{\mu}$, where $\overline{\mu}(f) = \int f\, d\mu$ for any f in $C_0(S)$ [1, Section 14]. Consider the continuous dual $M(S)^*$ of $M(S)$. An element M in $M(S)^{**}$ is called a mean on $M(S)$, if $M(1) = 1$ and $M(F) \geq 0$, whenever $F \geq 0$. An equivalent definition for a mean is that

$$\inf\{F(\mu) : \mu \in M_0(S)\} \leq M(F) \leq \sup\{F(\mu) : \mu \in M_0(S)\}, \tag{1.1}$$

for any F in $M(S)^*$. We also note that $M \in M(S)^{**}$ is a mean if and only if $\|M\| = M(1) = 1$. Each probability measure μ in $M_0(S)$ is a mean on $M(S)^{**}$ if we put $\mu(F) = F(\mu)$, for any F in $M(S)^*$. An application of Hahn-Banach separation theorem shows that $M_0(S)$ is weak* dense in the set of all means on $M(S)^*$.

Under point-wise operations and the supremum norm, $C_0(S)$ becomes a Banach algebra. Arens product can thus be defined in $C_0(S)^{**}$. In particular, we have the following defining formulas for any f, g in $C_0(S)$, m in $C_0(S)^*$, and θ, ϕ in $C_0(S)^{**}$:

$$\begin{aligned}(m \odot f)(g) &= m(fg),\\ (\phi \odot m)(f) &= \phi(m \odot f),\\ (\theta \odot \phi)(m) &= \theta(\phi \odot m).\end{aligned} \tag{1.2}$$

This product induces a multiplication in $M(S)^*$ via the identification $M(S) \cong C_0(S)^*$. Since $M(S)$ is a set of measures beside being the continuous dual of $C_0(S)$, this multiplication in $M(S)^*$ is richer in content than just a generic Arens product in the second dual of a Banach algebra, and it is different from the point-wise multiplication in $C_0(S)$.

For F, G in $M(S)^*$, we denote the multiplication of F and G by $F \times G$. In [2], it is shown that $F \times G$ is defined via the following three steps.

(a) For any $\mu \in M(S)$ and $f \in C_0(S)$, the measure $\mu_f \in M(S)$ is defined by

$$\int g \, d\mu_f = \int gf \, d\mu, \quad \forall g \in C_0(S). \tag{1.3}$$

(b) For any $\mu \in M(S)$ and $F \in M(S)^*$, the measure $F \times \mu \in M(S)$ is defined by

$$\int f \, d(F \times \mu) = F(\mu_f), \quad \forall f \in C_0(S). \tag{1.4}$$

(c) For any $F, G \in M(S)^*$, the functional $F \times G \in M(S)^*$ is defined by

$$(F \times G)(\mu) = F(G \times \mu), \quad \forall \mu \in M(S). \tag{1.5}$$

We can conclude that $M(S)^*$ becomes a commutative Banach algebra with an identity [2].

For the topological semigroup S, we define

$$(r_s f)(t) = f(ts), \qquad (l_s f)(t) = f(st), \tag{1.6}$$

where $s \in S$, $t \in S$, and $f \in m(S)$. Hence, r_s and l_s are the operators defined on $m(S)$ onto $m(S)$. A subset X of $m(S)$ is called a left (right) translation invariant, if $l_s X \subseteq X$ ($r_s X \subseteq X$) for all $s \in S$. It is well known that both $\mathrm{BM}(S)$ and $\mathrm{CB}(S)$ are left and right translation invariants [1]. Let S be a topological semigroup, which for each compact subset E of S and $a \in S$,

$$a^{-1}E = \{s \in S : as \in E\}, \qquad (Ea^{-1} = \{s \in S : sa \in E\}) \tag{1.7}$$

is compact. Then, $C_0(S)$ is a left (right) translation invariant.

Let X be a left (right) translation invariant subspace of $m(S)$ containing the constant function 1. A mean m on X is called a left (right) invariant, if

$$m(l_s f) = m(f) \qquad (m(r_s f) = m(f)), \tag{1.8}$$

for every $s \in S$ and $f \in X$ [1]. If $m(S)$ has a left invariant mean, then S is said to be left amenable [3]. If $m(S)$ has a multiplicative left-invariant mean, then S is said to be extremely left amenable (see [4, 5] for more details).

Suppose that S is a locally compact semigroup, then for $\mu, \nu \in M(S)$, the convolution $\mu * \nu$ is defined by

$$\int f \, d\mu * \nu = \iint f(st) d\mu(s) d\nu(t) = \iint f(st) d\nu(t) d\mu(s), \tag{1.9}$$

where $f \in C_0(S)$. Hence, $M(S)$ with a convolution $\mu * \nu$ as multiplication is a Banach algebra.

Now, for $F \in M(S)^*$ and $\mu \in M(S)$, the linear functional $l_\mu : M(S)^* \to M(S)^*$ is defined by

$$(l_\mu F)(\nu) = F(\mu * \nu), \quad \forall \nu \in M(S). \tag{1.10}$$

We denote $l_\mu F$ by $\mu \odot F$. Similarly, the $r_\mu F = F \odot \mu$ is defined by

$$(r_\mu F)(\nu) = (F \odot \mu)(\nu) = F(\nu * \mu), \quad \forall \nu \in M(S). \tag{1.11}$$

A mean M on $M(S)^*$ is called left invariant (LIM) if $M(l_{\varepsilon_a} F) = M(\varepsilon_a \odot F) = M(F)$ for all $F \in M(S)^*$ and for all $a \in S$. Also, a mean M on $M(S)^*$ is called topological left invariant (TLIM) if $M(l_\mu F) = M(\mu \odot F) = M(F)$ for all $F \in M(S)^*$ and for all $\mu \in M_0(S)$. A topological left invariant mean M on $M(S)^*$ is called a multiplicative topological left-invariant mean (MTLIM) if $M(F \times G) = M(F)M(G)$ for all $F, G \in M(S)^*$. If there is an MTLIM on $M(S)^*$, we say that S is extremely topological left amenable (ETLA). For results concerning ETLA and ELA semigroups, see [1, 6].

