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Research Article

The Attractors of the Common Differential Operator Are Determined by Hyperbolic and Lacunary Functions

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For analytic functions, we investigate the limit behavior of the sequence of their derivatives by means of Taylor series, the attractors are characterized by ω-limit sets. We describe four different classes of functions, with empty, finite, countable, and uncountable attractors. The paper reveals that Erdelyiés *hyperbolic functions of higher order* and *lacunary functions* play an important role for orderly or chaotic behavior. Examples are given for the sake of confirmation.

1. Historical Remarks

In 1952, MacLane [1] presented a strongly pioneering article, which studied sequences of derivatives for holomorphic functions and their limit behavior. He acted with sequences in a function space, generated by the common differential operator. When describing convergent and periodic behaviors, he found functions which Erdélyi et al. [2] have called *hyperbolic functions of higher order*. Besides he constructed a function whose limit behavior nowadays is called chaotic.

Lacunary functions, that is, *Lücken-functions* have been studied already by Hadamard 1892, he proved his Lacuna-theorem, see [3].

Li and Yorke [4] introduced the idea of chaos in the theory of dynamical systems 1975; they described periodic and chaotic behaviors of orbits in finite-dimensional systems. In 1978, Marotto [5] introduced snap-back repellers, the so-called *homocline orbits*, to enrich dynamics by a sufficient criterion for chaos. In 1989, Devaney gave a topological characterization of chaos by introducing *sensitivy*, *transitivy*, and the notion *dense periodical points*.

Parallel to these, in operator theory, a lot of investigations concerning iterated linear operators appeared. In 1986, Beauzamy characterized *hypercyclic operators* by a property very near to the definition of homocline orbits. In 1991, Godefroy and Shapiro [6] connected these two lines. Based on results of Rolewicz [7], they proved that common integral- and differential operators are hypercyclic. A widespread research activity followed. A quite good survey on the theory of hypercyclic operators has been given in 1999 by Grosse-Erdmann [8] and too in the conference report of the Congress of Mathematics in Zaragoza 2007, see [9].

In 1999, respectively, 2000 the author of the present article verified the chaos properties of Devaney and of Li and Yorke for the common differential operator, see [10, 11].

This paper continues the article of MacLane [1]. It gives more insight into the limit behavior of sequences of derivatives characterizing them by convergence properties of their Taylor coefficients. It gives predictions for their attractors, describing these by means of the concept *Omega-limit sets*, see Alligood et al. [12].

For our investigations, we choose the supremum norm, although in the topology of that norm the differential operator is discontinuous. By this, we can prove convergence properties very easily. Moreover, we focus our attention only on the cardinality of the Omega-limit sets.

2. Introduction

We investigate the dynamical system generated by the common differential operator D which maps a function f to its first derivative f'. Let its domain be the function space $\mathcal{A}$ of all functions which are analytic on the complex unit disc $E := \{x \in \mathbb{C} : |x| \leq 1\}$. An analytic function f means in complex analysis that the Taylor series of f exists and is absolutely convergent for all $x \in E$. Thus, all derivatives of f are contained in $\mathcal{A}$ too. They are continuous and differentiable. For $f \in \mathcal{A}$ and $D : \mathcal{A} \to \mathcal{A}$, $D(f) = f'$, we consider the sequence of functions

$$f^{(0)} := f, \qquad f^{(n+1)} := D\left(f^{(n)}\right) \quad \text{for } n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}. \tag{2.1}$$

Hence, we have for $f \in \mathcal{A}$, $f(x) = \sum_{i=0}^{\infty} a_i (x^i / i!)$, $a_i \in \mathbb{C}$, the relation

$$f^{(n)}(x) = \sum_{i=0}^{\infty} a_{n+i} \frac{x^i}{i!} \tag{2.2}$$

holds. The sequence $(a_i)_{i \in \mathbb{N}_0}$ of coefficients of the Taylor series $\sum_{i=0}^{\infty} a_i (x^i / i!)$ we call *Taylor sequence*. Equipped with the supremum norm $\|f\| := \max_{x \in E}\{|f(x)|\}$ the set $(\mathcal{A}, \|\cdot\|)$ is a normed linear space, and in the topology of this norm, (2.1) is a regular dynamical system with the linear operator D. For each function $f \in \mathcal{A}$, there is an orbit $(f^{(n)})_{n \in \mathbb{N}_0}$ of this dynamical system (2.1).

Due to results in [6, 10], we conclude that the common differential operator D is chaotic in the sense of Devaney [13], and from [11] in the sense of Li and Yorke [4].

3. Hyperbolic functions

For the reason of self-containedness, we inform on hyperbolic functions. Exponential functions of the type

$$e_\alpha : E \longrightarrow \mathbb{C}, \quad e_\alpha(x) := \alpha e^x, \quad \alpha \in \mathbb{C} \tag{3.1}$$

are fixed points or fixed elements of the dynamical system (2.1), whereas the so-called *hyperbolic functions of order* n are periodic elements of system (2.1).

With $n \in \mathbb{N}$, $i := \sqrt{-1}$, $\omega := e^{(2\pi/n)i}$ Erdélyi et al. defined in [2] $H_n : \mathbb{C} \to \mathbb{C}$

$$H_n(x) := \frac{1}{n}\sum_{\nu=1}^{n} e^{\omega^\nu x} = \frac{1}{n}\left(e^{\omega x} + e^{\omega^2 x} + e^{\omega^3 x} + \cdots + e^{\omega^n x}\right) \tag{3.2}$$

$$= \sum_{\nu=0}^{\infty} \frac{x^{\nu n}}{(\nu n)!} = 1 + \frac{x^n}{(n)!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \cdots. \tag{3.3}$$

For the real part h_n of H_n, we know from [14] that $h_n := \mathrm{Re}(H_n) : \mathbb{R} \to \mathbb{R}$ (for real x), using the abbreviation $\alpha_\nu := (2\pi/n)\nu$,

$$h_n(x) := \frac{1}{n}\sum_{\nu=0}^{n-1} e^{x\cos\alpha_\nu} \cdot \cos(x \sin\alpha_\nu) \tag{3.4}$$

$$= \sum_{i=0}^{\infty} \frac{x^{\nu n}}{(\nu n)!} = 1 + \frac{x^n}{(n)!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \cdots. \tag{3.5}$$

The Taylor series (3.3) and (3.5) reveal that hyperbolic functions coincide with their nth derivative, that is,

$$H_n^{(n)} = H_n, \qquad h_n^{(n)} = h_n. \tag{3.6}$$

With (3.1), (3.2), and (3.4), we find that

$$H_1 = e_1, \qquad H_2 = \cosh, \qquad H_4 = \frac{1}{2}(\cosh + \cos) \tag{3.7}$$

and, see [14],

$$\begin{aligned} h_3(x) &= \frac{1}{3}\left(e^x + 2e^{-(1/2)x}\cos\left(\frac{\sqrt{3}}{2}x\right)\right), \\ h_5(x) &= \frac{1}{5}\left(e^x + 2e^{x\cos(2\pi/5)}\cos\left(x\sin\frac{2\pi}{5}\right) + 2e^{x\cos(4\pi/5)}\cos\left(x\sin\frac{4\pi}{5}\right)\right), \\ h_6(x) &= \frac{1}{3}\left(\cosh x + 2\cos\left(\frac{\sqrt{3}}{2}x\right)\cosh\left(\frac{1}{2}x\right)\right). \end{aligned} \tag{3.8}$$

We should note that we consider the function H_n for complex x, and h_n for real x. It is easy to prove.

