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Research Article

Elliptic Equations in Weighted Sobolev Spaces on Unbounded Domains

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We study in this paper a class of second-order linear elliptic equations in weighted Sobolev spaces on unbounded domains of $\mathbb{R}^n$, $n \geq 3$. We obtain an a priori bound, and a regularity result from which we deduce a uniqueness theorem.

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1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 3$. Assign in Ω the uniformly elliptic second-order linear differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a, \tag{1.1}$$

with coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega), i, j = 1, \ldots, n,$ and consider the associate Dirichlet problem:

$$\begin{aligned} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f, \qquad f \in L^p(\Omega), \end{aligned} \tag{1.2}$$

where $p \in]1, +\infty[$.

It is well known that if Ω is a bounded and sufficiently regular set, the above problem has been widely investigated by several authors under various hypotheses of discontinuity on the leading coefficients, in the case $p = 2$ or p sufficiently close to 2. In particular, some $W^{2,p}$-bounds for the solutions of the problem (1.2) and related existence and uniqueness results have been obtained. Among the other results on this subject, we quote here those proved in [1], where the author assumed that a_{ij}'s belong to $W^{1,n}(\Omega)$ (and considered the case $p = 2$) and in [2–4] (where the coefficients belong to some classes wider than $W^{1,n}(\Omega)$). More recently, a relevant contribution has been given in [5–8], where the coefficients a_{ij} are assumed to be in the class VMO and $p \in]1, +\infty[$; observe here that VMO contains the space $W^{1,n}(\Omega)$.

If the set Ω is unbounded and regular enough, under assumptions similar to those required in [1], problem (1.2) has for instance been studied in [9–11] with $p = 2$, and in [12] with $p \in]1, +\infty[$. Instead, in [13, 14] the leading coefficients satisfy restrictions similar to those in [5, 6].

In this paper, we extend some results of [13, 14] to a weighted case. More precisely, we denote by ρ a weight function belonging to a suitable class such that

$$\inf_{\Omega} \rho > 0, \qquad \lim_{|x| \to +\infty} \rho(x) = +\infty, \tag{1.3}$$

and consider the Dirichlet problem:

$$\begin{gathered} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega), \\ Lu = f, \quad f \in L_s^p(\Omega), \end{gathered} \tag{1.4}$$

where $s \in \mathbb{R}$, $W_s^{2,p}(\Omega)$, $\overset{\circ}{W}{}_s^{1,p}(\Omega)$, and $L_s^p(\Omega)$ are some weighted Sobolev spaces and the weight functions are a suitable power of ρ. We obtain an a priori bound for the solutions of (1.4). Moreover, we state a regularity result that allows us to deduce a uniqueness theorem for the problem (1.4). A similar weighted case was studied in [15] with the leading coefficients satisfying hypotheses of Miranda's type and when $p = 2$.

2. Weight functions and weighted spaces

Let G be any Lebesgue measurable subset of $\mathbb{R}^n$ and let $\Sigma(G)$ be the collection of all Lebesgue measurable subsets of G. If $F \in \Sigma(G)$, denote by $|F|$ the Lebesgue measure of F, by χ_F the characteristic function of F, by $F(x, r)$ the intersection $F \cap B(x, r)$ ($x \in \mathbb{R}^n$, $r \in \mathbb{R}_+$)—where $B(x, r)$ is the open ball of radius r centered at x—and by $\mathfrak{D}(F)$ the class of restrictions to F of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ with $\overline{F} \cap \operatorname{supp} \zeta \subseteq F$. Moreover, if $X(F)$ is a space of functions defined on F, we denote by $X_{\mathrm{loc}}(F)$ the class of all functions $g : F \to \mathbb{R}$, such that $\zeta g \in X(F)$ for any $\zeta \in \mathfrak{D}(F)$.