In this paper, we demonstrate that the Arens product and multiplication on $M(S)^*$ defined by (1.3), (1.4), and (1.5) are associative (see Lemma 2.4). In [7], it is proved that

$$(F \times G) \odot \mu = (F \odot \mu) \times (G \odot \mu) \tag{1.12}$$

is valid, for all $\mu \in M_0(S)$ and $F, G \in M(S)^*$. This means that Arens product $\odot$ distributes over multiplication $\times$ on $M(S)^*$ from right, for all $\mu \in M_0(S)$ and $F, G \in M(S)^*$. Note that the multiplication in $M(S)^*$ is different from the point-wise multiplication in $C_0(S)$. We show that Arens product $\odot$ distributes over multiplication $\times$ on $M(S)^*$ from left, when $F, G \in M(S)^*$, and μ is a Dirac measure (see Lemma 2.2(ii)). Also, it is shown that if S is a locally compact semigroup with a right identity and Arens product $\odot$ distributes from left over multiplication $\times$, then μ must be a Dirac measure (see Theorem 3.1). In the rest of this paper, we give some characterizations on ETLA of locally compact semigroups.

2. Preliminaries

In this section, we offer some results which are useful in the sequel. For more details, refer to [1, 2, 8]. Let $|\mu|$ be the total variation of μ, where $\mu \in M(S)$ and $L_\infty(S,|\mu|) = \mathrm{BM}(S)/N(\mu)$are the quotient Banach algebra with a quotient norm $\|\cdot\|_{\mu,\infty}$; $N(\mu)$ is the closed ideal of $\mathrm{BM}(S)$ consisting of all locally $|\mu|$-null functions. Consider the product linear space

$$\prod\{L_\infty(S,|\mu|) : \mu \in M(S)\}. \tag{2.1}$$

An element $f = \{f_\mu\}_{\mu\in M(S)}$ is called a generalized function on S, if the following conditions are satisfied:

(a) $\|f\| = \sup\{\|f_\mu\|_{\mu,\infty} : \mu \in M(S)\} < \infty$,

(b) for μ, ν in $M(S)$ with $\mu \ll \nu$, we have $f_\mu = f_\nu |\mu|$-a.e.

Let $\mathrm{GL}(S)$ be the linear space of all generalized functions on S. It is known that $\mathrm{GL}(S)$ is a Banach space with the norm defined by the formula (a) and that $\mathrm{GL}(S) \cong M(S)^*$ via the isometric Banach space isomorphism, $\varphi : \mathrm{GL}(S) \to M(S)^*$, where $(\varphi f)(\mu) = \int f_\mu \, d\mu$ for any μ in $M(S)$ and f in $\mathrm{GL}(S)$ (see [8, 9]). A function f in $\mathrm{BM}(S)$ can be treated as an element in $\mathrm{GL}(S)$ with $f_\mu = f$ for all μ in $M(S)$. The space $\mathrm{BM}(S)$ is thus a subspace of $\mathrm{GL}(S)$.

For $f \in \mathrm{BM}(S)$, $\mu \in M(S)$, and $\nu \in M(S)$, we define $f \odot \mu$ and $\mu \odot f$ in $L_\infty(S,|\nu|)$ by

$$\begin{aligned} f \odot \mu(s) &= \int f(st)d\mu(t) = \int l_s f \, d\mu \quad |\nu|\text{-a.e.}, \\ \mu \odot f(s) &= \int f(ts)d\mu(t) = \int r_s f \, d\mu \quad |\nu|\text{-a.e.} \end{aligned} \tag{2.2}$$

It is shown that $f \odot \mu$, $\mu \odot f \in \mathrm{GL}(S)$ [8]. If $f \in \mathrm{CB}(S)$, then the above equalities hold everywhere, and $f \odot \mu$ and $\mu \odot f$ are in $\mathrm{CB}(S)$. Also, if $f \in \mathrm{BM}(S)$ and $a \in S$, then

$$f \odot \varepsilon_a(s) = \int f(st)d\varepsilon_a(t) = f(sa) = (r_a f)(s), \tag{2.3}$$

$$\varepsilon_a \odot f(s) = \int f(ts)d\varepsilon_a(t) = f(as) = (l_a f)(s). \tag{2.4}$$

Hence, $f \odot \varepsilon_a$ and $\varepsilon_a \odot f$ belong to $\mathrm{BM}(S)$.

Lemma 2.1. *The map $\varphi : GL(S) \to M(S)^*$ defined by $(\varphi f)(\mu) = \int f_\mu \, d\mu$ for any μ in $M(S)$ and f in $GL(S)$ satisfies the following statements.*

(i) *For any $f \in BM(S)$ and $\mu \in M(S)$,*

$$\varphi(\mu \odot f) = \mu \odot \varphi f, \qquad \varphi(f \odot \mu) = \varphi f \odot \mu. \tag{2.5}$$

(ii) *For any $a \in S$ and $f \in BM(S)$,*

$$\varphi(l_a f) = \varepsilon_a \odot \varphi f, \qquad \varphi(r_a f) = \varphi f \odot \varepsilon_a. \tag{2.6}$$

Proof. We have $(\mu \odot f)_\nu = \mu \odot f_{\mu*\nu}$ for any $\nu \in M(S)$ [8]. Hence,

$$\begin{aligned}\varphi(\mu \odot f)(\nu) &= \int (\mu \odot f)_\nu d\nu \\ &= \int (\mu \odot f_{\mu*\nu})(s) d\nu(s) \\ &= \int\int f_{\mu*\nu}(ts) d\mu(t) d\nu(s) \\ &= \int f_{\mu*\nu} d\mu*\nu \\ &= \varphi f(\mu*\nu) \\ &= (\mu \odot \varphi f)(\nu).\end{aligned} \tag{2.7}$$

Thus, $\varphi(\mu \odot f) = \mu \odot \varphi f$. Similarly, $\varphi(f \odot \mu) = \varphi f \odot \mu$. This proves (i). From (i) and (2.4), part of (ii) is trivial. □

Lemma 2.2. *For each $a \in S$ and $F, G \in M(S)^*$, we have*

(i) $(F \times G) \odot \varepsilon_a = (F \odot \varepsilon_a) \times (G \odot \varepsilon_a)$,

(ii) $\varepsilon_a \odot (F \times G) = (\varepsilon_a \odot F) \times (\varepsilon_a \odot G)$.