Proposition 3.1. *The statements (A) and (B) are equivalent.*

(A) $p \in \mathcal{A}$ *is a linear combination of* H_n *and* $H_n^{(1)}, H_n^{(2)}, \ldots, H_n^{(n-1)}$:

$$p = \sum_{\nu=0}^{n-1} \alpha_\nu H_n^{(\nu)} = \alpha_0 H_n + \alpha_1 H_n^{(1)} + \cdots + \alpha_{n-1} H_n^{(n-1)}, \qquad \alpha_\nu \in \mathbb{C}. \tag{3.9}$$

(B) *The sequence* $(p^{(n)})_{n\in\mathbb{N}_0}$ *is a periodic orbit of the dynamical system* (2.1).

Hence, the orbit of p in (3.9) move in circles planet-like in the function space $\mathcal{A}$.

4. Preliminaries

Let $(X, \|\cdot\|)$ be a normed linear space and $(x_n)_{n\in\mathbb{N}_0}$ a sequence in X.

Definition 4.1 (Li/Yorke-property). One calls $(x_n)_{n\in\mathbb{N}_0}$ an *aperiodic sequence* or a *chaotic orbit* if it is bounded but not asymptotically periodic, that is, for each periodic sequence $(y_n)_{n\in\mathbb{N}_0} \subset X$,one has

$$0 < \limsup_{n\to\infty} \|x_n - y_n\| < \infty. \tag{4.1}$$

Hence, an aperiodic sequence has at least two cluster points.

According to Alligood et al. [12], one defines attractors of an orbit by the *Omega-limit set* $\omega(f)$ of an element $f \in \mathcal{A}$. It contains all cluster points of the orbit $(f^{(n)})_{n\in\mathbb{N}_0}$. Thus, for the functions (3.1) and (3.7),

$$\omega(e_\alpha) = \{e_\alpha\}, \qquad \omega(\cosh) = \{\cosh, \sinh\}, \qquad \omega(\sin) = \{\sin, \cos, -\sin, -\cos\}. \tag{4.2}$$

Proposition 4.2. *The* ω*-operator is linear in the following sense. Let* $f \in \mathcal{A}$ *and* e_α *defined in* (3.1)*. Then*

$$\omega(\alpha f) = \alpha\omega(f), \qquad \omega(f + e_\alpha) = e_\alpha + \omega(f). \tag{4.3}$$

For Taylor sequences $(a_i)_{i\in\mathbb{N}_0}$ of type

$$(\ldots, \alpha, \alpha, \alpha, \ldots, \alpha, \alpha, \alpha, \alpha, \ldots, \alpha, \alpha, \alpha, \alpha, \alpha, \ldots), \tag{4.4}$$

one introduces the concept of *lacuna cluster.*

Definition 4.3. Let the Taylor sequence $(a_i)_{i\in\mathbb{N}_0}$ of $f \in \mathcal{A}$ have the cluster point $\alpha \in \mathbb{C}$, and let $I \subset \mathbb{N}_0$ be the index set defined by $I := \{n_i : a_{n_i} \neq \alpha\}$. Then α is called *lacuna cluster* of $(a_i)_{i\in\mathbb{N}_0}$, if the sequence $(n_{i+1} - n_i)_{n_i\in I}$ is unbounded.

For $\alpha = 0$, this definition coincides with the classical definition of lacunary functions used by Hadamard, Polya, and so on. Thus, an analytic function is lacunary function, if its Taylor sequence has the lacuna cluster 0. Hence, the flutter function $\Phi : E \to \mathbb{C}$, introduced in [11],

$$\Phi(x) := 1 + \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^9}{9!} + \frac{x^{16}}{16!} + \frac{x^{25}}{25!} + \frac{x^{36}}{36!} + \cdots \tag{4.5}$$

is lacunary function, its Taylor sequence has the lacuna cluster 0.

Lacunary functions have been already discussed by Weierstraß and Hadamard; Polya (1939) proved that functions of this type possess no extension to any point on their periphery, see [3]. In recent time, lacunary functions with unbounded Taylor sequence play a role in complex analysis again.

Next, we introduce for each sequence $(a_i)_{i\in\mathbb{N}_0}$ its *cluster sequence* $(\overline{a}_i)_{i\in\mathbb{N}_0}$ by identifying elements of convergent subsequences by their limit point. Note that $(a_i)_{i\in\mathbb{N}_0}$ is bounded, thus cluster points exist (Bolzano-Weierstraß).

Definition 4.4. Let $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ be the set of cluster points of the sequence $(a_i)_{i\in\mathbb{N}_0}$. One constructs inductively a mapping $a_i \to \overline{a_i} \in \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$.

(1) Due to α_0 is cluster point, there is a subsequence $(a_i)_{i\in I_0}$, $I_0 \subset \mathbb{N}_0$, converging to α_0. For $i \in I_0$, we define $\overline{a_i} := \alpha_0$.

(2) If $\mathbb{N}_0 \setminus I_0$ is a finite set, one defines for $i \in \mathbb{N}_0 \setminus I_0$ $\overline{a_i} := \alpha_0$. Otherwise there is a cluster point $\alpha_1 \neq \alpha_0$ and a subset $(a_i)_{i\in I_1}$, $I_1 \subset \mathbb{N}_0 \setminus I_0$, converging to α_1. For $i \in I_1$,one defines $\overline{a_i} := \alpha_1$.

(3) Continuing inductively for $k = 0, 1, 2, 3, \ldots$.

If $\mathbb{N}_0 \setminus \bigcup_{j=0}^{k} I_j$ is a finite set, one defines $\overline{a_i} := \alpha_k$, otherwise there is a cluster point α_{k+1} different from α_j for $j = 0, 1, 2, \ldots, k$, and a subset $(a_i)_{i\in I_{k+1}}$, $I_{k+1} \subset \mathbb{N}_0 \setminus \bigcup_{j=0}^{k} I_j$, converging to α_{k+1}. For $i \in I_{k+1}$,one defines $\overline{a_i} := \alpha_{k+1}$.

Note that the set of indices $\{I_k : k = 0, 1, 2, \ldots\}$ are pairwise disjoint and their union is $\mathbb{N}_0$.

The cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$ reveals the asymptotical behavior of an orbit $(f^{(n)})_{n\in\mathbb{N}_0}$, Property (C) will be very useful.