We introduce a class of weight functions defined on an open subset Ω of $\mathbb{R}^n$. Denote by $\mathcal{A}(\Omega)$ the set of all measurable functions $\rho : \Omega \to \mathbb{R}_+$, such that

$$\gamma^{-1} \rho(y) \le \rho(x) \le \gamma \rho(y) \quad \forall y \in \Omega,\ \forall x \in \Omega\big(y, \rho(y)\big), \tag{2.1}$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y. Examples of functions in $\mathcal{A}(\Omega)$ are the function

$$x \in \mathbb{R}^n \longrightarrow 1 + a|x|, \quad a \in]0, 1[, \tag{2.2}$$

and, if $\Omega \neq \mathbb{R}^n$ and S is a nonempty subset of $\partial\Omega$, the function

$$x \in \Omega \longrightarrow a \operatorname{dist}(x, S), \quad a \in]0, 1[. \tag{2.3}$$

For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \{z \in \partial\Omega : \lim_{x\to z} \rho(x) = 0\}. \tag{2.4}$$

It is known that

$$\rho \in L^\infty_{\text{loc}}(\overline{\Omega}), \qquad \rho^{-1} \in L^\infty_{\text{loc}}(\overline{\Omega} \setminus S_\rho) \tag{2.5}$$

(see [16, 17]).

We assign an unbounded open subset Ω of $\mathbb{R}^n$.

Let ρ_1 be a function, such that $\rho_1 \in \mathcal{A}(\mathbb{R}^n)$ and

$$\inf_\Omega \rho_1 > 0, \qquad \lim_{|x|\to+\infty} \rho_1(x) = +\infty. \tag{2.6}$$

We put

$$\rho = \rho_{1|\Omega}. \tag{2.7}$$

For any $a \in]0,1]$ and $x \in \mathbb{R}^n$, we set

$$I_a(x) = \Omega(x, a\rho_1(x)). \tag{2.8}$$

If $k \in \mathbb{N}_0$, $1 \le p < +\infty$, $s \in \mathbb{R}$, and $\rho \in \mathcal{A}(\Omega)$, consider the space $W^{k,p}_s(\Omega)$ of distributions u on Ω, such that $\rho^s \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$, equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha|\le k} \|\rho^s \partial^\alpha u\|_{L^p(\Omega)}. \tag{2.9}$$

Moreover, denote by $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$. A more detailed account of properties of the above defined spaces can be found, for instance, in [18].

From [15, Lemmas 1.1 and 2.1], we deduce the following two lemmas, respectively.

Lemma 2.1. *For any $p \in [1,+\infty[$, $s \in \mathbb{R}$, and $a \in]0,1]$, $g \in L^p_s(\Omega)$ if and only if $g \in L^p_{loc}(\overline{\Omega})$ and the function $x \in \mathbb{R}^n \to \rho_1^{s-n/p}(x)\|g\|_{L^p(I_a(x))}$ belongs to $L^p(\mathbb{R}^n)$. Moreover, there exist $c_1, c_2 \in \mathbb{R}_+$, such that*

$$c_1\|g\|^p_{L^p_s(\Omega)} \le \int_{\mathbb{R}^n} \rho_1^{sp-n}(x)\|g\|^p_{L^p(I_a(x))}dx \le c_2\|g\|^p_{L^p_s(\Omega)} \quad \forall g \in L^p_s(\Omega), \tag{2.10}$$

where c_1 and c_2 depend on n, p, s, a, and ρ.

Lemma 2.2. *If Ω has the segment property, then for any $k \in \mathbb{N}_0$, $p \in [1,+\infty[$, and $s \in \mathbb{R}$ one has*

$$W^{k,p}_s(\Omega) \cap \overset{\circ}{W}{}^{k,p}_{loc}(\overline{\Omega}) = \overset{\circ}{W}{}^{k,p}_s(\Omega). \tag{2.11}$$

3. Some embedding lemmas

We now recall the definitions of the function spaces in which the coefficients of the operator will be chosen. If Ω has the property

$$\left|\Omega(x,r)\right| \geq Ar^n \quad \forall x \in \Omega,\ \forall r \in]0,1], \tag{3.1}$$

where A is a positive constant independent of x and r, it is possible to consider the space $\mathrm{BMO}(\Omega,\tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\mathrm{loc}}(\overline{\Omega})$ such that

$$[g]_{\mathrm{BMO}\,(\Omega,\tau)} = \sup_{\substack{x\in\Omega \\ r\in]0,\tau]}} ⨍_{\Omega(x,r)} \left| g - ⨍_{\Omega(x,r)} g \right| < +\infty, \tag{3.2}$$

where

$$⨍_{\Omega(x,r)} g = \left|\Omega(x,r)\right|^{-1} \int_{\Omega(x,r)} g. \tag{3.3}$$