Proof. (i) For each $\mu \in M(S)$, from (2.3), we have

$$\begin{aligned}((F \times G) \odot \varepsilon_a)(\mu) &= r_{\varepsilon_a}(F \times G)(\mu) \\ &= (F \times G)(\mu*\varepsilon_a) \\ &= F(G \times (\mu*\varepsilon_a)).\end{aligned} \tag{2.8}$$

But, for $g \in C_0(S)$, from (1.4) and (1.9), we have

$$\begin{aligned}\int g d(G \times (\mu*\varepsilon_a)) &= G((\mu*\varepsilon_a)_g) \quad (\text{by } (1.4)) \\ &= G(\mu_{r_a g}*\varepsilon_a) \\ &= (G \odot \varepsilon_a)(\mu_{r_a g}) \\ &= \int (r_a g)(y) d((G \odot \varepsilon_a) \times \mu)(y) \quad (\text{by } (1.4)) \\ &= \int g(ya) d((G \odot \varepsilon_a) \times \mu)(y) \\ &= \int\int g(yx) d((G \odot \varepsilon_a) \times \mu)(y) d\varepsilon_a(x) \quad (\text{by } (1.9)) \\ &= \int g d(((G \odot \varepsilon_a) \times \mu)*\varepsilon_a).\end{aligned} \tag{2.9}$$

Hence, by the Riesz representation theorem, $G \times (\mu * \varepsilon_a) = ((G \odot \varepsilon_a) \times \mu) * \varepsilon_a$. Thus,

$$\begin{aligned} ((F \times G) \odot \varepsilon_a)(\mu) &= F(G \times (\mu * \varepsilon_a)) \\ &= F(((G \odot \varepsilon_a) \times \mu) * \varepsilon_a) \\ &= (F \odot \varepsilon_a)((G \odot \varepsilon_a) \times \mu) \\ &= ((F \odot \varepsilon_a) \times (G \odot \varepsilon_a))(\mu). \end{aligned} \tag{2.10}$$

Therefore, $(F \times G) \odot \varepsilon_a = (F \odot \varepsilon_a) \times (G \odot \varepsilon_a)$.

(ii) For each $\mu \in M(S)$, equality (2.4) implies that

$$\begin{aligned} (\varepsilon_a \odot (F \times G)) &= l_{\varepsilon_a}(F \times G)(\mu) \\ &= (F \times G)(\varepsilon_a * \mu) \\ &= F(G \times (\varepsilon_a * \mu)). \end{aligned} \tag{2.11}$$

Now, for $g \in C_0(S)$, from (1.4) and (1.9), we obtain

$$\begin{aligned} \int g d(G \times (\varepsilon_a * \mu)) &= G((\varepsilon_a * \mu)_g) \\ &= G(\varepsilon_a * \mu_{l_a g}) \\ &= (\varepsilon_a \odot G)(\mu_{l_a g}) \\ &= \int (l_a g)(y) d((\varepsilon_a \odot G) \times \mu)(y) \\ &= \int g(ay) d((\varepsilon_a \odot G) \times \mu)(y) \\ &= \int\int g(xy) d\varepsilon_a(x) d((\varepsilon_a \odot G) \times \mu)(y) \\ &= \int g d(((\varepsilon_a \odot G) \times \mu) * \varepsilon_a). \end{aligned} \tag{2.12}$$

Hence, by the Riesz representation theorem, $G \times (\varepsilon_a * \mu) = ((\varepsilon_a \odot G) \times \mu) * \varepsilon_a$. Thus

$$\begin{aligned} (\varepsilon_a \odot (F \times G))(\mu) &= F(G \times (\varepsilon_a * \mu)) \\ &= F(((\varepsilon_a \odot G) \times \mu) * \varepsilon_a) \\ &= (\varepsilon_a \odot F)((\varepsilon_a \odot G) \times \mu) \\ &= ((\varepsilon_a \odot F) \times (\varepsilon_a \odot G))(\mu). \end{aligned} \tag{2.13}$$

Therefore, $\varepsilon_a \odot (F \times G) = (\varepsilon_a \odot F) \times (\varepsilon_a \odot G)$. □

Remarks 2.3. (a) In the proof of Lemma 2.2, we use the equalities $(\mu*\varepsilon_a)_g = \mu_{r_a g}*\varepsilon_a$ and $(\varepsilon_a*\mu)_g = \varepsilon_a*\mu_{l_a g}$. For $f \in C_0(S)$, we have

$$\begin{aligned}
\int f d(\mu*\varepsilon_a)_g &= \int g f d(\mu*\varepsilon_a) \quad (\text{by } (1.3)) \\
&= \iint (gf)(xy) d\mu(x) d\varepsilon_a(y) \quad (\text{by } (1.9)) \\
&= \int (gf)(xa) d\mu(x) \\
&= \int g(xa) f(xa) d\mu(x) \\
&= \int (r_a g)(r_a f) d\mu \quad (\text{by } (1.6)) \\
&= \int (r_a f)(x) d\mu_{r_a g}(x) \quad (\text{by } (1.3)) \\
&= \int f(xa) d\mu_{r_a g}(x) \quad (\text{by } (1.9)) \\
&= \iint f(xy) d\mu_{r_a g}(x) d\varepsilon_a(y) \quad (\text{by } (1.6)) \\
&= \int f d(\mu_{r_a g}*\varepsilon_a).
\end{aligned} \tag{2.14}$$

Hence, by the Riesz representation theorem, $(\mu*\varepsilon_a)_g = \mu_{r_a g}*\varepsilon_a$. Similarly, $(\varepsilon_a*\mu)_g = \varepsilon_a*\mu_{l_a g}$.

