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Research Article

Affine Anosov Diffeomorphims of Affine Manifolds

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We show that a compact affine manifold endowed with an affine Anosov transformation is finitely covered by a complete affine nilmanifold. This is a partial answer of a conjecture of Franks for affine manifolds.

1. Introduction

An n-affine manifold (M, ∇_M) is an n-differentiable manifold M endowed with a locally flat connection ∇_M, that is a connection ∇_M whose curvature and torsion forms vanish identically. The connection ∇_M defines on M an atlas (affine) whose coordinates change is locally affine maps. The pull-back $\nabla_{\widehat{M}}$ of the connection ∇_M to the universal cover $\widehat{M}$ of M is a locally flat connection. The affine structure of $\widehat{M}$ is defined by a local diffeomorphism $D_M : \widehat{M} \to \mathbb{R}^n$ called the developing map. The developing map gives rise to a representation $h_M : \pi_1(M) \to \mathrm{Aff}(\mathbb{R}^n)$ called the holonomy. The linear part $L(h_M)$ of h_M is the linear holonomy. The affine manifold (M, ∇_M) is complete if and only if D_M is a diffeomorphism. This means also that the connection ∇_M is geodesically complete.

A diffeomorphism f of M is called an Anosov diffeomorphism f if and only if there exists a norm $\|\cdot\|$ on M associated to a differentiable metric $\langle \cdot, \cdot \rangle$, a real number $0 < \lambda < 1$ such that the tangent bundle TM of M is the direct summand of two bundles TM^s and TM^u called, respectively, the stable bundle and the unstable bundle such that

$$\begin{aligned} \| df^m(v) \| &\le c\lambda^m \|v\|, \quad v \in TM^s, \\ \|df^m(w)\| &\ge c\lambda^{-m}\|w\|, \quad w \in TM^u, \end{aligned} \tag{1.1}$$

where d is the usual differential, c is a positive real number, and m is a positive integer.

The stable distribution TM^s (resp., the unstable distribution TM^u) is tangent to a topological foliation $\mathcal{F}^s$ (resp., $\mathcal{F}^u$).

The property for a diffeomorphism to be Anosov is independent of the choice of the differentiable metric if M is compact. In this case we can suppose that c is 1.

A conjecture of Franks asserts that an Anosov diffeomorphism f defined on a compact manifold M is C^0-conjugated to an hyperbolic infranilautomorphism. This conjecture is proved in [1] with the assumptions that f is topologically transitive, the stable and unstable foliations of f are C^∞, and f preserves a symplectic form or a connection.

The goal of this paper is to characterize compact affine manifolds endowed with affine Anosov transformations. More precisely, we show the following.

Theorem 1.1. *Let (M, ∇_M) be a compact affine manifold, and f an affine Anosov transformation of M, then (M, ∇_M) is finitely covered by a complete affine nilmanifold.*

2. The proof of the main theorem

The main goal of this part is to show Theorem 1.1. In the sequel, (M, ∇_M) will be an n-compact affine manifold endowed with an affine Anosov diffeomorphism f. The stable foliation $\mathcal{F}^s$ (resp., the unstable foliation $\mathcal{F}^u$) pulls-back on the universal cover $\widehat{M}$ to a foliation $\widehat{\mathcal{F}}^s$, (resp., $\widehat{\mathcal{F}}^u$).

Let (U, ϕ) be an affine chart of M, we say that the restriction of a differentiable metric of M to U is an Euclidean metric adapted to the affine structure of U if this restriction is the pull-back by ϕ to U, of the restriction to $\phi(U)$ of an Euclidean metric of $\mathbb{R}^n$.

Proposition 2.1. *The stable and the unstable distributions of f define on (M, ∇_M) foliations whose leaves are immersed affine submanifolds.*

Proof. Let x be an element of M, $\| \ \|$ a norm associated to a differentiable metric $\langle , \rangle$ of M. Let v be an element of T_xM^s the subspace of T_xM tangent to $\mathcal{F}^s$; we have $\|df^m(v)\| \leq \lambda^m \|v\|$ where $0 < \lambda < 1$ and $m \in \mathbb{N}$. Let (U, ϕ) be an affine chart which contains an accumulation point of the sequence $(f^p(x))_{p\in\mathbb{N}}$. We can suppose that the restriction of $\langle , \rangle$ to U is Euclidean and adapted to the affine structure. Let $p > p'$ such that $f^p(x)$ and $f^{p'}(x)$ are elements of U, and $v \in T_{f^{p'}(x)}M^s$; we have $\|df^{p-p'}(v)\| \leq \lambda^{p-p'}\|v\|$. This implies that every element $y = f^{p'}(x) + w$, such that $w \in T_{f^{p'}(x)}M^s$ and $f^{p-p'}(y) \in U$ is an element of the stable leaf of $f^{p'}(x)$ since the distance between $f^{n_q}(x)$ and $f^{n_q}(y)$ converges towards zero for a subsequence $n_q > p$ such that $f^{n_q}(x)$ is an element of U. We deduce that this leaf is an immersed submanifold. The result for the unstable foliation is deduced by considering f^{-1}. □