If $g \in \mathrm{BMO}(\Omega) = \mathrm{BMO}(\Omega,\tau_A)$, where

$$\tau_A = \sup\left\{ \tau \in \mathbb{R}_+ \ :\ \sup_{\substack{x\in\Omega \\ r\in]0,\tau]}} \frac{r^n}{\left|\Omega(x,r)\right|} \leq \frac{1}{A} \right\}, \tag{3.4}$$

we will say that $g \in \mathrm{VMO}(\Omega)$ if $[g]_{\mathrm{BMO}(\Omega,\tau)} \to 0$ for $\tau \to 0^+$. A function $\eta[g] :]0,1] \to \mathbb{R}_+$ is called a *modulus of continuity* of g in $\mathrm{VMO}(\Omega)$ if

$$[g]_{\mathrm{BMO}\,(\Omega,\tau)} \leq \eta[g](\tau) \quad \forall \tau \in]0,1],\ \lim_{\tau\to 0^+} \eta[g](\tau) = 0. \tag{3.5}$$

For $t \in [1,+\infty[$ and $\lambda \in [0,n[$, we denote by $M^{t,\lambda}(\Omega)$ the set of all functions g in $L^t_{\mathrm{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{t,\lambda}(\Omega)} = \sup_{\substack{r\in]0,1] \\ x\in\Omega}} r^{-\lambda/t} \|g\|_{L^t(\Omega(x,r))} < +\infty, \tag{3.6}$$

endowed with the norm defined by (3.6). Then, we define $M^{t,\lambda}_\circ(\Omega)$ as the closure of $C^\infty_\circ(\Omega)$ in $M^{t,\lambda}(\Omega)$. In particular, we put $M^t(\Omega) = M^{t,0}(\Omega)$, and $M^t_\circ(\Omega) = M^{t,0}_\circ(\Omega)$. In order to define the modulus of continuity of a function g in $M^{t,\lambda}_\circ(\Omega)$, recall first that for a function $g \in M^{t,\lambda}(\Omega)$ the following characterization holds:

$$g \in M^{t,\lambda}_\circ(\Omega) \Longleftrightarrow \lim_{\tau\to 0^+} \left(p_g(\tau) + \|(1-\zeta_{1/\tau}) g\|_{M^{t,\lambda}(\Omega)} \right) = 0, \tag{3.7}$$

where

$$p_g(\tau) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_{x\in\Omega}|E(x,1)|\leq\tau}} \left\|\chi_E g\right\|_{M^{t,\lambda}(\Omega)}, \quad \tau \in \mathbb{R}_+, \tag{3.8}$$

and ζ_r, $r \in \mathbb{R}_+$, is a function in $C_\circ^\infty(\mathbb{R}^n)$ such that

$$0 \le \zeta_r \le 1, \qquad \zeta_{r|B_r} = 1, \qquad \operatorname{supp}\zeta_r \subset B_{2r}, \tag{3.9}$$

with the position $B_r = B(0,r)$. Thus, the *modulus of continuity* of $g \in M_\circ^{t,\lambda}(\Omega)$ is a function

$$\sigma_\circ[g] : \,]0,1] \longrightarrow \mathbb{R}_+, \tag{3.10}$$

such that

$$p_g(\tau) + \left\|(1-\zeta_{1/\tau})g\right\|_{M^{t,\lambda}(\Omega)} \le \sigma_\circ[g](\tau) \quad \forall \tau \in]0,1], \; \lim_{\tau\to 0^+}\sigma_\circ[g](\tau) = 0. \tag{3.11}$$

A more detailed account of properties of the above defined function spaces can be found in [9, 19, 20].

We consider the following condition:

(h_0) Ω has the cone property, $p \in]1,+\infty[$, $s \in \mathbb{R}$, k, h, t are numbers such that

$$k \in \mathbb{N}, \; h \in \{0,1,\ldots,k-1\}, \; t \ge p, \; t > p \quad \text{if } p = \frac{n}{k-h}, \; g \in M^t(\Omega). \tag{3.12}$$

From [21, Theorem 3.1] we have the following.

