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Research Article

Kurosh-Amitsur Right Jacobson Radical of Type 0 for Right Near-Rings

Ravi Srinivasa Rao,[1] K. Siva Prasad,[2] and T. Srinivas[3]

[1] *Department of Mathematics, P.G.Centre, P.B. Siddhartha College of Arts and Science, Vijayawada-520010, Andhra Pradesh, India*
[2] *Department of Mathematics, Chalapathi Institute of Engineering and Technology, Chalapathi Nagar, Lam, Guntur-522034, Andhra Pradesh, India*
[3] *Department of Mathematics, Kakatiya University, Warangal-506009, Andhra Pradesh, India*

Correspondence should be addressed to Ravi Srinivasa Rao, dr_rsrao@yahoo.com

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By a near-ring we mean a right near-ring. J_0^r, the right Jacobson radical of type 0, was introduced for near-rings by the first and second authors. In this paper properties of the radical J_0^r are studied. It is shown that J_0^r is a Kurosh-Amitsur radical (KA-radical) in the variety of all near-rings R, in which the constant part R_c of R is an ideal of R. So unlike the left Jacobson radicals of types 0 and 1 of near-rings, J_0^r is a KA-radical in the class of all zero-symmetric near-rings. J_0^r is not s-hereditary and hence not an ideal-hereditary radical in the class of all zero-symmetric near-rings.

1. Introduction

R denotes a right near-ring and all near-rings considered are right near-rings and not necessarily zero-symmetric.

In [1, 2], the first author studied the structure of near-rings in terms of right ideals, and showed that as in rings, matrix units determined by right ideals identify matrix near-rings. To show the importance of the right Jacobson radicals of near-rings in the extension of a form of the Wedderburn-Artin theorem of rings involving the matrix rings to near-rings, the right Jacobson radicals of type ν were introduced and studied by the first and second authors in [3–6], $\nu \in \{0, 1, 2, s\}$. In [6], Wedderburn-Artin theorem was extended to near-rings, and some generalizations of it were presented.

In this paper, properties of the right Jacobson radical of type 0 are studied. It is known that the left Jacobson radicals of types 0 and 1 are not KA-radicals in the class of all zero-symmetric near-rings, and only the left Jacobson radicals of types 2 and 3 are KA-radicals in the class of all zero-symmetric near-rings. Surprisingly, J_0^r, the right Jacobson radical of type 0, is a KA-radical in the class of all zero-symmetric near-rings. It is also shown that J_0^r is a KA-radical even in a bigger class of near-rings, namely, in the variety of all near-rings R, in which the constant part of R is an ideal of R. Moreover, J_0^r is not s-hereditary, and hence not an ideal-hereditary radical in the class of all zero-symmetric near-rings.

2. Preliminaries

Near-rings considered are right near-rings and not necessarily zero-symmetric. Unless otherwise specified, R stands for a right near-ring. Near-ring notions not defined here can be found in [7].

R_0 and R_c denote the zero-symmetric part and the constant part of R, respectively.

$\mathcal{F}$ denotes the class of near-rings R, in which the constant part R_c of R is an ideal of R. In [8], Fuchs has shown that the class of near-rings $\mathcal{F}$ is a variety. Obviously, $\mathcal{F}$ contains all zero-symmetric, constant, and abstract affine near-rings. Now we give here some definitions and results of [3], which will be used later.

An element $a \in R$ is called *right quasiregular* if and only if the right ideal of R generated by the set $\{x - ax \mid x \in R\}$ is R. A right ideal (left ideal, ideal, subset) K of R is called a *right quasiregular right ideal (left ideal, ideal, subset)* of R if each element of K is right quasiregular.

A right ideal K of R is called *right modular* if there is an element $e \in R$ such that $x-ex \in K$ for all $x \in R$. In this case, we say that K is *right modular by* e.

A maximal right modular right ideal of R is called a *right 0-modular right ideal* of R.

$J_{1/2}^r(R)$ is the intersection of all right 0-modular right ideals of R, and if R has no right 0-modular right ideals, then $J_{1/2}^r(R) = R$.

The largest ideal of R contained in $J_{1/2}^r(R)$ is denoted by $J_0^r(R)$ and called the *right Jacobson radical of R of type 0*.