Proposition 4.5. *For $f \in \mathcal{A}$, $f(x) = \sum_{i=0}^{\infty} a_i (x^i/i!)$ and $\overline{f}(x) := \sum_{i=0}^{\infty} \overline{a}_i (x^i/i!)$,one has*

$$\begin{aligned} &\text{(A)} \quad |a_i - \overline{a}_i| \longrightarrow 0 \quad \textit{for } i \longrightarrow \infty. \\ &\text{(B)} \quad \left\| f^{(n)} - \overline{f}^{(n)} \right\| \longrightarrow 0 \quad \textit{for } n \longrightarrow \infty. \\ &\text{(C)} \quad \omega(f) = \omega(\overline{f}). \end{aligned} \tag{4.6}$$

5. Finite attractors

Like the oracle of Delphi in ancient Greece informed people about their future, our theorems will show that the Taylor sequence $(a_i)_{i\in\mathbb{N}_0}$ predicts the asymptotical behavior of an orbit

$(f^{(n)})_{n\in\mathbb{N}_0}$ for $n \to \infty$. The following theorem deals with empty and finite attractors, it reveals the role of Erdelyi's hyperbolic functions H_n for the attractors $\omega(f)$ of the differential operator.

Theorem 5.1. *Let $f \in \mathcal{A}$, $f(x) = \sum_{i=0}^{\infty} a_i(x^i/i!)$ for $x \in E$.*

(A) *If $(a_i)_{i\in\mathbb{N}_0}$ is unbounded and contains no lacuna cluster, then $(f^{(n)})_{n\in\mathbb{N}_0}$ is unbounded too and $\omega(f)$ is empty.*

(B) *If $(a_i)_{i\in\mathbb{N}_0}$ is convergent to $\alpha \in \mathbb{C}$, then $(f^{(n)})_{n\in\mathbb{N}_0}$ converges to e_α and $\omega(f) = \{e_\alpha\}$.*

(C) *If $(a_i)_{i\in\mathbb{N}_0}$ is asymptotically periodic to $\{\beta_0, \beta_1, \dots, \beta_{n-1}\} \subset \mathbb{C}$, then $(f^{(n)})_{n\in\mathbb{N}_0}$ is asymptotically periodic too, and*

$$\omega(f) = \{p, p^{(1)}, p^{(2)}, \dots, p^{(n-1)}\}, \quad \text{where } p := \sum_{k=0}^{n-1} \beta_k H_n^{(n-k)}. \tag{5.1}$$

We give some examples as follows.

(1) The function $\tan + 1/\cos \in \mathcal{A}$ possess for $|x| < \pi/2$ with the Bernoulli numbers B_ν and the Euler numbers E_ν the Taylor series

$$\begin{aligned} \tan(x) + \frac{1}{\cos(x)} &= \sum_{\nu=1}^{\infty} \frac{4^\nu(4^\nu - 1)}{2\nu} B_\nu \frac{x^{2\nu-1}}{(2\nu-1)!} + 1 + \sum_{\nu=1}^{\infty} E_\nu \frac{x^{2\nu}}{(2\nu)!} \\ &= 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!} + 272\frac{x^7}{7!} + 1385\frac{x^8}{8!} + \cdots. \end{aligned} \tag{5.2}$$

Its Taylor sequence $\{1, 1, 1, 2, 5, 16, 61, 272, 1385, \dots\}$ is unbounded without any cluster point, hence $\omega(\tan + 1/\cos) = \varnothing$.

(2) For each polynomial $q \in \mathcal{A}$, we have $\omega(q) = \{e_0\} = \{0\}$.

(3) Let $f : E \to \mathbb{C}$ defined by

$$f(x) := \begin{cases} 1, & \text{if } x = 0, \\ \sin x\left(1 + \dfrac{1}{x}\right), & \text{else.} \end{cases} \tag{5.3}$$

Then

$$f(x) = 1 + x - \frac{x^2}{3\cdot 2!} - \frac{x^3}{3!} + \frac{x^4}{5\cdot 4!} + \frac{x^5}{5!} - \frac{x^6}{7\cdot 6!} - \frac{x^7}{7!} + \frac{x^8}{9\cdot 8!} + \frac{x^9}{9!} + \cdots. \tag{5.4}$$

Its Taylor sequence $(1, 1, -1/3, -1, 1/5, 1, -1/7, -1, 1/9, 1, -1/11, \dots)$ is asymptotically periodic to the periodic sequence $\{0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \dots\}$. Using (5.1), (3.7), and $\sin = H_4^{(3)} - H_4^{(1)}$, its attractor becomes $\omega(f) = \{\sin, \cos, -\sin, -\cos\}$.

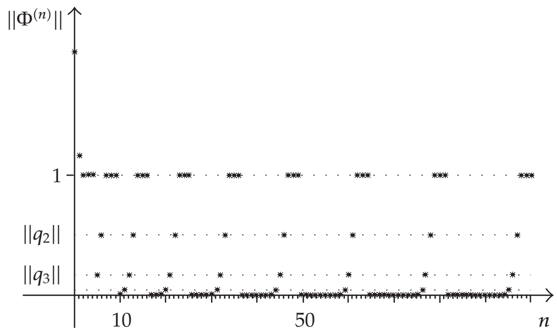

Figure 1: The sequence $(\|\Phi^{(n)}\|)_{n\in\mathbb{N}_0}$.

6. Countable attractors

We now consider chaotic orbits of the differential operator. The next theorem shows that these are characterized by aperiodic Taylor sequences.

Theorem 6.1. *Let $f \in \mathcal{A}$, $f(x) = \sum_{i=0}^{\infty} a_i(x^i/i!)$ for $x \in E$. Then the statements (A) and (B) are equivalent as follows.*

(A) *The Taylor sequence $(a_i)_{i\in\mathbb{N}_0}$ is aperiodic.*

(B) *The sequence of derivatives $(f^{(n)})_{n\in\mathbb{N}_0}$ is a chaotic orbit of the system* (2.1).

Figure 1 presents the sequence $(\|\Phi^{(n)}\|)_{n\in\mathbb{N}_0}$ of the flutter function Φ, see (4.5), graphically. Imagine a chicken that wants to escape the kitchen. It flutters up to a window one meter high, it bumps against the window and crashes down to the bottom. Then it starts the same procedure again, but it has lost energy, so it needs a longer way to flutter up again. There is no periodicity, the time difference between "downs" and "ups" increases. This fluttering upward and crashing down may be seen in Figure 1.

The next theorems reveal the part of lacunary functions and exponential functions e_α for chaotic orbits of the differential operator and its attractor.