Lemma 3.1. *If the assumption (h_0) holds, then for any $u \in W_s^{k,p}(\Omega)$ one has $g\partial^h u \in L_s^p(\Omega)$ and*

$$\left\|g\partial^h u\right\|_{L_s^p(\Omega)} \le c\|g\|_{M^t(\Omega)}\|u\|_{W_s^{k,p}(\Omega)}, \tag{3.13}$$

with c dependent only on Ω, n, k, h, p, and t.

From [21, Theorem 3.2] it follows Lemma 3.2.

Lemma 3.2. *If the assumption (h_0) is satisfied and in addition $g \in M_\circ^t(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ there exist a constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open set $\Omega_\varepsilon \subset\subset \Omega$, with the cone property, such that*

$$\left\|g\partial^h u\right\|_{L_s^p(\Omega)} \le \varepsilon\|u\|_{W_s^{k,p}(\Omega)} + c(\varepsilon)\|u\|_{L^p(\Omega_\varepsilon)} \quad \forall u \in W_s^{k,p}(\Omega), \tag{3.14}$$

where $c(\varepsilon)$, Ω_ε depend on ε, Ω, n, k, h, p, t, ρ, s, and $\sigma_\circ[g]$.

4. An a priori bound

Assume that Ω is an unbounded open subset of $\mathbb{R}^n$, $n \ge 3$, with the uniform $C^{1,1}$-regularity property, and let ρ be the function defined by (2.7). Moreover, let $p \in]1,+\infty[$ and $s \in \mathbb{R}$. Consider in Ω the differential operator:

$$L = -\sum_{i,j=1}^n a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \tag{4.1}$$

with the following conditions on the coefficients:

(h_1)

$$a_{ij} = a_{ji} \in L^\infty(\Omega) \cap \mathrm{VMO}_{\mathrm{loc}}(\overline{\Omega}), \quad i,\ j = 1,\dots,n,$$
$$\exists \nu > 0\ :\ \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2 \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^n, \tag{4.2}$$

there exist functions e_{ij}, $i, j = 1,\dots,n$, g and $\mu \in \mathbb{R}_+$ such that

(h_2)

$$e_{ij} = e_{ji} \in L^\infty(\Omega), \qquad (e_{ij})_{x_h} \in M_0^{t,n-t}(\Omega), \quad \text{with } t \in]2,n],\ i,j,h = 1,\dots,n,$$
$$\sum_{i,j=1}^{n} e_{ij}\xi_i\xi_j \ge \mu|\xi|^2 \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^n,$$
$$g \in L^\infty(\Omega), \qquad g_0 = \operatorname*{ess\,inf}_{\Omega} g > 0, \tag{4.3}$$
$$\lim_{r\to+\infty} \sum_{i,j=1}^{n} \|e_{ij} - g a_{ij}\|_{L^\infty(\Omega\setminus B_r)} = 0,$$

(h_3)

$$a_i \in M_\circ^{t_1}(\Omega), \quad i = 1,\dots,n,$$
$$a = a' + a'', \quad a' \in M_\circ^{t_2}(\Omega),\ a'' \in L^\infty(\Omega),\ \operatorname*{ess\,inf}_{\Omega} a'' = a_0'' > 0, \tag{4.4}$$

where

$$t_1 \ge n \quad \text{if } p < n, \qquad t_1 > n \quad \text{if } p = n, \qquad t_1 = p \quad \text{if } p > n,$$
$$t_2 \ge n/2 \quad \text{if } p < n/2, \qquad t_2 > n/2 \quad \text{if } p = n/2, \qquad t_2 = p \quad \text{if } p > n/2. \tag{4.5}$$

Observe that under the assumptions (h_1)–(h_3), it follows that the operator $L : W_s^{2,p}(\Omega) \to L_s^p(\Omega)$ is bounded from Lemma 3.1.

Theorem 4.1. *If the hypotheses (h_1), (h_2), and (h_3) are verified, then there exist a constant $c \in \mathbb{R}_+$ and a bounded open subset $\Omega_0 \subset\subset \Omega$, with the cone property, such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \le c\left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L^p(\Omega_0)}\right), \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega), \tag{4.6}$$

with c and Ω_0 depending on n, p, ρ, s, Ω, ν, μ, g_0, a_0'', t, t_1, t_2, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0} a_{ij}]$, $\sigma_\circ[(e_{ij})_x]$, $\sigma_\circ[a_i]$, $\sigma_\circ[a']$, where $r_0 \in \mathbb{R}_+$ depends on n, p, Ω, μ, g_0, a_0'', t, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\sigma_\circ[(e_{ij})_x]$.