The largest ideal contained in a right 0-modular right ideal of R is called a *right 0-primitive ideal* of R. R is called a *right 0-primitive near-ring* if $\{0\}$ is a right 0-primitive ideal of R.

A group $(G, +)$ is called a *right R-group* if there is a mapping $((g, r) \to gr)$ of $G \times R$ into G such that (i) $(g+h)r = gr + hr$ and (ii) $g(rs) = (gr)s$ for all $g,h \in G$ and $r,s \in R$. A subgroup (normal subgroup) H of a right R-group G is called an *R-subgroup (ideal)* of G if $hr \in H$ for all $h \in H$ and $r \in R$.

Let G be a right R-group. An element $g \in G$ is called a *generator* of G if $gR = G$ and $g(r+s) = gr + gs$ for all $r,s \in R$. G is said to be *monogenic* if G has a generator.

G is said to be *simple* if $G \neq \{0\}$, and G and $\{0\}$ are the only ideals of G.

A monogenic right R-group G is said to be a *right R-group of type 0* if G is simple.

The *annihilator* of a right R-group G, denoted by $(0 : G)$, is defined as $(0 : G) = \{a \in R \mid Ga = \{0\}\}$.

Lemma 2.1. *The constant part of R is right quasiregular.*

Lemma 2.2. *A nilpotent element of R is right quasiregular.*

Theorem 2.3. $J_{1/2}^r(R)$ *is the largest right quasiregular right ideal of R.*

Theorem 2.4. $J_0^r(R)$ *is the largest right quasiregular ideal of R.*

Theorem 2.5. *$J_0^r(R)$ is the intersection of all right 0-primitive ideals of R.*

Theorem 2.6. *Let P be an ideal of R. P is a right 0-primitive ideal of R if and only if R/P is a right 0-primitive near-ring.*

Proposition 2.7. *Let G be a right R-group of type 0 and g_0 a generator of G. Then $(0 : g_0) := \{r \in R \mid g_0 r = 0\}$ is a right 0-modular right ideal of R.*

Proposition 2.8. *Let G be a right R-group. G is a right R-group of type 0 if and only if there is a maximal right modular right ideal K of R such that G is R-isomorphic to R/K.*

Proposition 2.9. *Let P be an ideal of a zero-symmetric near-ring R. P is right 0-primitive if and only if P is the largest ideal of R contained in $(0 : G)$ for some right R-group G of type 0.*

Let Q be a mapping which assigns to each near-ring R an ideal $Q(R)$ of R. Such mappings are called ideal-mappings. We consider the following properties which Q may satisfy:

(H1) $h(Q(R)) \subseteq Q(h(R))$ for all homomorphisms h of R;

(H2) $Q(R/Q(R)) = \{0\}$ for all R;

Q is *r-hereditary* if $I \cap Q(R) \subseteq Q(I)$ for all ideals I of R;

Q is *s-hereditary* if $Q(I) \subseteq I \cap Q(R)$ for all ideals I of R;

Q is *ideal-hereditary* if it is both r-hereditary and s-hereditary, that is, if $Q(I) = I \cap Q(R)$ for all ideals I of R;

Q is *idempotent* if $Q(Q(R)) = Q(R)$ for all R;

Q is *complete* if $Q(I) = I$ and I is an ideal of R that implies $I \subseteq Q(R)$.

With Q we associate two classes of near-rings $\mathbb{R}_Q$ and $\mathbb{S}_Q$ defined by $\mathbb{R}_Q := \{R \mid Q(R) = R\}$, $\mathbb{S}_Q := \{R \mid Q(R) = \{0\}\}$, and are called a Q-radical class and a Q-semisimple class, respectively.

An ideal-mapping Q is a *Hoehnke* radical (H-radical) if it satisfies conditions (H1) and (H2).

An ideal-mapping Q is a *Kurosh-Amitsur* radical (KA-radical) if it is a complete idempotent H-radical.

Let $\mathbb{M}$ be a class of near-rings. Classes of near-rings are always assumed to be abstract, that is, they contain the one element near-ring and are closed under isomorphic copies. With every near-ring R, we associate two ideals of R, depending on $\mathbb{M}$. These ideals are defined by the following:

$$\begin{aligned}\mathbb{M}(R) &:= \Sigma\{I \mid I \text{ is an ideal of } R,\ I \in \mathbb{M}\},\\ (R)\mathbb{M} &:= \cap\{I \mid I \text{ is an ideal of } R,\ R/I \in \mathbb{M}\}.\end{aligned} \tag{2.1}$$

The mapping P defined by $P(R) := (R)\mathbb{M}$ is always an H-radical and is called the H-radical corresponding to $\mathbb{M}$.