Proposition 2.2. *The affine structure induced by ∇_M on a leaf of $\mathcal{F}^s$ is geodesically complete.*

Proof. Let x be an element of M; since M is compact, the sequence $(f^m(x))_{m\in\mathbb{N}}$ has an accumulation point y. Let U_y be an open set containing y such that there is a strictly positive number r, such that for every $z \in U_y$, $v \in T_zM$ whose norm is less than r for a given differentiable metric, a (affine) geodesic from x whose derivative at 0 is v is defined at 1. Let w be an element of T_xM^s such that $\|df^p(w)\| \leq r$. Without loss of generality, we can suppose that $f^p(x) \in U_y$, the (affine) geodesic from $f^p(x)$ whose derivative is $df^p(w)$ at 0 is defined at 1. This implies that the geodesic from x whose derivative at 0 is w is defined at 1, since f^{-1} is an affine map. This shows the result. □

It is a well-known fact that an Anosov's diffeomorphism of a compact manifold as a periodic fixed point. We can replace f by an iterated f^p, $p \in \mathbb{N}$ and suppose that f has a fixed point x. This implies the existence of a map $F : \widehat{M} \to \widehat{M}$ over f which fixed the element $\hat{x}$ over x.

Proposition 2.3. *Let $\hat{y}$ and $\hat{z}$ be two elements of $\widehat{\mathcal{F}}^u_{\hat{t}}$, where $\hat{t}$ is an element of $\widehat{M}$, then the images of $\widehat{\mathcal{F}}^s_{\hat{y}}$ and $\widehat{\mathcal{F}}^s_{\hat{z}}$ by the developing map are parallel affine subspaces.*

Proof. Let $\hat{y}$ be an element of $\widehat{\mathcal{F}}^u_{\hat{t}}$, it is sufficient to show that $D(\widehat{\mathcal{F}}^s_{\hat{t}})$ and $D(\widehat{\mathcal{F}}^s_{\hat{y}})$ are parallel affine subspaces.

We know that the tangent bundle of a simply connected affine manifold is trivial. We have $T\widehat{M} = \widehat{M} \times T_{\hat{t}}\widehat{M}$. Let $\langle,\rangle'$ be differentiable metric on M which pulls back on $\widehat{M}$ is $\langle,\rangle$. The map F is Anosov relatively to $\langle,\rangle$. Without restricting the generality, we can assume that the distributions tangent to $\widehat{\mathcal{F}}^s$ and $\widehat{\mathcal{F}}^u$ are orthogonal.

Let w be a vector of $T\widehat{M}_{\hat{y}}$ tangent to $\widehat{\mathcal{F}}^s_{\hat{y}}$. We can write $w = s + u$ where s is a vector of $T\widehat{M}_{\hat{t}}$ tangent to $\widehat{\mathcal{F}}^s$ and u is a vector of $T\widehat{M}_{\hat{t}}$ tangent to $\widehat{\mathcal{F}}^u$. The vector u is also an element of $T\widehat{M}_{\hat{y}}$ tangent to $\widehat{\mathcal{F}}^u_{\hat{y}}$ since the unstable foliation is affine (we can identify canonically the vector spaces $T\widehat{M}_{\hat{y}}$ and $T\widehat{M}_{\hat{t}}$ since $T\widehat{M}$ is trivial). This implies that $\|dF^l_{\hat{y}}(u)\| \geq \lambda^{-l}\|u\|$ for $0 < \lambda < 1, l \in \mathbb{N}$. But on the other hand we have $\|dF^l_{\hat{y}}(s+u)\| \leq \lambda^l\|(s+u)\|$, which implies that the limit of $\|dF^l_{\hat{y}}(s+u)\|$ is zero when l converges towards the infinity. We have supposed that the distributions tangent to $\widehat{\mathcal{F}}^s$ and $\widehat{\mathcal{F}}^u$ are orthogonal, this implies that $\|dF_{\hat{y}}(s+u)\| = \|dF_{\hat{y}}(s)\| + \|dF_{\hat{y}}(u)\|$. We deduce that $u = 0$. □

The images by D_M of the leaf $\widehat{\mathcal{F}}^s_{\hat{t}}$ of $\widehat{\mathcal{F}}^s$ and $\widehat{\mathcal{F}}^u_{\hat{t}}$ of $\widehat{\mathcal{F}}^u$ are affine subspaces of $\mathbb{R}^n$ whose direction are supplementary subspaces of $\mathbb{R}^n$. Since for every element z of $\widehat{\mathcal{F}}^s_{\hat{t}}$ the leaf of $\widehat{\mathcal{F}}^u$ passing by z is complete, we deduce that the developing map is surjective.