Proof. We consider a function $\phi \in C_\circ^\infty(\mathbb{R}^n)$, such that

$$\phi_{|B_{1/2}} = 1, \quad \operatorname{supp}\phi \subset B_1,$$
$$\sup_{\mathbb{R}^n}\left|\partial^\alpha \phi\right| \le c_\alpha \quad \forall \alpha \in \mathbb{N}_0^n, \tag{4.7}$$

where $c_\alpha \in \mathbb{R}_+$ depends only on α, fix $y \in \mathbb{R}^n$ and put

$$\psi = \psi_y : x \in \mathbb{R}^n \longrightarrow \phi\left(\frac{x-y}{\rho_1(y)}\right). \tag{4.8}$$

Clearly we have

$$\begin{gathered} \psi_{|B(y,(1/2)\rho_1(y))} = 1, \qquad \operatorname{supp}\psi \subset B(y,\rho_1(y)), \\ \sup_{\mathbb{R}^n}\left|\partial^\alpha\psi\right| \le c_\alpha\rho_1^{-|\alpha|}(y) \quad \forall\alpha\in\mathbb{N}_0^n. \end{gathered} \tag{4.9}$$

Now, we put

$$L_0 = -\sum_{i,j=1}^n a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} \tag{4.10}$$

and fix $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)$. Since $\psi u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$, from [14, Theorem 3.3], it follows that there exist $c_1 \in \mathbb{R}_+$ and a bounded open subset $\Omega_1 \subset\subset \Omega$, with the cone property, such that

$$\|\psi u\|_{W^{2,p}(\Omega)} \le c_1\left(\left\|(L_0 + a'')(\psi u)\right\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)}\right), \tag{4.11}$$

with c_1 and Ω_1 depending on n, p, Ω, ν, μ, g_0, a_0'', t, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0}a_{ij}]$, $\sigma_\circ[(e_{ij})_x]$, where $r_0 \in \mathbb{R}_+$ depends on n, p, Ω, μ, g_0, a_0'', t, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\sigma_\circ[(e_{ij})_x]$. Since

$$L_0(\psi u) = \psi L_0 u - 2\sum_{i,j=1}^n a_{ij}\psi_{x_i}u_{x_j} - \sum_{i,j=1}^n a_{ij}\psi_{x_ix_j}u, \tag{4.12}$$

from (4.11) and (4.12), we have

$$\begin{aligned} &\|\psi u\|_{W^{2,p}(\Omega)} \\ &\quad\le c_2\left(\|\psi(L_0+a'')u\|_{L^p(\Omega)} + \sum_{i,j=1}^n\left\|\psi_{x_i}u_{x_j}\right\|_{L^p(\Omega)} + \sum_{i,j=1}^n\left\|\psi_{x_ix_j}u\right\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)}\right), \end{aligned} \tag{4.13}$$

with c_2 dependent on the same parameters of c_1.

On the other hand, since $\rho \in L^\infty_{\mathrm{loc}}(\overline{\Omega})$, we have that

$$\|\psi u\|_{L^p(\Omega_1)} \le c_3\rho_1^{-2}(y)\|u\|_{L^p(I_1(y))}, \tag{4.14}$$

where $c_3 \in \mathbb{R}_+$ depends only on ρ.

Therefore, by (4.13) and (4.14), we deduce the bound:

$$\begin{aligned} \|u\|_{W^{2,p}(I_{1/2}(y))} &\le \|\psi u\|_{W^{2,p}(\Omega)} \\ &\le c_4\left(\left\|L_0u + a''u\right\|_{L^p(I_1(y))} + \rho_1^{-1}(y)\|u_x\|_{L^p(I_1(y))} + \rho_1^{-2}(y)\|u\|_{L^p(I_1(y))}\right), \end{aligned} \tag{4.15}$$

where $c_4 \in \mathbb{R}_+$ depends on the same parameters of c_2 and on ρ.