From Theorems 2.5 and 2.6, we have the following.

Proposition 2.10. *J_0^r is an H-radical corresponding to the class of all right 0-primitive near-rings.*

3. Properties of the radical J_0^r

If $(A, +)$ is a group and T is a subset of A, then the subgroup (normal subgroup) of A generated by T is denoted by $\langle T\rangle_s(\langle T\rangle_n)$.

Remark 3.1. Let G be a right R-group. It is clear that $H = \{g \in G \mid gR = \{0\}\}$ is an ideal of G. So if G is simple and $gR = \{0\}$, then $g = 0$ provided $GR \neq \{0\}$.

Theorem 3.2. *Let G be a right R-group of type 0. Suppose that S is an invariant subnear-ring and a right ideal of R. If $GS \neq \{0\}$, then G is also a right S-group of type 0.*

Proof. Suppose that $GS \neq \{0\}$. Clearly, G is a right S-group. Let $g \in G$ and $gS := \{gs \mid s \in S\} \subseteq G$. Consider the normal subgroup $\langle gS \rangle_n$ of $(G,+)$. Let $r \in R$, $h \in \langle gS \rangle_n$. Now $h = (x_1 + \delta_1(gs_1) - x_1) + (x_2 + \delta_2(gs_2) - x_2) + \cdots + (x_k + \delta_k(gs_k) - x_k)$, $s_i \in S$, $x_i \in G$, $\delta_i \in \{1,-1\}$. Since $SR \subseteq S$, $hr = (x_1r + \delta_1(g(s_1r)) - x_1r) + (x_2r + \delta_2(g(s_2r)) - x_2) + \cdots + (x_kr + \delta_k(g(s_kr)) - x_kr) \in \langle gS \rangle_n$. So $\langle gS \rangle_n$ is an ideal of the right R-group G, and hence it is also an ideal of the right S-group G. Let $0 \neq h \in G$. Suppose that $hS = \{0\}$. Since $hR \neq \{0\}$, $\langle hR \rangle_n$ is a nonzero ideal of the right R-group G. Since G is a simple right R-group, $\langle hR \rangle_n = G$. So $GS = \langle hR \rangle_n S \subseteq \langle hS \rangle_n = \{0\}$, a contradiction to $GS \neq \{0\}$. Therefore, $hS \neq \{0\}$. Let g_0 be a generator of the right R-group G. So g_0 is a distributive element of the right R-group G and $g_0R = G$. Clearly, g_0 is a distributive element of the right S-group G and hence g_0S is a subgroup of $(G,+)$. We have $(g_0S)R = g_0(SR)g_0S$. So g_0S is an R-subgroup of G. Let g $\in G$ and s $\in S$. Since $g_0R = G$, $g = g_0r$ for some $r \in R$. So $g + g_0s - g = g_0r + g_0s - g_0r = g_0(r + s - r) \in g_0S$, as S is a normal subgroup of $(R,+)$. Therefore, g_0S is an ideal of the right R-group G and hence $g_0S = G$. So g_0 is also a generator of the right S-group G. Let K be a nonzero ideal of the right S-group G. Let $0 \neq y \in K$. As seen above, $\langle yS \rangle_n$ is a nonzero ideal of the right R-group G, and hence $\langle yS \rangle_n = G$. Since $G = \langle yS \rangle_n \subseteq K$, $G = K$. Therefore, $\{0\}$ and G are the only ideals of the right S-group G and hence G is a right S-group of type 0. □

Proposition 3.3. *Let G be a right R-group of type 0 and let T be a right quasiregular invariant subnear-ring of R. If T is a right ideal of R, then $GT = \{0\}$.*

Proof. Suppose that T is a right ideal of R and g_0 is a generator of G. So $g_0(r + s) = g_0r + g_0s$ for all $r, s \in R$ and $g_0R = G$. Now $L := (0 : g_0) = \{r \in R \mid g_0r = 0\}$ is a right 0-modular right ideal of R. Therefore, L contains the largest right quasiregular right ideal of R. Since T is a right quasiregular right ideal of R, $T \subseteq L$, that is, $g_0T = \{0\}$. Let $g \in G$ and $t \in T$. Now $g = g_0r$ for some $r \in R$. $gt = g_0(rt) = 0$, as $rt \in T$. Therefore, $GT = \{0\}$. □

Since R_c is right quasiregular in R, we have the following.