(b) The statement (i) of Lemma 2.2 has a general form as replacing a Dirac measure by $\mu \in M_0(S)$ [7]. It is natural to ask for which μ in $M_0(S)$, the equality

$$\mu \odot (F \times G) = (\mu \odot F) \times (\mu \odot G), \quad \forall F, G \in M(S)^* \tag{2.15}$$

is valid?

Now, we demonstrate that the multiplication on $M(S)^*$ defined by (1.3), (1.4), and (1.5) is associative.

Lemma 2.4. *The multiplication* $\times$ *defined by* (1.3), (1.4), *and* (1.5) *on* $M(S)^*$ *is associative.*

Proof. We know that the Arens product $\odot$ is associative [3, Lemma 1, page 527]. Let $\pi : C_0(S)^* \to M(S)$ be isometric order-preserving linear space isomorphism in [1, Theorem 14.10, page 170], namely, for any $m \in C_0(S)^*$ and $f \in C_0(S)$,

$$\int f d\pi(m) = m(f). \tag{2.16}$$

Now, let $f, g \in C_0(S)^*$, $m \in C_0(S)^*$, and $F, G \in M(S)^*$, then (1.3) implies that

$$\begin{aligned}\int g d\pi(m)_f &= \int f g d\pi(m) \\ &= m(fg) \\ &= (m \odot f)(g) \\ &= \int g d\pi(m \odot f).\end{aligned} \tag{2.17}$$

Thus, $\pi(m)_f = \pi(m \odot f)$. Also, from (1.4), we have

$$\begin{aligned}\int f d\big(F \times \pi(m)\big) &= F\big(\pi(m)_f\big) \\ &= F\big(\pi(m \odot f)\big) \\ &= \pi^*(F)(m \odot f) \\ &= \big(\pi^*(F) \odot m\big)(f) \\ &= \int f d\pi\big(\pi^*(F) \odot m\big).\end{aligned} \tag{2.18}$$

Hence, $F \times \pi(m) = \pi(\pi^*(F) \odot m)$. Also, from (1.5), we have

$$\begin{aligned}\big(\pi^*(F \times G)\big)(m) &= (F \times G)\big(\pi(m)\big) \\ &= F\big(G \times \pi(m)\big) \\ &= F\big(\pi\big(\pi^*(G) \odot m\big)\big) \\ &= \big(\pi^*(F)\big)\big(\pi^*(G) \odot m\big) \\ &= \big(\pi^*(F) \odot \pi^*(G)\big)(m).\end{aligned} \tag{2.19}$$

Therefore, $\pi^*(F \times G) = \pi^*(F) \odot \pi^*(G)$. Now, for any $F, G, H \in M(S)^*$,

$$\begin{aligned}\pi^*\big((F \times G) \times H\big) &= \pi^*\big((F \times G) \odot \pi^*(H)\big) \\ &= \big(\pi^*(F) \odot \pi^*(G)\big) \odot \pi^*(H) \\ &= \pi^*(F) \odot \big(\pi^*(G) \odot \pi^*(H)\big) \\ &= \pi^*(F) \odot \big(\pi^*(G \times H)\big) \\ &= \pi^*\big(F \times (G \times H)\big).\end{aligned} \tag{2.20}$$

So, $(F \times G) \times H = F \times (G \times H)$, and thus the multiplication of $\times$ is associative. □

Remark 2.5. We note that one can go through a process analogous to Day's proof [3] and establish the associativity of $\times$ via the demonstration of the following identities one by one.

(i) For any $\mu \in M(S)$ and $f, g \in C_0(S)$,

$$\big(\mu_f\big)_g = \mu_{fg}. \tag{2.21}$$

(ii) For any $F \in M(S)^*$, $\mu \in M(S)$ and $f \in C_0(S)$,

$$F \times (\mu_f) = (F \times \mu)_f. \tag{2.22}$$

(iii) For any $F, G \in M(S)^*$ and $\mu \in M(S)$,

$$(F \times G) \times \mu = F \times (G \times \mu). \tag{2.23}$$

(iv) For any $F, G, H \in M(S)^*$,

$$F \times (G \times H) = (F \times G) \times H. \tag{2.24}$$

The proofs of (i), (ii), and (iii) use the Riesz representation theorem and the relations (1.3), (1.4), and (1.5). The proof of (iv) follows from (iii) using definition.

3. Main results

Each probability measure μ in $M_0(S)$ is a mean on $M(S)^*$, if we put $\mu(F) = F(\mu)$ for any F in X. We give a partial answer to the question: For which $\mu \in M_0(S)$, is the equality

$$\mu \odot (F \times G) = (\mu \odot F) \times (\mu \odot G), \quad \forall F, G \in M(S)^* \tag{3.1}$$

valid?

Let $f \in \mathrm{GL}(S)$, from the isometric Banach space isomorphism $\varphi : \mathrm{GL}(S) \to M(S)^*$, we have $\varphi f = F$ which is in $M(S)^*$, where $F(\mu) = (\varphi f)(\mu) = \int f_\mu d\mu$, for any μ in $M(S)$. For $\mu \in M(S)$ and $g \in C_0(S)$, we have

$$\begin{aligned} \int g d(F \times \mu) &= F(\mu_g) \\ &= (\varphi f)(\mu_g) \\ &= \int f_{\mu_g} d\mu_g \\ &= \int f_\mu d\mu_g \\ &= \int f_\mu g d\mu \\ &= \int g d\mu_{f_\mu}. \end{aligned} \tag{3.2}$$

Therefore, $F \times \mu = \varphi f \times \mu = \mu_{f_\mu}$. In particular, $\varphi f \times \mu \ll \mu$ and so $1 \times \mu_1 = \mu_1 = \mu$. In view of (1.4), if $F \in M(S)^*$ and $\mu \in M(S)$, then

$$(F \times \mu)(S) = \int d(F \times \mu) = F(\mu_1) = F(\mu). \tag{3.3}$$

Hence, if $F \geq 0$ and $\mu \geq 0$, then

$$\|F \times \mu\| = (F \times \mu)(S) = F(\mu). \tag{3.4}$$

Also, since $\|G\|\ 1 - G \geq 0$, we have

$$(\|G\|\ 1 - G) \times \mu \geq 0, \tag{3.5}$$

and hence, $G \times \mu \leq \|G\|\ 1 \times \mu = \|G\|\ \mu$ whenever $\mu \geq 0$.