From (4.15) it follows

$$\int_{\mathbb{R}^n} \rho_1^{ps-n}(y)\|u\|^p_{W^{2,p}(I_{1/2}(y))}dy \le c_5\Big(\int_{\mathbb{R}^n} \rho_1^{ps-n}(y)\|L_0u + a''u\|^p_{L^p(I_1(y))}dy + \int_{\mathbb{R}^n} \rho_1^{ps-n-p}(y)\|u_x\|^p_{L^p(I_1(y))}dy + \int_{\mathbb{R}^n} \rho_1^{ps-n-2p}(y)\|u\|^p_{L^p(I_1(y))}dy\Big), \tag{4.16}$$

where $c_5 \in \mathbb{R}_+$ depends on the same parameters of c_4.

Since

$$L^p_s(\Omega) \hookrightarrow L^p_{s-1}(\Omega), \qquad L^p_s(\Omega) \hookrightarrow L^p_{s-2}(\Omega), \tag{4.17}$$

from (4.16) and from Lemma 2.1 we have that

$$\|u\|_{W^{2,p}_s(\Omega)} \le c_6\big(\|L_0u + a''u\|_{L^p_s(\Omega)} + \|u_x\|_{L^p_{s-1}(\Omega)} + \|u\|_{L^p_{s-2}(\Omega)}\big), \tag{4.18}$$

with $c_6 \in \mathbb{R}_+$ dependent on the same parameters of c_5 and also on s.

Moreover, from Lemma 3.2 it follows that for any $\varepsilon \in \mathbb{R}_+$, there exist $c'(\varepsilon), c''(\varepsilon) \in \mathbb{R}_+$, and two bounded open sets $\Omega'_\varepsilon, \Omega''_\varepsilon \subset\subset \Omega$, both with the cone property, such that

$$\begin{aligned} \|u_x\|_{L^p_{s-1}(\Omega)} + \|u\|_{L^p_{s-2}(\Omega)} &\le \varepsilon\|u\|_{W^{2,p}_s(\Omega)} + c'(\varepsilon)\|u\|_{L^p(\Omega'_\varepsilon)}, \\ \Big\|\sum_{i=1}^n a_i u_{x_i} + a'u\Big\|_{L^p_s(\Omega)} &\le \varepsilon\|u\|_{W^{2,p}_s(\Omega)} + c''(\varepsilon)\|u\|_{L^p(\Omega''_\varepsilon)}, \end{aligned} \tag{4.19}$$

where $c'(\varepsilon)$, Ω'_ε depend on $\varepsilon, \Omega, n, p, \rho, s$, and $c''(\varepsilon)$, Ω''_ε depend on $\varepsilon, \Omega, n, p, t_1, t_2, \rho, s, \sigma_\circ[a_i]$, and $\sigma_\circ[a']$.

From (4.18) and (4.19) it follows (4.6) and then we have the result. □

5. A uniqueness result

In this section, we will prove our uniqueness theorem. We begin to prove a regularity result.

Lemma 5.1. *Suppose that the assumptions* (h_1), (h_2), *and* (h_3) *(with* $t_1 > n$ *and* $t_2 > n/2$*) hold, and let* u *be a solution of the problem*

$$\begin{gathered} u \in W^{2,q}_{loc}(\overline{\Omega}) \cap \overset{\circ}{W}{}^{1,q}_{loc}(\overline{\Omega}) \cap L^p_m(\Omega), \\ Lu \in L^p_s(\Omega), \end{gathered} \tag{5.1}$$

where $q \in]1, p]$ *and* $m \in \mathbb{R}$. *Then,* u *belongs to* $W^{2,p}_s(\Omega)$.

Proof. By [13, Lemma 4.1] we have

$$u \in W^{2,p}_{\mathrm{loc}}(\overline{\Omega}) \cap \overset{\circ}{W}{}^{1,p}_{\mathrm{loc}}(\overline{\Omega}). \tag{5.2}$$

We choose $r, r' \in \mathbb{R}_+$, with $r < r' < 1$, and a function $\phi \in C^\infty_\circ(\mathbb{R}^n)$, such that

$$\begin{gathered} \phi_{|B_r} = 1, \qquad \operatorname{supp}\phi \subset B_{r'}, \\ \sup_{\mathbb{R}^n} \left|\partial^\alpha \phi\right| \le c_\alpha \left(r' - r\right)^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}^n_0, \end{gathered} \tag{5.3}$$

where $c_\alpha \in \mathbb{R}_+$ depends only on α.