Theorem 6.2. *Let $f \in \mathcal{A}$, $f(x) = \sum_{i=0}^{\infty} a_i(x^i/i!)$ for $x \in E$ and $(a_i)_{i\in\mathbb{N}_0}$ aperiodic. Then*

(A) *if $(a_i)_{i\in\mathbb{N}_0}$ possesses only a finite number of cluster points, then the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$ contains at least one lacuna cluster.*

(B) *If $\alpha \in \mathbb{C}$ is a lacuna cluster of the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$, then for the exponential function e_α, one has $e_\alpha \in \omega(f)$.*

We introduce abbreviations splitting the exponential function e_1 into a Taylor polynomial T_n and its remainder R_n:

$$T_n(x) := \sum_{i=0}^{n} \frac{x^i}{i!}, \qquad R_n(x) := \sum_{i=n+1}^{\infty} \frac{x^i}{i!}, \qquad q_n(x) := \frac{x^n}{n!}. \tag{6.1}$$

Thus, $e_1 = T_n + R_n$ for each $n \in \mathbb{N}$. We will use it for constructing a stairway $\beta T_n + \gamma R_n$ between exponential functions e_β and e_γ in the function space $\mathcal{A}$.

Theorem 6.3. *Let $f \in \mathcal{A}$, $f(x) = \sum_{i=0}^{\infty} a_i(x^i/i!)$ for $x \in E$ and the Taylor sequence $(a_i)_{i\in\mathbb{N}_0}$ be aperiodic. Then*

(A) *for each lacuna cluster β in the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$, infinitely often followed by a lacuna cluster $\gamma \neq \beta$, one has with $U_n := \beta T_n + \gamma R_n$,*

$$\{e_\beta, e_\gamma\} \cup \{U_n : n \in \mathbb{N}_0\} \subset \omega(f). \tag{6.2}$$

(B) *For each tupel $(b_0, b_1, \ldots, b_{k-1}) \subset \mathbb{C}^k$ in the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$, which appears infinitely often between the lacuna clusters β and γ, one has with $S_n := \beta T_{n-1} + \sum_{j=0}^{k-1} b_j q_{n+j} + \gamma R_{n+k-1}$*

$$\{e_\beta, e_\gamma\} \cup \{S_n : n \in \mathbb{N}_0\} \subset \omega(f). \tag{6.3}$$

(C) *If the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$ contains arbitrary many lacuna clusters but only a finite number of nonlacuna clusters, then the attractor $\omega(f)$ is a countably infinite set.*

Example for statement (A)

To define the function $\Lambda \in \mathcal{A}$, we choose $a_i(= \overline{a}_i)$ according to the rule

$$a_i := \begin{cases} 0, & \text{if for } k \in \mathbb{N}_0 : \dfrac{k}{2}(3k+3) \leq i \leq \dfrac{k}{2}(3k+5), \\ 1, & \text{if } \dfrac{k}{2}(3k+5) < i < \dfrac{k+1}{2}(3k+4), \\ 2, & \text{otherwise.} \end{cases} \tag{6.4}$$

Then

$$\begin{aligned} (a_i)_{i\in\mathbb{N}_0} &= (0,1,2,0,0,1,1,2,2,0,0,0,1,1,1,2,2,2,0,0,0,0,1,1,1,1,2,\ldots), \\ \Lambda(x) &:= \frac{x}{1!} + 2\frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^6}{6!} + 2\frac{x^7}{7!} + 2\frac{x^8}{8!} + \frac{x^{12}}{12!} + \frac{x^{13}}{13!} + \frac{x^{14}}{14!} + 2\frac{x^{15}}{15!} + \cdots. \end{aligned} \tag{6.5}$$

We see three lacuna clusters $0, 1, 2$. The attractor of Λ becomes

$$\omega(\Lambda) = \{e_0, e_1, e_2\} \cup \{R_n : n \in \mathbb{N}_0\} \cup \{T_n + 2R_n : n \in \mathbb{N}_0\} \cup \{2T_n : n \in \mathbb{N}_0\}. \tag{6.6}$$

Figure 2 presents the sequence $(\|\Lambda^{(n)}\|)_{n\in\mathbb{N}_0}$ graphically, Figure 3 shows schematically the orbit $(\Lambda^{(n)})_{n\in\mathbb{N}_0}$ and its attractor $\omega(\Lambda)$ in the function space $\mathcal{A}$. In both figures, we see the stairways up from e_0 to e_1 and from e_1 to e_2, and the stairway down from e_2 to e_0.

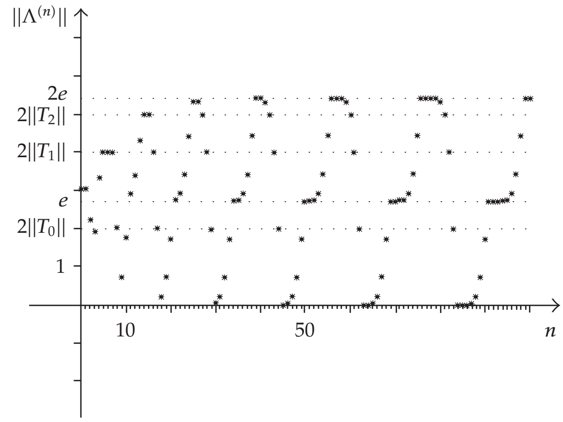

Figure 2: The sequence $(\|\Lambda^{(n)}\|)_{n\in\mathbb{N}_0}$ of the lacunary function Λ.

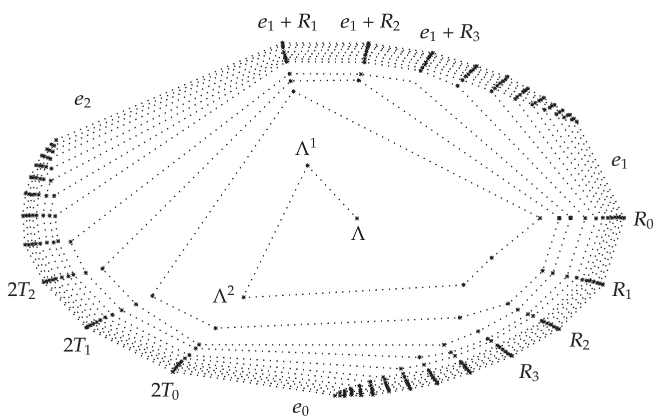

Figure 3: Orbit and ω-limit set of the lacunary function Λ.

Example for statement (B)

Is given by the flutter function Φ defined in (4.5), with $k = 1,\ b_0 = 1,\ \beta = \gamma = 0$. It leads to the attractor of Φ:

$$\omega(\Phi) = \{e_0\} \cup \{q_n : n \in \mathbb{N}_0\}. \tag{6.7}$$

Figure 1 shows the stairway $\{q_n : n \in \mathbb{N}_0\}$.