Theorem 3.1. *Let S be a locally compact semigroup with a right identity and that $0 < \mu \in M(S)$. If $\mu \odot (F \times G) = (\mu \odot F) \times (\mu \odot G)$ for any $F, G \in \varphi(C_0(S))$, then μ is a Dirac measure.*

Proof. For $f, g \in C_0(S)$, we have

$$\begin{aligned}\varphi(\mu \odot (fg)) &= \mu \odot \varphi(fg) \quad \text{(by Lemma 2.1(i))} \\ &= \mu \odot (\varphi f \times \varphi g) \\ &= (\mu \odot \varphi f) \times (\mu \odot \varphi g) \\ &= \varphi(\mu \odot f) \times \varphi(\mu \odot g) \quad \text{(by Lemma 2.1(i))} \\ &= \varphi((\mu \odot f)(\mu \odot g)).\end{aligned} \tag{3.6}$$

Thus, for any $f, g \in C_0(S)$, from (3.6), we have

$$\mu \odot (fg) = (\mu \odot f)(\mu \odot g). \tag{3.7}$$

Now, let e_r be a right identity of S, that is, $se_r = s$ for any $s \in S$, then for any $f \in C_0(S)$, we have

$$(\mu \odot f)(e_r) = \int f(te_r)\,d\mu(t) = \int f(t)\,d\mu(t). \tag{3.8}$$

Hence, for each $f, g \in C_0(S)$,

$$\begin{aligned}\int fg\,d\mu &= (\mu \odot (fg))(e_r) \\ &= (\mu \odot f)(e_r)(\mu \odot g)(e_r) \\ &= \left(\int f\,d\mu\right)\left(\int g\,d\mu\right).\end{aligned} \tag{3.9}$$

In (3.9), we put $f = g$, then for any $f \in C_0(S)$,

$$\int f^2 d\mu = \left(\int f\,d\mu\right)^2, \tag{3.10}$$

so for each $f, g \in C_0(S)$,

$$\left(\int fg d\mu\right)^2 = \left[\left(\int f d\mu\right)\left(\int g d\mu\right)\right]^2 = \left(\int f d\mu\right)^2\left(\int g d\mu\right)^2 = \left(\int f^2 d\mu\right)\left(\int g^2 d\mu\right), \tag{3.11}$$

and by Holder's inequality, there exist real numbers α and β, not being zero, such that

$$\alpha f^2 = \beta g^2 \quad \text{a.e. } (\mu). \tag{3.12}$$

Now, if A and B are the disjoint compact subsets of S with $\mu(A) > 0$ and $\mu(B) > 0$, by the Urysohn's lemma, there exist f and g in $C_{00}(S)$ such that

$$f(A) = 0 = g(B), \qquad f(B) = 1 = g(A). \tag{3.13}$$

But from (3.12) and $\mu(A) > 0$, there is $x_0 \in A$ such that

$$\alpha f(x_0)^2 = \beta g(x_0)^2. \tag{3.14}$$

So, $0 = \beta g(x_0)^2 = \beta$. Also, $\mu(B) > 0$ follows that there is $y_0 \in B$, such that

$$\alpha f(y_0)^2 = \beta g(y_0)^2, \tag{3.15}$$

and therefore $0 = \alpha f(y_0)^2 = \alpha$. This contradicts the fact that α and β are not both zero. Hence, if A is a compact subset of S with $\mu(A) > 0$ and B is another compact subset of S disjointed from A, then we must have $\mu(B) = 0$. Therefore, $\mu(A^c) = 0$, that is, $\mu(A) = \mu(S)$. This proves that if A is a compact subset S, then either $\mu = 0$ or $\mu(A) = \mu(S)$.

Now, the regularity of μ follows that for each Borel subset B of S, $\mu(B) = 0$ or $\mu(B) = \mu(S)$. Hence, either $\mu = 0$ or the measure $\mu/\mu(S)$ is a Dirac measure, say $\mu/\mu(S) = \varepsilon_a$ [2].

Now, put $\mu = \mu(S)\varepsilon_a$, we have

$$\mu(S) = \int d\mu = \int 1^2 d\mu = \left(\int 1 d\mu\right)^2 = \left(\int d\mu\right)^2 = \mu(S)^2, \tag{3.16}$$

and so $\mu(S) = 1$. Thus, $\mu = \varepsilon_a$. □

Remark 3.2. If S is a discrete semigroup, then $C_0(S)^* = \ell_1(S)$, and so $M_0(S)^* = C_0(S)^{**} = \ell_1(S)^* = m(S)$ [1]. In this case, the multiplication on $M(S)^*$ is just the point-wise multiplication as in $m(S)$. Let e be the right identity of S, then

$$\begin{aligned}
(\mu \odot (F \times G))(\varepsilon_e) &= (F \times G)(\mu * \varepsilon_e) \\
&= (F \times G)(\mu) \\
&= \mu(F \times G), \\
((\mu \odot F) \times (\mu \odot G))(\varepsilon_e) &= (\mu \odot F)(\varepsilon_e)(\mu \odot G)(\varepsilon_e) \\
&= \mu(F)\mu(G),
\end{aligned} \tag{3.17}$$

since both $\mu \odot F$ and $\mu \odot G$ are in $M(S)^* = m(S)$. Hence, $\mu \in M(S)$ is multiplicative, if the condition of Theorem 3.1 is satisfied. Therefore, μ must be either 0 or a Dirac measure. But, when S is a topological semigroup, the multiplication in $M(S)^*$ defined by (1.3), (1.4), and (1.5) is just a generic Arens product in the second dual of a Banach algebra, which is different from the point-wise multiplication.