We fix $y \in \mathbb{R}^n$ and put

$$\psi = \psi_y : x \in \mathbb{R}^n \longrightarrow \phi\left(\frac{x-y}{\rho_1(y)}\right). \tag{5.4}$$

Clearly we have

$$\begin{gathered} \psi_{|B(y, r\rho_1(y))} = 1, \qquad \operatorname{supp}\psi \subset B\left(y, r'\rho_1(y)\right), \\ \sup_{\mathbb{R}^n} \left|\partial^\alpha \psi\right| \le c_\alpha \rho_1^{-|\alpha|}(y)\left(r' - r\right)^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}^n_0. \end{gathered} \tag{5.5}$$

Since $\psi u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$, from [14, Theorem 3.3] it follows that there exist $c_1 \in \mathbb{R}_+$ and a bounded open subset $\Omega_1 \subset\subset \Omega$, with the cone property, such that

$$\|\psi u\|_{W^{2,p}(\Omega)} \le c_1\left(\left\|L(\psi u)\right\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)}\right), \tag{5.6}$$

with c_1 and Ω_1 depending on n, p, Ω, ν, μ, g_0, a''_0, t, t_1, t_2, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0} a_{ij}]$, $\sigma_\circ[(e_{ij})_x]$, $\sigma_\circ[a_i]$, $\sigma_\circ[a']$, where $r_0 \in \mathbb{R}_+$ depends on n, p, Ω, μ, g_0, a''_0, t, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\sigma_\circ[(e_{ij})_x]$.

Since

$$\begin{aligned} L(\psi u) &= -\sum_{i,j=1}^n a_{ij}(\psi u)_{x_i x_j} + \sum_{i=1}^n a_i(\psi u)_{x_i} + a\psi u \\ &= \psi L u - 2\sum_{i,j=1}^n a_{ij}\left(\psi_{x_i} u\right)_{x_j} + \sum_{i,j=1}^n a_{ij}\psi_{x_i x_j} u + \sum_{i=1}^n a_i \psi_{x_i} u, \end{aligned} \tag{5.7}$$

from (5.6) and (5.7), we have

$$\begin{aligned} &\|\psi u\|_{W^{2,p}(\Omega)} \\ &\le c_2\left(\|\psi L u\|_{L^p(\Omega)} + \sum_{i,j=1}^n \left\|\left(\psi_{x_i} u\right)_{x_j}\right\|_{L^p(\Omega)} + \sum_{i,j=1}^n \left\|\psi_{x_i x_j} u\right\|_{L^p(\Omega)} + \sum_{i=1}^n \left\|a_i \psi_{x_i} u\right\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)}\right), \end{aligned} \tag{5.8}$$

with c_2 dependent on the same parameters of c_1.

From Lemma 3.1 with $s = 0$, we have that

$$\|a_i \varphi_{x_i} u\|_{L^p(\Omega)} \leq c_3 \|a_i\|_{M^{t_1}(\Omega)} \left(\|\varphi_{x_i} u\|_{L^p(\Omega)} + \|(\varphi_{x_i} u)_x\|_{L^p(\Omega)} \right), \tag{5.9}$$

with c_3 dependent on Ω, n, p, and t_1.

Using [22, Corollary 4.5], we can obtain the following interpolation estimate:

$$\|\varphi_{x_i} u\|_{L^p(\Omega)} + \|(\varphi_{x_i} u)_{x_j}\|_{L^p(\Omega)} \leq c_4 \left(\|(\varphi_{x_i} u)_{xx}\|^{1/2}_{L^p(\Omega)} \|\varphi_{x_i} u\|^{1/2}_{L^p(\Omega)} + \|\varphi_{x_i} u\|_{L^p(\Omega)} \right), \tag{5.10}$$

where the constant c_4 depends on Ω, n, p.

Thus, by (5.8)–(5.10), with easy computations, we deduce the bound:

$$\begin{aligned} \|u\|_{W^{2,p}(I_r(y))} &\leq \|\varphi u\|_{W^{2,p}(\Omega)} \leq c_5 (r' - r)^{-2} \\ &\quad \times \left(\|Lu\|_{L^p(I_{r'}(y))} + \|u\|^{1/2}_{W^{2,p}(I_{r'}(y))} \left(\rho_1^{-1}(y) \|u\|_{L^p(I_{r'}(y))} \right)^{1/2} + \rho_1^{-1}(y) \|u\|_{L^p(I_{r'}(y))} \right), \end{aligned} \tag{5.11}$$

where $c_5 \in \mathbb{R}_+$ depends on n, p, ρ, Ω, ν, μ, g_0, a_0'', t, t_1, t_2, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|a''\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0} a_{ij}]$, $\sigma_\circ[(e_{ij})_x]$, $\|a_i\|_{M^{t_1}(\Omega)}$, $\sigma_\circ[a_i]$, $\sigma_\circ[a']$.