Example for statement (C)

Is given by the sequence $(1/n)_{n\in\mathbb{N}}$ for the construction of a Taylor sequence with infinitely many lacuna clusters:

$$
\begin{aligned}
(a_n)_{n\in\mathbb{N}_0} := \Bigl(&1,\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},1,1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{4},1,1,1,1,\\
&\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},1,1,1,1,1,\ldots\Bigr).
\end{aligned}
\tag{6.8}
$$

Mathematical research on lacunary functions deals usually with unbounded coefficients. In addition to Theorem 5.1(A), we give an example of a lacunary function with unbounded Taylor sequence, whose ω-limit set is nonempty. For $f : E \to \mathbb{C}$, given by

$$
\begin{aligned}
f(x) &:= 1+\frac{x}{1!}+1!\frac{x^4}{4!}+3!\frac{x^9}{9!}+5!\frac{x^{16}}{16!}+7!\frac{x^{25}}{25!}+9!\frac{x^{36}}{36!}+\cdots\\
&= 1+x+\sum_{i=2}^{\infty}(2i-3)!\frac{x^{i^2}}{(i^2)!},
\end{aligned}
\tag{6.9}
$$

we show $e_0 \in \omega(f)$. The k^2th derivative of f is

$$
f^{(k^2)}(x) = \sum_{i=k}^{\infty}(2i-3)!\frac{x^{i^2-k^2}}{(i^2-k^2)!} = (2k-3)!+\sum_{i=k+1}^{\infty}(2i-3)!\frac{x^{i^2-k^2}}{(i^2-k^2)!}.
\tag{6.10}
$$

Because $0 \in E$, we have $\|f^{(k^2)}\| \ge (2k-3)!$, which means that the orbit is unbounded.

We consider its successor $f^{(k^2+1)}$ and $\|f^{(k^2+1)}\|$:

$$
\begin{aligned}
f^{(k^2+1)}(x) &= \sum_{i=k+1}^{\infty}(2i-3)!\frac{x^{i^2-k^2-1}}{(i^2-k^2-1)!}\\
&= (2k-1)!\frac{x^{2k}}{(2k)!}+\sum_{i=k+2}^{\infty}(2i-3)!\frac{x^{i^2-k^2-1}}{(i^2-k^2-1)!}\\
&= \frac{x^{2k}}{2k}+\sum_{i=k+2}^{\infty}\frac{x^{i^2-k^2-1}}{\prod_{j=2i-2}^{i^2-k^2-1} j},\\
\|f^{(k^2+1)}\| &= \frac{1}{2k}+\sum_{i=k+2}^{\infty}\frac{1}{\prod_{j=2i-2}^{i^2-k^2-1} j} < \frac{1}{2k}+\sum_{i=k+2}^{\infty}\frac{1}{(2i-2)^{(i-1)^2-k^2}}.
\end{aligned}
\tag{6.11}
$$

Hence, $\|f^{(k^2+1)}\| \to 0$ with $k \to \infty$. We conclude the exponential function $e_0 \in \omega(f)$.

7. Uncountable attractors

Finally, we demonstrate that not only lacunary functions may have chaotic orbits. We use the Cantor sequence

$$
(c_i)_{i\in\mathbb{N}_0} = \Bigl(\frac{1}{1},\frac{1}{2},\frac{2}{2},\frac{1}{3},\frac{2}{3},\frac{3}{3},\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{4}{4},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{5}{5},\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},\frac{6}{6},\frac{1}{7},\ldots\Bigr)
\tag{7.1}
$$

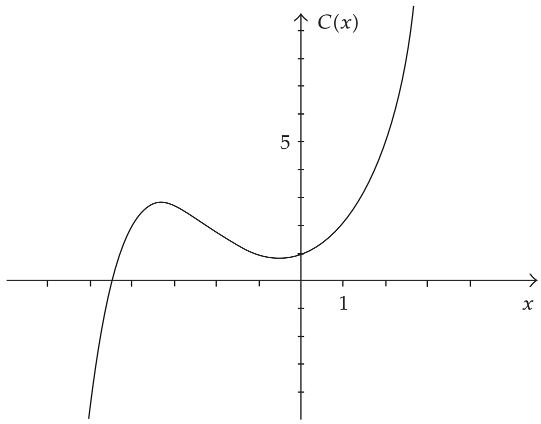

Figure 4: Graph of the Cantor function C for real arguments.

to define an analytic function $C \in \mathscr{A}$. It has countably infinitely many cluster points, but no lacuna cluster:

$$C(x) := \sum_{i=0}^{\infty} c_i \frac{x^i}{i!} = 1 + \frac{1}{2}x + \frac{x^2}{2!} + \frac{1}{3}\frac{x^3}{3!} + \frac{2}{3}\frac{x^4}{4!} + \frac{x^5}{5!} + \frac{1}{4}\frac{x^6}{6!} + \frac{1}{2}\frac{x^7}{7!} + \frac{3}{4}\frac{x^8}{8!} + \cdots . \tag{7.2}$$

With the abbreviation

$$s_n := \frac{n}{2}(n+1), \quad n \in \mathbb{N}_0, \tag{7.3}$$

the elements of the Cantor sequence $(c_i)_{i \in \mathbb{N}_0}$ maybe given by

$$c_i := \frac{i + 1 - s_n}{n+1} \quad \text{for } s_n \le i < s_{n+1}. \tag{7.4}$$

In Figure 4, we see the graph of C for real values. Figure 5 shows the sequence $(\|C^{(n)}\|)_{n \in \mathbb{N}}$ of the orbit of C. It is bounded from below by 0 and from above by the Euler number e. It increases apparently linear in some subintervals, followed by a descent at the values

$$1 = \|T_0\|, \qquad 2 = \|T_1\|, \qquad 2.5 = \|T_2\|, \qquad 2.\overline{6} = \|T_3\|, \qquad 2.708\overline{3} = \|T_4\|, \ldots . \tag{7.5}$$

Figure 6 shows $\omega(C)$ and a subset of the orbit schematically. Like a squirrel runs up a tree, the orbit runs up along the stick $\{e_\alpha : \alpha \in [0;1]\}$. After that the orbit jumps to a Taylor polynomial T_m (the squirrel jumps to a branch), and to another one, lower one T_{m-1}, and then to $T_{m-2}, \ldots, T_0$. Then it starts again to run upward along the stick, with one step more than before, at each circulation it reaches a higher level. It climbs up nearer and nearer to the top of the stick e_1.

We describe the properties of the orbit of C in a theorem, using Taylor polynomials T_n, remainders R_n (6.1), exponential functions (3.1), and s_n (7.3).

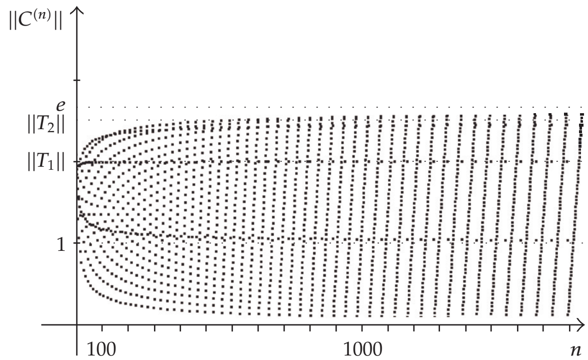

Figure 5: Sequence $(\|C^{(n)}\|)_{n\in\mathbb{N}}$ of the Cantor function C.