It is known that $M_0(S)$ is weak* dense in the set of all means on $M(S)^*$. We give some characterizations theorems for the extreme amenability of locally compact semigroup.

Lemma 3.3. *Let $M(S)^*$ be TLA. The following statements are equivalent:*

(i) *$M(S)^*$ is ETLA,*

(ii) *for every $F \in M(S)^*$ and $\mu \in M_0(S)$, there exists a mean M on $M(S)^*$ such that $M(F \times F) = M(\mu \odot F)^2$,*

(iii) *for every $F \in M(S)^*$ and $\mu \in M_0(S)$, there exists a mean M on $M(S)^*$ such that $M(F \times F) = M(F)M(\mu \odot F)$,*

(iv) *for every $F \in M(S)^*$ and $\mu \in M_0(S)$, there exists a mean M on $M(S)^*$ such that $M(\mu \odot (F \times F)) = M(F)^2$.*

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious.

(iv) ⇒ (i). Suppose that $F, G \in M(S)^*$ and $\mu \in M_0(S)$. For $F + G$ by (iv), there exists a mean M on $M(S)^*$ such that

$$M(\mu \odot ((F+G) \times (F+G)) = M(F+G)^2) = M(F)^2 + 2M(F)M(G) + M(G)^2, \tag{3.18}$$

and to expand the right-hand side, we get

$$M(\mu \odot (F \times G)) = M(F)M(G). \tag{3.19}$$

Since $M(S)^*$ is topological left invariant, hence, $M(F \times G) = M(F)M(G)$. Therefore, $M(S)^*$ is ETLA. □

Theorem 3.4. *Let M be a topological left invariant mean on $M(S)^*$. The following statements are equivalent:*

(i) *M is a multiplicative,*

(ii) *there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that for any μ in $M_0(S)$ and G in $M(S)^*$,*

$$w^* - \lim_\alpha \big(G \times (\mu * \mu_\alpha) - M(G)\mu_\alpha\big) = 0. \tag{3.20}$$

Proof. (i) $\Rightarrow$ (ii). Let M be a multiplicative topological left invariant mean on $M(S)^*$. By Lemma 3.3, for any $F \in M(S)^*$ and $\mu \in M_0(S)$,

$$M\big(\mu \odot (F \times F)\big) = M(F)^2. \tag{3.21}$$

Let $F, G \in M(S)^*$, then

$$M\big(\mu \odot ((F+G) \times (F+G))\big) = M(F+G)^2. \tag{3.22}$$

We have

$$M\big(\mu \odot (F \times F + 2F \times G + G \times G)\big) = \big(M(F) + M(G)\big)^2. \tag{3.23}$$

So,

$$M\big(\mu \odot (F \times F)\big) + 2M\big(\mu \odot (F \times G)\big) + M\big(\mu(G \times G)\big) = M(F)^2 + M(G)^2 + 2M(F)M(G). \tag{3.24}$$

Thus,

$$M\big(\mu \odot (F \times G)\big) = M(F)M(G). \tag{3.25}$$

Note that we apply the commutativity of $\times$ in $M(S)^*$. Since M is a mean on $M(S)^*$ and $M_0(S)$ is weak* dense in the set of all means on $M(S)^*$, hence, there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $M = w^* - \lim_\alpha \mu_\alpha$ in $M(S)^{**}$. Now for $F \in M(S)^*$,

$$\begin{aligned} w^* - \lim_\alpha F\big(G \times (\mu * \mu_\alpha) - M(G)\mu_\alpha\big) &= w^* - \lim_\alpha \big(\mu_\alpha\big(\mu \odot (F \times G)\big) - M(G)\mu_\alpha(F)\big) \\ &= M\big(\mu \odot (F \times G)\big) - M(F)M(G) \\ &= 0. \end{aligned} \tag{3.26}$$

Thus,

$$F\Big(w^* - \lim_\alpha \big(G \times (\mu * \mu_\alpha) - M(G)\mu_\alpha\big)\Big) = 0, \tag{3.27}$$

that is,

$$w^* - \lim_{\alpha}\left(G \times (\mu * \mu_\alpha) - M(G)\mu_\alpha\right) = 0. \tag{3.28}$$

(ii) ⇒ (i). Since M is a topological left invariant mean on $M(S)^*$, there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $M = w^* - \lim_\alpha \mu_\alpha$ in $M(S)^{**}$. If $\mu \in M_0(S)$,

$$\begin{aligned}
M(F \times G) - M(F)M(G) &= M(\mu \odot (F \times G)) - M(F)M(G) \\
&= w^* - \lim_{\alpha} \mu_\alpha(\mu \odot (F \times G)) - M(G)\Big(w^* - \lim_{\alpha} \mu_\alpha(F)\Big) \\
&= w^* - \lim_{\alpha}\big((F \times G)(\mu * \mu_\alpha) - M(G)F(\mu_\alpha)\big) \\
&= w^* - \lim_{\alpha}\big(F(G \times (\mu * \mu_\alpha)) - F(M(G)\mu_\alpha)\big) \\
&= w^* - \lim_{\alpha} F\big(G \times (\mu * \mu_\alpha) - M(G)\mu_\alpha\big) \\
&= F\Big(w^* - \lim_{\alpha}\big(G \times (\mu * \mu_\alpha) - M(G)\mu_\alpha\big)\Big) \\
&= F(0) \\
&= 0.
\end{aligned} \tag{3.29}$$

Therefore, $M(F \times G) = M(F)M(G)$, that is, $M(S)^*$ is extremely topological left amenable (ETLA). □