Proposition 2.4. *The affine manifold (M, ∇_M) is complete.*

Proof. Let $\hat{t}$ be an element of $\widehat{M}$, and $\hat{E}_{\hat{t}}$ the set of elements y of $\widehat{M}$ such that there is and element z in $\widehat{\mathcal{F}}^u_{\hat{t}}$ such that y is an element of $\widehat{\mathcal{F}}^s_z$. The image of $\hat{E}_{\hat{t}}$ by D_M is $\mathbb{R}^n$, and the restriction of D_M to $\hat{E}_{\hat{t}}$ is injective. The set $\{\hat{E}_{\hat{t}}, \hat{t} \in \widehat{M}\}$ is a partition of $\widehat{M}$ by disjoint open sets. It has only one element since $\widehat{M}$ is connected. We deduce that (M, ∇_M) is complete. □

Remark 2.5. The existence of an affine Anosov transformation $\hat{f}$ on the universal cover of a compact affine manifold (M, ∇_M) which pushes forward to a diffeomorphism f of M does not imply that f is an Anosov diffeomorphism as the following example shows.

Let N_n be the quotient of $\mathbb{R}^n/\{0\}$ by an homothetic transformation h_λ whose ratio λ is such that $0 < \lambda < 1$. It is a compact affine manifold. Every homothetic transformation h_c which ratio c is positive and different from 1 and λ is an Anosov diffeomorphism of $\mathbb{R}^n$ endowed with an Euclidean metric. But the push forward of h_c to N_n is an isometry of N_n endowed with the push forward of the differentiable metric of $\mathbb{R}^n/\{0\}$ defined by

$$\frac{1}{\|x\|^2}\langle u, v\rangle, \tag{2.1}$$

where x is an element of $\mathbb{R}^n/\{0\}$, and u, v are elements of its tangent space.

Proof of Theorem 1.1. First, we show that the module of the eigenvalues of the elements of the linear holonomy of (M, ∇_M) is 1.

Let (A, a) be an element of $\pi_1(M)$ such that A has an eigenvector u associated to an eigenvalue b (which may be a complex number) whose norm is different from 1.

The argument used before Proposition 2.3, allows us to suppose that f has a fixed point x, and there exists a map F over f which has also a fixed point $\widehat{x}$. Since (M, ∇_M) is complete, we can assume without restricting the generality that $\widehat{x} = 0$.

Consider a differentiable metric $\langle , \rangle'$ on M whose restriction to an affine neighborhood N of x is Euclidean adapted to the affine structure. Expressing on N the fact that F is an Anosov diffeomorphism using the metric $\langle , \rangle'$, one obtains that $\mathbb{R}^n = U \oplus V$, where U and V are two subvector spaces such that there exists a number $0 < \lambda < 1$ such that

$$\begin{aligned} \left\| F^l(u) \right\| &\leq \lambda^l \|u\|, \quad u \in U, \\ \left\| F^l(v) \right\| &\geq \lambda^{-l} \|u\|, \quad v \in V, \end{aligned} \tag{2.2}$$

where u and v are, respectively, elements of U and V and $\| \ \|$ is a norm associated to an Euclidean metric $\langle , \rangle$ of $\mathbb{R}^n$. The images of the subvector spaces U and V by the covering map are, respectively, T_xM^s and T_xM^u. We will assume that they are orthogonal with respect to $\langle , \rangle$. The vectors spaces U and V are stable by the linear holonomy (see Proposition 2.3).

Put $u = u_1 + u_2$, where u_1 and u_2 are, respectively, elements of U' and V', the complexified vectors spaces, respectively, associated to U and V.

Without restricting the generality, one can assume after eventually having changed $\gamma = (A, a)$ by γ^{-1} and (or) f by f^{-1} that u_1 is not zero and that the norm of b is strictly superior to 1.

Let q be a positive integer. We denote by $\|A^q\| = \sup_{\{\|x\|=1\}} \|A^q(x)\|$ the norm of the linear operator A^q associated to the Euclidean metric $\langle , \rangle$. For every element $u \in U$ and every integer n, since A preserves U, we have

$$\left\| F^n\left(A^q(u)\right) \right\| \leq \lambda^n \left\| A^q(u) \right\| \leq \lambda^n \left\| A^q \right\| \cdot \|u\|. \tag{2.3}$$

Since $\lambda < 1$, there exists an integer n_0 such that for every $n > n_0$, $\|F^n(A^q(u))\| < \|u\|$ for every element u in U.