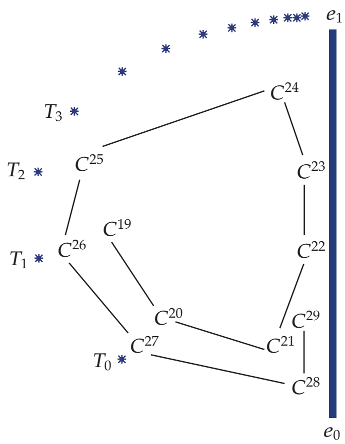

Figure 6: Orbit and ω-limit set of the Cantor function C.

Theorem 7.1. *For the orbit $(C^{(n)})_{n\in\mathbb{N}_0}$ of the Cantor function C,one has*

(A) $\|C^{(s_n)}\| \to 0$ *for* $n \to \infty$;

(B) *for* m *with* $0 \le m < n$: $\|C^{(s_n-m-1)} - T_m\| \to 0$ *for* $n \to \infty$;

(C) *for each* $\alpha \in [0;1]$, *there is a subsequence* $(C^{(n_j)})_{n_j\in I}$, $I \subset \mathbb{N}$, *of the orbit* $(C^{(n)})_{n\in\mathbb{N}}$ *with* $\|C^{(n_j)} - e_\alpha\| \to 0$ *for* $j \to \infty$;

(D) $\limsup_{n\to\infty}\|C^{(n)}\| = e^1$, $\liminf_{n\to\infty}\|C^{(n)}\| = 0$;

(E) $\omega(C) = \{T_n : n \in \mathbb{N}_0\} \cup \{e_\alpha : \alpha \in [0;1]\}$, *containing uncountably many different elements.*

Properties (A), (B), and (D) can be seen in Figure 5, property (E) in Figure 6.

8. Proofs

8.1. Proof of Theorem 5.1

Property (A)

Using (2.2),

$$\|f^{(n)}\| = \max_{x\in E}\left|\sum_{i=0}^{\infty} a_{n+i}\frac{x^i}{i!}\right| = \max_{x\in E}\left|a_n + \sum_{i=1}^{\infty} a_{n+i}\frac{x^i}{i!}\right| \geq |a_n| \tag{8.1}$$

because $0 \in E$. Thus, $(|a_n|)_{n\in\mathbb{N}_0}$ is an unbounded minorizing sequence.

Property (B)

It follows from (C) with $n = 1$ and $\beta_0 = \alpha$.

Property (C)

By assumption the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$ of the Taylor sequence becomes $\overline{a}_i = \beta_{i \bmod n}$. Consider $p \in \mathcal{A}$ defined by

$$\begin{aligned} p(x) &:= \sum_{i=0}^{\infty} \overline{a}_i \frac{x^i}{i!} = \sum_{i=0}^{\infty} \beta_{i \bmod n}\frac{x^i}{i!} = \sum_{k=0}^{n-1} \beta_k \sum_{i=0}^{\infty} \frac{x^{in+k}}{(in+k)!} \\ &= \sum_{k=0}^{n-1} \beta_k H_n^{(n-k)}. \end{aligned} \tag{8.2}$$

Proposition 3.1 implies that the orbit $(p^{(n)})_{n\in\mathbb{N}_0}$ is a periodic orbit. Using (4.6), the orbit $(f^{(n)})_{n\in\mathbb{N}_0}$ is asymptotically periodic to its attractor $\omega(f) = \{p, p^{(1)}, p^{(2)}, \ldots, p^{(n-1)}\}$.

8.2. Proof of Theorem 6.1

Using the Li/Yorke-property (4.1).

(A) $\Rightarrow$ (B) Let $(b_i)_{i\in\mathbb{N}_0} \subset \mathbb{C}$ be a periodic sequence. Then Theorem 5.1(C) implies that the sequence $(p^{(n)})_{n\in\mathbb{N}}$, defined by $p(x) = \sum_{i=0}^{\infty} b_i (x^i/i!)$, is a periodic orbit of system (2.1).

Let $\delta := \limsup_{i\to\infty} |a_i - b_i|$, assuming $\delta > 0$. Using (2.2),

$$\begin{aligned} \|f^{(n)} - p^{(n)}\| &= \max_{x\in E}\left|\sum_{i=0}^{\infty} a_{n+i}\frac{x^i}{i!} - \sum_{i=0}^{\infty} b_{n+i}\frac{x^i}{i!}\right| = \max_{x\in E}\left|\sum_{i=0}^{\infty}(a_{n+i} - b_{n+i})\frac{x^i}{i!}\right| \\ &\geq |a_n - b_n| \end{aligned} \tag{8.3}$$

because $0 \in E$. For infinitely many n, we have $|a_n - b_n| > \delta/2$. Thus,

$$\limsup_{n\to\infty} \|f^{(n)} - p^{(n)}\| \geq \frac{\delta}{2} > 0, \tag{8.4}$$

where $(f^{(n)})_{n\in\mathbb{N}}$ and $(p^{(n)})_{n\in\mathbb{N}}$ are bounded, and hence $(\|f^{(n)} - p^{(n)}\|)_{n\in\mathbb{N}_0}$ is bounded too.

(B) $\Rightarrow$ (A) Let $(p^{(n)})_{n\in\mathbb{N}_0}$ be a periodic orbit of system (2.1). Because $(\|f^{(n)}\|)_{n\in\mathbb{N}_0}$ and $(\|p^{(n)}\|)_{n\in\mathbb{N}_0}$ are bounded, $(\|f^{(n)} - p^{(n)}\|)_{n\in\mathbb{N}_0}$ is bounded too, hence we have the right part of relation (4.1). The left part follows indirectly with Theorem 5.1(C).

8.3. Proof of Theorem 6.2

Statement (A)

It can be proved directly from elementary combinatorics.

Statement (B)

(1) For $\alpha = 0$, let $I \subset \mathbb{N}_0$ be the set of indices, where a block of zeros in the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$ starts. Then for $n \in I$,

$$\overline{a}_i = 0 \quad \text{for } n \leq i < n + k_n,\ k_n \in \mathbb{N}, \qquad k_n \longrightarrow \infty \quad \text{for } n \longrightarrow \infty. \tag{8.5}$$

Using (2.2) and (4.6), we have for $n \in I$,

$$\|\overline{f}^{(n)}\| = \max_{x\in E}\left|\sum_{i=0}^{\infty} \overline{a}_{n+i}\frac{x^i}{i!}\right| = \max_{x\in E}\left|\sum_{i=k_n}^{\infty} \overline{a}_{n+i}\frac{x^i}{i!}\right| \leq \sum_{i=k_n}^{\infty}|\overline{a}_{n+i}|\frac{1}{i!}. \tag{8.6}$$

The Taylor sequence is bounded by $a \in \mathbb{R}^+$. Thus, (8.5) guarantees that it is the case for its cluster sequence too. The latter estimate

$$\leq a\sum_{i=k_n}^{\infty}\frac{1}{i!} \longrightarrow 0 \quad \text{for } n \longrightarrow \infty. \tag{8.7}$$

Hence, $\overline{f}^{(n)} \to e_0$ for $n \to \infty$ and $e_0 \in \omega(\overline{f}) = \omega(f)$.