Lemma 3.5. *If M is a multiplicative topological left invariant mean on $M(S)^*$, then there is a net $\{\mu_\beta\}$ in $M_0(S)$ such that for any μ in $M_0(S)$ and F in $M(S)^*$,*

$$\lim_{\beta}\left\| F \times (\mu * \mu_\beta) - M(F)\mu_\beta \right\| = 0. \tag{3.30}$$

Proof. We consider $M(S)$ with the norm topology. Let $\mathcal{D} = M(S)^{M(S)^* \times M_0(S)}$ with the product of the norm topologies, where $M(S)^* \times M_0(S)$is the set theoretic cartesian product. Then, $\mathcal{D}$ is a locally convex topological vector space [10]. Now, by Theorem 3.4 corresponding to M, there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that for any μ in $M_0(S)$ and F in $M(S)^*$,

$$w^* - \lim_{\alpha}\left(F \times (\mu * \mu_\alpha) - M(F)\mu_\alpha\right) = 0. \tag{3.31}$$

We define a linear map $\mathbf{P} : M(S) \to \mathcal{D}$ by

$$\mathbf{P}(\nu)(F, \mu) = F \times (\mu * \nu) - M(F)\nu, \tag{3.32}$$

for all $(F, \mu) \in M(S)^* \times M_0(S)$. Hence,

$$w^* - \lim_{\alpha} \mathbf{P}(\mu_\alpha)(F, \mu) = 0, \tag{3.33}$$

that is, $\mathbf{P}(\mu_\alpha) \to 0$ in the product of weak topologies [10]. Therefore, 0 lies in the weak closure of the convex set $\mathbf{P}(M_0(S))$, and so is in the closure of $\mathbf{P}(M_0(S))$ in the original topology of $\mathcal{D}$. So, there is a net $\{\mu_\beta\}$ in $M_0(S)$ such that for all $(F,\mu) \in M(S)^* \times M_0(S)$,

$$\lim_\beta \|\mathbf{P}(\mu_\beta)(F,\mu)\| = 0, \tag{3.34}$$

that is,

$$\lim_\beta \|F \times (\mu * \mu_\beta) - M(F)\mu_\beta\| = 0, \tag{3.35}$$

and the proof is complete. □

Theorem 3.6. *Let S be a locally compact semigroup. The following statements are equivalent:*

(i) *$M(S)^*$ is extremely topological left amenable,*

(ii) *there exists a net $\{\mu_\beta\}$ in $M_0(S)$ such that for any μ in $M_0(S)$ and F in $M(S)^*$,*

$$\lim_\beta \|\mu * \mu_\beta - \mu_\beta\| = 0, \qquad \lim_\beta \|F \times \mu_\beta - F(\mu_\beta)\mu_\beta\| = 0, \tag{3.36}$$

(iii) *there exists a net $\{\mu_\gamma\}$ in $M_0(S)$ such that for any μ in $M_0(S)$ and F in $M(S)^*$,*

$$w^* - \lim_\gamma (\mu * \mu_\gamma - \mu_\gamma) = 0, \tag{3.37}$$

$$w^* - \lim_\gamma (F \times \mu_\gamma - F(\mu_\gamma)\mu_\gamma) = 0. \tag{3.38}$$

Proof. (i) ⇒ (ii). Let M be a multiplicative left invariant mean on $M(S)^*$. Theorem 3.4 implies that there exists a net $\{\mu_\alpha\}$ in $M_0(S)$ such that for any μ in $M_0(S)$ and F in $M(S)^*$,

$$w^* - \lim_\alpha (F \times (\mu * \mu_\alpha) - M(F)\mu_\alpha) = 0. \tag{3.39}$$

By Lemma 3.5, there exists a net $\{\mu_\beta\}$ in $M_0(S)$ such that for any μ in $M_0(S)$ and F in $M(S)^*$,

$$\lim_\beta \|F \times (\mu * \mu_\beta) - M(F)\mu_\beta\| = 0. \tag{3.40}$$

Without the loss of generality, we may assume that $\mu_\beta \to M_1$ $\sigma(M(S)^{**}, M(S)^*)$ for some mean M_1 in $M(S)^{**}$. Therefore, for any F, G in $M(S)^*$ and μ in $M_0(S)$, we have

$$\begin{aligned} M_1(\mu \odot (G \times F)) - M(F)M_1(G) &= \lim_\beta \{ (\mu \odot (G \times F))(\mu_\beta) - M(F)M_1(G) \} \\ &= \lim_\beta [G(F \times (\mu * \mu_\beta)) - G(M(F)\mu_\beta)] \\ &= \lim_\beta G(F \times (\mu * \mu_\beta) - M(F)\mu_\beta) \\ &= 0. \end{aligned} \tag{3.41}$$

In (3.41), we put $F = 1 = G$, then

$$\begin{gathered} M_1(\mu \odot (1 \times 1)) = M(1) M_1(1), \\ M_1(\mu \odot 1) = M(1) M_1(1), \\ M(1) = 1. \end{gathered} \tag{3.42}$$

Also, for $F = 1$ and G in $M(S)^*$,

$$M_1(\mu \odot G) = M_1(G), \tag{3.43}$$

and for $G = 1$ and F in $M(S)^*$,

$$M_1(\mu \odot F) = M_1(F). \tag{3.44}$$

Therefore, for any F in $M(S)^*$, we have

$$M(F) = M_1(\mu \odot F) = M_1(F). \tag{3.45}$$

Now, from (3.30) for $F = 1$, we get

$$\lim_\beta \|\mu * \mu_\beta - \mu_\beta\| = 0. \tag{3.46}$$

Also, let F in $M(S)^*$ and $\varepsilon > 0$ be given. Since

$$M(F) = M_1(F), \qquad F(\mu_\beta) \longrightarrow M(F), \quad \|\mu_\beta\| \le 1, \tag{3.47}$$

it follows from (3.30) that for any μ in $M_0(S)$,

$$\lim_\beta \|F \times (\mu * \mu_\beta) - M_1(F)\mu_\beta\| = 0. \tag{3.48}$$

Now fix an arbitrary $\mu \in M_0(S)$. This together with (3.46) implies that there exists a β_0 such that

$$\begin{gathered} \|\mu * \mu_\beta - \mu_\beta\| < \frac{\varepsilon}{3(\|F\| + 1)}, \quad \forall \beta \ge \beta_0, \\ \|F \times (\mu * \mu_\beta) - M_1(F)\mu_\beta\| < \frac{\varepsilon}{3}. \end{gathered} \tag{3.49}$$