By a well-known lemma of monotonicity of Miranda (see [23, Lemma 3.1]), it follows from (5.11) that

$$\|u\|_{W^{2,p}(I_{1/2}(y))} \leq c_6 \left(\|Lu\|_{L^p(I_1(y))} + \rho_1^{-1}(y) \|u\|_{L^p(I_1(y))} + \left(\rho_1^{-1}(y) \|u\|_{L^p(I_1(y))} \right)^{1/2} \|u\|^{1/2}_{W^{2,p}(I_{1/2}(y))} \right), \tag{5.12}$$

and then, using Young's inequality, we deduce from (5.12) that

$$\|u\|_{W^{2,p}(I_{1/2}(y))} \leq c_7 \left(\|Lu\|_{L^p(I_1(y))} + \rho_1^{-1}(y) \|u\|_{L^p(I_1(y))} \right), \tag{5.13}$$

with $c_6, c_7 \in \mathbb{R}_+$ dependent on the same parameters of c_5.

From (5.13) it follows

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho_1^{ps-n}(y) \|u\|^p_{W^{2,p}(I_{1/2}(y))} dy \\ &\quad \leq c_8 \left(\int_{\mathbb{R}^n} \rho_1^{ps-n}(y) \|Lu\|^p_{L^p(I_1(y))} dy + \int_{\mathbb{R}^n} \rho_1^{ps-n-p}(y) \|u\|^p_{L^p(I_1(y))} dy \right), \end{aligned} \tag{5.14}$$

where $c_8 \in \mathbb{R}_+$ depends on the same parameters of c_7.

If $m \geq s - 1$, since

$$L^p_m(\Omega) \hookrightarrow L^p_{s-1}(\Omega), \tag{5.15}$$

from (5.14) and from Lemma 2.1 we have that

$$\|u\|_{W^{2,p}_s(\Omega)} \leq c_9 \left(\|Lu\|_{L^p_s(\Omega)} + \|u\|_{L^p_{s-1}(\Omega)} \right), \tag{5.16}$$

with $c_9 \in \mathbb{R}_+$ dependent on the same parameters of c_8 and on s. Therefore, u belongs to $W^{2,p}_s(\Omega)$.

If $m < s-1$, we denote by k the positive integer, such that

$$s - m - 1 \leq k < s - m. \tag{5.17}$$

Then, for $i = 1, \dots, k$, we have that

$$L^p_s(\Omega) \hookrightarrow L^p_{m+i}(\Omega). \tag{5.18}$$

Therefore, using (5.14) and (5.16) with $m+i$, $i = 1, \dots, k$, instead of s, we deduce that $u \in W^{2,p}_{m+1}(\Omega), \dots, u \in W^{2,p}_{m+k}(\Omega)$. On the other hand, we have that

$$W^{2,p}_{m+k}(\Omega) \hookrightarrow L^p_{s-1}(\Omega) \tag{5.19}$$

and then, since $u \in L^p_{s-1}(\Omega)$, (5.14) holds. Thus, u satisfies (5.16) and then $u \in W^{2,p}_s(\Omega)$. □

Theorem 5.2. *If conditions (h_1), (h_2), and (h_3) (with $t_1 > n$ and $t_2 > n/2$) hold, and $a \geq a_0 > 0$ a.e. in Ω, then the problem*

$$u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega), \quad Lu = 0, \tag{5.20}$$

admits only the zero solution.

Proof. Fix $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)$, such that $Lu = 0$. From Lemma 5.1 it follows that $u \in W^{2,p}(\Omega)$. On the other hand, since $u \in W^{1,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{\text{loc}}(\overline{\Omega})$, from Lemma 2.2 we have that $u \in \overset{\circ}{W}{}^{1,p}(\Omega)$. Thus, from [13, Theorem 5.2] we deduce that $u = 0$. □

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