(2) For $\alpha \neq 0$, the function $g := f - e_\alpha$ is lacunary function. Thus, its Taylor sequence has the lacuna cluster 0. Using (4.3) and the case $\alpha = 0$ above imply

$$e_0 \in \omega(g) = \omega(f - e_\alpha) = -e_\alpha + \omega(f) \Longleftrightarrow e_\alpha + e_0 = e_\alpha \in \omega(f). \tag{8.8}$$

8.4. Proof of Theorem 6.3

Statement (A)

Let $m \in \mathbb{N}$. For $U_m = \beta T_m + \gamma R_m$, we construct a subsequence $(\overline{f}^n)_{n\in I}, I \subset \mathbb{N}_0$, of the orbit $(\overline{f}^n)_{n\in\mathbb{N}_0}$, converging to U_m.

As assumed, the cluster sequence $(\overline{a}_i)_{i\in\mathbb{N}_0}$ of the Taylor sequence is of type

$$(\overline{a}_i)_{i\in\mathbb{N}_0} = (\ldots,\beta,\gamma,\ldots,\beta,\beta,\gamma,\gamma,\ldots,\beta,\beta,\beta,\gamma,\gamma,\gamma,\ldots). \tag{8.9}$$

The last β in a row of β's defines an index $n \in I$, where

$$I := \{n \in \mathbb{N} : \overline{a}_i = \beta \text{ for } n-m \leq i \leq n \text{ and } \overline{a}_i = \gamma \text{ for } n < i \leq n+k \text{ with } k \geq m\}. \tag{8.10}$$

Note that I is an infinite set. Using (2.2), we find for the derivative $\overline{f}^{(n-m)}$,

$$\begin{aligned}\left\|\overline{f}^{(n-m)} - U_m\right\| &= \max_{x\in E}\left|\sum_{i=0}^{\infty} \overline{a}_{n-m+i}\frac{x^i}{i!} - \beta\sum_{i=0}^{m}\frac{x^i}{i!} - \gamma\sum_{i=m+1}^{\infty}\frac{x^i}{i!}\right| \\ &= \max_{x\in E}\left|\sum_{i=n+k+1}^{\infty}(\overline{a}_{n-m+i}-\gamma)\frac{x^i}{i!}\right| \leq \sum_{i=n+k+1}^{\infty}\left|\overline{a}_{n-m+i}-\gamma\right|\frac{1}{i!}.\end{aligned} \tag{8.11}$$

As assumed, the sequence $(a_i)_{i\in\mathbb{N}_0}$ is bounded, thus $(|\overline{a}_i - \gamma|)_{i\in\mathbb{N}_0}$ is bounded by a real number $c \in \mathbb{R}^+$, thus,

$$\left\|\overline{f}^{(n-m)} - U_m\right\| \leq c\sum_{i=n+k+1}^{\infty}\frac{1}{i!} \longrightarrow 0 \quad \text{for } n \longrightarrow \infty,\ n \in I. \tag{8.12}$$

Hence, $\overline{f}^{(n-m)} \to U_m$ for $n \to \infty, n \in I$ and $U_m \in \omega(\overline{f}) = \omega(f)$.

Statement (B)

Let $m \in \mathbb{N}$. For S_m, we construct a subsequence $(\overline{f}^{(n)})_{n\in I}$, $I \subset \mathbb{N}_0$, of the orbit $(\overline{f}^{(n)})_{n\in\mathbb{N}_0}$, converging to S_m. At index n starts a row of β's, at index $n+m$ the tupel, and at $n+m+k$ a row of γ's, which has its end at index $n+m+k+M_n-1$. We define the set I by

$$\begin{aligned} I := \{n \in \mathbb{N}_0 : \ &\overline{a}_{n+i} = \beta \text{ for } 0 \leq i < m,\ \ \overline{a}_{n+i} = b_{i-m} \text{ for } m \leq i < m+k, \\ &\overline{a}_{n+i} = \gamma \text{ for } m+k \leq i < m+k+M_n, \\ &\overline{a}_{n+i} \neq \gamma \text{ for } i = m+k+M_n\}, \end{aligned} \tag{8.13}$$

where I contains infinitely many elements. Because γ is a lacuna cluster, we have

$$M_n \longrightarrow \infty \quad \text{for } n \longrightarrow \infty. \tag{8.14}$$

For $n \in I$, we consider the nth derivative $\overline{f}^{(n)}$, using (2.2) and the abbreviation $M := M_n$,

$$\begin{aligned}
\overline{f}^{(n)} &= \sum_{i=0}^{\infty} \overline{a}_{n+i} q_i \\
&= \beta \sum_{i=0}^{m-1} q_i + \sum_{i=m}^{m+k-1} b_{i-m} q_i + \gamma \sum_{i=m+k}^{m+k+M-1} q_i + \sum_{i=m+k+M}^{\infty} \overline{a}_{n+i} q_i \\
&= \beta T_{m-1} + \sum_{j=0}^{k-1} b_j q_{m+j} + \gamma (R_{m+k-1} - R_{m+k+M-1}) + \sum_{i=m+k+M}^{\infty} \overline{a}_{n+i} q_i.
\end{aligned} \tag{8.15}$$

To S_m, it has the distance

$$\begin{aligned}
\left\| \overline{f}^{(n)} - S_m \right\| &= \left\| \sum_{i=m+k+M}^{\infty} \overline{a}_{n+i} q_i - \gamma R_{m+k+M-1} \right\| \\
&= \left\| \sum_{i=m+k+M}^{\infty} (\overline{a}_{n+i} - \gamma) q_i \right\| \leq \sum_{i=m+k+M}^{\infty} |\overline{a}_{n+i} - \gamma| \| q_i \| = \sum_{i=m+k+M}^{\infty} |\overline{a}_{n+i} - \gamma| \frac{1}{i!}.
\end{aligned} \tag{8.16}$$

The sequence $(|\overline{a}_i - \gamma|)_{i \in \mathbb{N}}$ is bounded by $c \in \mathbb{R}^+$, thus the latter term

$$\leq c \sum_{i=m+k+M}^{\infty} \frac{1}{i!}. \tag{8.17}$$

Because of (8.14), the sum converges to 0 for $n \to \infty$. Thus,

$$\left\| \overline{f}^{(n)} - S_m \right\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty, \ n \in I, \ S_m \in \omega(\overline{f}) = \omega(f). \tag{8.18}$$

Statement (C)

If there is only one lacuna cluster in the cluster sequence, then statement (B) implies with $\beta = \gamma$ that the attractor is countably infinite.