Also, we may assume that

$$|F(\mu_\beta) - M_1(F)| < \frac{\varepsilon}{3}. \tag{3.50}$$

Consequently,

$$\begin{aligned}
&\left\|F \times \mu_\beta - F(\mu_\beta)\mu_\beta\right\| \\
&\quad \le \left\|F \times \mu_\beta - F \times (\mu * \mu_\beta)\right\| + \left\|F \times (\mu * \mu_\beta) - M_1(F)\mu_\beta\right\| + \left\|M_1(F)\mu_\beta - F(\mu_\beta)\mu_\beta\right\| \\
&\quad \le \|F\|\left\|\mu * \mu_\beta - \mu_\beta\right\| + \left\|F \times (\mu * \mu_\beta) - M_1(F)\mu_\beta\right\|\left\|\mu_\beta\right\|\left|M_1(F) - F(\mu_\beta)\right| \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\quad = \varepsilon, \quad \forall \beta \ge \beta_0.
\end{aligned} \tag{3.51}$$

Obviously, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Since $M_0(S)$ is weak* dense in the set of all means on $M(S)^*$, by passing to a subnet if necessary, we may assume that $\mu_\gamma \to M$ weakly* in $M(S)^{**}$ for some mean M. Thus, the assertion of (3.37) implies that M is a topological left invariant mean. Also, (3.38) implies that M is multiplicative because for any F, G in $M(S)^*$ and μ in $M_0(S)$,

$$\begin{aligned}
M(G \times F) - M(G)M(F) &= w^* - \lim_\gamma\left\{(G \times F)(\mu_\gamma) - G(\mu_\gamma)F(\mu_\gamma)\right\} \\
&= w^* - \lim_\gamma\left\{G(F \times \mu_\gamma) - G\left(F(\mu_\gamma)\mu_\gamma\right)\right\} \\
&= w^* - \lim_\gamma G\left(F \times \mu_\gamma - F(\mu_\gamma)\mu_\gamma\right) \\
&= G(0) = 0.
\end{aligned} \tag{3.52}$$

Therefore, $M(G \times F) = M(G)M(F)$, that is, $M(S)^*$ is extremely left amenable. □

Remark 3.7. The conclusions of Theorem 3.6 are different from the classical characterizations of extremely left amenable discrete semigroups [4, Theorem 2]. This difference in the two situations is that any multiplicative mean on $m(S)$ is the weak* limit of evaluation functionals, while taking weak* limits of all convergent nets of Dirac measures in $M(S)^*$ does not exhaust all multiplicative means on $M(S)^*$ [2, Theorem 2.7].

Theorem 3.8. *Let S be a locally compact semigroup. Define a function $\mathbf{T} : M(S)^* \to m(S)$ by*

$$\left(\mathbf{T}(F)\right)(a) = F \odot \varepsilon_a, \quad \forall a \in S. \tag{3.53}$$

Then,

(i) $\mathbf{T}$ *is bounded and linear,*

(ii) $\mathbf{T}(1) = 1$,

(iii) $\mathbf{T}(F) \ge 0$ *whenever* $F \ge 0$,

(iv) $\mathbf{T}(F \times G) = \mathbf{T}(F)\mathbf{T}(G)$ *for all F and G in $M(S)^*$,*

(v) $l_b(\mathbf{T}(F)) = \mathbf{T}(\varepsilon_b \odot F)$ *for all $b \in S$ and $F \in M(S)^*$.*

Proof. (i), (ii), and (iii) are obvious.

(iv) For any $a \in S$, we have

$$\begin{aligned}(\mathbf{T}(F \times G))(a) &= (F \times G) \odot \varepsilon_a \\ &= (F \odot \varepsilon_a) \times (G \odot \varepsilon_a) \quad \text{(by Lemma 2.2(i))} \\ &= \mathbf{T}(F)(a) \times \mathbf{T}(G)(a) \\ &= (\mathbf{T}(F) \times \mathbf{T}(G))(a) \\ &= (\mathbf{T}(F)\mathbf{T}(G))(a).\end{aligned} \tag{3.54}$$

In the final equality, we use the fact that multiplication in $m(S)$ is a point-wise multiplication, see Remark 3.2 of Theorem 3.1.

(v) Let $a \in S$ and $F \in M(S)^*$, then

$$\begin{aligned}l_b(\mathbf{T}(F))(a) &= (\mathbf{T}(F))(ba) = F \odot \varepsilon_{ba} \\ &= F(\varepsilon_b * \varepsilon_a) = (\varepsilon_b \odot F)(\varepsilon_a) \\ &= (\mathbf{T}(\varepsilon_b \odot F))(a).\end{aligned} \tag{3.55}$$

So, $l_b(\mathbf{T}(F)) = \mathbf{T}(\varepsilon_b \odot F)$. □

Remark 3.9. From Theorem 3.8, it follows that the map $\mathbf{T}^* : m(S)^* \rightarrow M(S)^{**}$ carries means to means, multiplicative means to multiplicative means, left invariant means to left invariant means, and multiplicative left invariant means to multiplicative left invariant means. But $\mathbf{T}^*$ does not carry a type of means in $m(S)^*$ onto the same type of means in $M(S)^{**}$. Indeed, if M is a multiplicative topological left invariant mean which is not weak* limit of all convergent nets of Dirac measures in $M(S)^*$, then M does not belong to $\mathbf{T}^*(\mathcal{M})$, where $\mathcal{M}$ is the set of all means on $m(S)$.

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