We know that A is invertible; let l be the dimension of V; since the $l-1$-dimensional sphere is compact, there exists a strictly positive real number a_q such that $\inf_{\{v \in V, \|v\|=1\}} \|A^q(v)\| \geq a_q$. For every element v in V, and every integer n, we have

$$\left\| F^n\left(A^q(v)\right) \right\| \geq \lambda^{-n} \left\| A^q(v) \right\| \geq \lambda^{-n} a_q \|v\|. \tag{2.4}$$

This implies the existence of integer n_1, such that for every $n > n_1$, $\|F^{n_q}(A^q(v))\| > \|v\|$. This implies the existence of an integer n_q such that $F^{n_q} \circ \gamma^q$ has a fixed point $\widehat{m}_q$; take for example $n = \sup(n_0, n_1)$. In the sequel, we suppose that n_q is the smallest integer such that $F^{n_q} \circ \gamma^q$ has a fixed point.

Let y be a point of accumulation of the sequence $p(\widehat{m}_q)$ which exists since M is compact. Up to the replacement of $(\widehat{m}_q)$ by a subsequence and the replacement of $\widehat{m}_q$ by another element over m_q, we can suppose that the sequence $(\widehat{m}_q)$ converges to $\widehat{y}$ over y. This can be done by considering a neighborhood U of $\widehat{y}$ in the universal cover of M on which the restriction of the covering map p is bijective onto its image; there exists q_0 such that $q > q_0$ implies that m_q is an element of $p(U)$. We can then lift m_q to U for $q > q_0$ to obtain the desired sequence. The linear part of the elements over f^{n_q} which fix the elements over m_q has the same eigenvalues since they are conjugated. Consider a differentiable metric of M which pullback on $\mathbb{R}^n$ coincides on a neighborhood of $\widehat{y}$ with an Euclidean metric.

(1) The sequence (n_q) is bounded

The restriction of the linear part of the map over f^{n_q} conjugated to $F^{n_q}\circ\gamma^q$ which fixes $\widehat{m}_q$ at U is a contracting map (for the last Euclidean metric) with ratio λ^{n_q} $(0<\lambda<1)$, since it coincides on the neighborhood of $\widehat{y}$ with the lift of a metric on M, and f is an Anosov map. This is not possible since the restriction of the linear part of this element to U' (the tensor product of U with the complex line) has the same eigenvalues than the restriction of $F^{n_q}\circ A^q$ to U', and the limit when q goes to infinity of the norm of $F^{n_q}\circ A^q(u_1)$ for the Hermitian metric associated to the last Euclidean metric is infinity, since the sequence n_q is bounded and the norm of $b>1$ (recall that b is an eigenvalue of A).

(2) The sequence (n_q) is not bounded

Up to the change of (n_q) to a subsequence, we can suppose that (n_q) goes to infinity. The linear map $F^{n_q-1}\circ\gamma^q$ does not have a fixed point. Its linear part has the eigenvalue 1 associated to the eigenvector v_q. Write $v_q=v_{1q}+v_{2q}$ where v_{1q} and v_{2q} are, respectively, elements of U and V. If v_{1q} is not zero, then $\|F^{n_q}\circ A^q(v_{1q})\|=\|F(v_{1q})\|$ (the norm considered is the precedent Euclidean norm); the restriction of $F^{n_q}\circ A^q$ cannot be contracting with ratio λ^{n_q} for q big enough since the sequence (n_q) goes to infinity.

If v_{2q} is not zero, then $\|F^{n_q}\circ A^q(v_{2q})\|=\|F(v_{2q})\|$, the restriction of $F^{n_q}\circ A^q$ to V cannot be dilating (for the last Euclidean metric) with ratio λ^{-n_q} for q big enough since n_q goes to infinity. There is a contradiction since the eigenvalues of the restriction of $F^{n_q}\circ A^q$ to U' (resp., to V') coincide with the eigenvalues of the restriction to U' of the linear part of the conjugated to $F^{n_q}\circ\gamma^q$ which fixes $\widehat{m}_q$ (resp., the eigenvalues of the restriction to V' of the linear part of the element conjugated to $F^{n_q}\circ\gamma^q$ which fixes $\widehat{m}_q$). This implies that the eigenvalues of A have norm 1. We deduce that the linear holonomy of M is distal (see [2, paragraph 5.2]), it follows from [3, Theorem 3] that M is finitely covered by a nilmanifold, see also [4, Theorem 2]. □

Remark 2.6. Let (M',∇') be the finite cover of (M,∇)whose fundamental group is G, and f' the pulls-back of f to M'. The map $H^n(M',\mathbb{R})\rightarrow H^n(M',\mathbb{R})$, $\alpha\rightarrow(f'^2)^*\alpha$ is the identity since f'^2 is a diffeomorphism which preserves the orientation. We deduce that f'^2 preserves the parallel volume form of (M',∇'); then a well-known result of Anosov implies that f'^2 is ergodic and its periodic points are dense.

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