If there are infinitely many lacuna clusters $\alpha_1, \alpha_2, \alpha_3, \ldots$ in the cluster sequence, then statement (A) implies for each couple α_j, α_{j+1} countably infinitely many elements of the attractor. Using *countable* × *countable* = *countable*, we conclude (C).

8.5. Proof of Theorem 7.1

From (7.4), we find for $k = 1, 2, \ldots, n$

$$c_{s_n} = \frac{1}{n+1}, \qquad c_{s_n+k} = \frac{k+1}{n+1}, \qquad c_{s_n-k} = \frac{n+1-k}{n}. \tag{8.19}$$

Statement (A)

Using (8.19) and $0 < c_i \le 1$, we have for sufficiently large n,

$$\begin{aligned}\left\|C^{(s_n)}\right\| &= \left\|\sum_{i=0}^{n}\frac{i+1}{n+1}\frac{x^i}{i!} + \sum_{i=n+1}^{\infty} c_{s_n+i}\frac{x^i}{i!}\right\| \\ &\le \frac{1}{n+1}\left\|\sum_{i=0}^{n}(i+1)\frac{x^i}{i!}\right\| + \left\|\sum_{i=n+1}^{\infty} c_{s_n+i}\frac{x^i}{i!}\right\| \\ &< \frac{1}{n+1}\sum_{i=0}^{\infty}(i+1)\frac{1}{i!} + \left\|\sum_{i=n+1}^{\infty}\frac{x^i}{i!}\right\| \\ &= \frac{2e^1}{n+1} + \|R_n\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty.\end{aligned} \tag{8.20}$$

Statement (B)

Let $n, m \in \mathbb{N}_0,\ m < n$. For T_m, see (6.1) and (7.5), using (2.2) and (8.19), we have

$$\begin{aligned}C^{(s_n-m-1)}(x) &= \sum_{i=0}^{\infty} c_{s_n-m-1+i}\frac{x^i}{i!} \\ &= \sum_{i=0}^{m}\frac{n-m+i}{n}\frac{x^i}{i!} + \sum_{i=m+1}^{n+m+1}\frac{i-m}{n+1}\frac{x^i}{i!} + \sum_{i=n+m+2}^{\infty} c_{s_n-m-1+i}\frac{x^i}{i!} \\ &= T_m(x) + \frac{1}{n}\sum_{i=0}^{m}(i-m)\frac{x^i}{i!} + \frac{1}{n+1}\sum_{i=m+1}^{n+m+1}(i-m)\frac{x^i}{i!} + \sum_{i=n+m+2}^{\infty} c_{s_n-m-1+i}\frac{x^i}{i!}.\end{aligned} \tag{8.21}$$

This leads to

$$\begin{aligned}\left\|C^{(s_n-m-1)} - T_m\right\| &\le \frac{1}{n}\left\|\sum_{i=0}^{m}(i-m)\frac{x^i}{i!}\right\| + \frac{1}{n+1}\left\|\sum_{i=m+1}^{n+m+1}(i-m)\frac{x^i}{i!}\right\| + \left\|\sum_{i=n+m+2}^{\infty} c_{s_n-m-1+i}\frac{x^i}{i!}\right\| \\ &\le \frac{1}{n}\sum_{i=0}^{m}\frac{|i-m|}{i!} + \frac{1}{n+1}\sum_{i=m+1}^{n+m+1}\frac{|i-m|}{i!} + \|R_{n+m+1}\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty.\end{aligned} \tag{8.22}$$

Statement (C)

Statement (A) implies the statement is valid for $\alpha = 0$. From statement (B), we conclude $T_m \in \omega(C)$ for each $m \in \mathbb{N}_0$. Because of $\lim_{m\to\infty} T_m = e_1$ and Theorem 5.1, we find $e_1 \in \omega(C)$. Thus, the statement is true for $\alpha = 1$.

Let $0 < \alpha < 1$ and $\varepsilon > 0$. Due to the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we find a rational number $p/q \in \mathbb{Q}$, $p < q$, and $|p/q - \alpha| < \varepsilon/3e$.

For $\varepsilon > 0$, we choose $k \in \mathbb{N}$ such that

$$\frac{e}{k} < \frac{\varepsilon}{3}, \qquad \|R_k\| < \frac{\varepsilon}{3}. \tag{8.23}$$

We define $m := kp - 1$ and $n := kq - 1$.

This leads to $n - m = k(q - p) \geq k$ and

$$\|R_{n-m}\| \leq \|R_k\| < \frac{\varepsilon}{3}. \tag{8.24}$$

Furthermore, we have

$$\left|\frac{m+1}{n+1} - \alpha\right| = \left|\frac{p}{q} - \alpha\right| < \frac{\varepsilon}{3e}. \tag{8.25}$$

Using (2.2), (8.19), (8.23), (8.24), (8.25), we deduce

$$\begin{aligned}
\left\|C^{(s_n+m)} - e_\alpha\right\| &= \left\|\sum_{i=0}^{\infty} c_{s_n+m+i}\frac{x^i}{i!} - \alpha\sum_{i=0}^{\infty}\frac{x^i}{i!}\right\| \\
&= \left\|\sum_{i=0}^{n-m}\frac{m+i+1}{n+1}\frac{x^i}{i!} - \alpha\sum_{i=0}^{n-m}\frac{x^i}{i!} + \sum_{i=n-m+1}^{\infty} c_{s_n+m+i}\frac{x^i}{i!} - \alpha\sum_{i=n-m+1}^{\infty}\frac{x^i}{i!}\right\| \\
&= \left\|\sum_{i=0}^{n-m}\left(\frac{m+1}{n+1} - \alpha\right)\frac{x^i}{i!} + \sum_{i=0}^{n-m}\frac{i}{n+1}\frac{x^i}{i!} + \sum_{i=n-m+1}^{\infty}\left(c_{s_n+m+i} - \alpha\right)\frac{x^i}{i!}\right\| \\
&\leq \sum_{i=0}^{n-m}\left|\frac{m+1}{n+1} - \alpha\right|\frac{1}{i!} + \sum_{i=1}^{n-m}\frac{1}{n+1}\frac{1}{(i-1)!} + \sum_{i=n-m+1}^{\infty}\left|c_{s_n+m+i} - \alpha\right|\frac{1}{i!} \\
&< \frac{\varepsilon}{3e}\sum_{i=0}^{n-m}\frac{1}{i!} + \frac{1}{kq}\sum_{i=1}^{n-m}\frac{1}{(i-1)!} + \sum_{i=n-m+1}^{\infty}\frac{1}{i!} < \frac{\varepsilon}{3e}e^1 + \frac{1}{kq}e^1 + \|R_{n-m}\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned} \tag{8.26}$$

Thus, we have proved statement (C). Statements (D) and (E) follow by using (A), (B), and (C).

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