Corollary 3.4. *If R_c is a normal subgroup of $(R,+)$, then $GR_c = \{0\}$ for all right R-groups G of type 0.*

Corollary 3.5. *Let $R \in \mathcal{F}$. If G is a right R-group of type 0, then $GJ_0^r(R) = \{0\}$.*

Proof. Let G be a right R-group of type 0. We have that $I := J_0^r(R)$ is the largest right quasiregular ideal of R. Since R_c is a right quasiregular ideal of R, $R_c \subseteq I$. So I is an invariant ideal of R. Therefore, by Proposition 3.3, $GI = \{0\}$. □

Proposition 3.6. *Let $R \in \mathcal{F}$. Let I be an ideal of R and $K := I + R_c$. If G is a right K-group of type 0, then G is a right I-group of type 0.*

Proof. Suppose that G is a right K-group of type 0 and g_0 is a generator of G. So g_0 is distributive over K and $g_0K = G$. Let K_c be the constant part of K. Since $K_c = R_c$ is a normal subgroup of K, by Corollary 3.4, $GR_c = \{0\}$. Clearly, G is a right I-group. Now $G = g_0K = g_0(I + R_c) = g_0I$, and hence g_0 is a generator of the right I-group G. Let H be a nonzero ideal of the right I-group G. Let $h \in H$ and $k \in K$. $k = i + r_c$, $i \in I$, $r_c \in R_c$ and $h = g_0 t$, $t \in I$. $hk = g_0 t(i + r_c) = g_0((t(i + r_c) - ti) + ti) = g_0(t(i + r_c) - ti) + g_0(ti) = 0 + (g_0 t)i = hi \in H$. Therefore, H is a nonzero ideal of the right K-group G and hence $H = G$. So G is a right I-group of type 0. □

We show now that the Hoehnke radical J_0^r is complete in the variety $\mathcal{F}$.

Theorem 3.7. *Let $R \in \mathcal{F}$. If I is an ideal of R and $J_0^r(I) = I$, then $I \subseteq J_0^r(R)$.*

Proof. Let I be an ideal of R and $J_0^r(I) = I$. Suppose that $I \not\subseteq J_0^r(R)$. So $K := I + R_c$ is an ideal of R and $K \not\subseteq J_0^r(R)$. We get a right R-group G of type 0 such that $GK \neq \{0\}$. Since K is an invariant ideal of R, by Theorem 3.2, G is a right K-group of type 0. Therefore, by Proposition 3.6, G is a right I-group of type 0. This is a contradiction to the fact that $J_0^r(I) = I$. Therefore, $I \subseteq J_0^r(R)$. □

Theorem 3.8. *J_0^r is a complete Hoehnke radical in the variety $\mathcal{F}$.*

Theorem 3.9. *J_0^r is a complete Hoehnke radical in the class of all zero-symmetric near-rings.*

Theorem 3.10. *Suppose that S is an invariant subnear-ring of R. If G is a right S-group of type 0, then G is also a right R-group of type 0.*

Proof. Suppose that G is a right S-group of type 0 and g_0 is a generator. We have that g_0 is distributive over S and $g_0 S = G$. For $g \in G$ and $r \in R$, define $gr := g_0(sr)$, if $g = g_0 s$, $s \in S$. We show now that this operation is well defined. Suppose that $g = g_0 s = g_0 t$, $s, t \in S$. Let $r \in R$ and $h := g_0(sr) - g_0(tr)$. Now $hk = (g_0(sr) - g_0(tr))k = g_0((sr)k) - g_0((tr)k) = g_0(s(rk)) - g_0(t(rk)) = g(rk) - g(rk) = 0$ for all $k \in S$. Therefore, $hS = \{0\}$, and hence $h = 0$, that is, $g_0(sr) = g_0(tr)$. We show that G is a right R-group of type 0. It is clear that G is a right R-group. $g_0 = g_0 e$ for some $e \in S$. Now $G \supseteq g_0 R = g_0(eR) \supseteq g_0(eS) = g_0 S = G$. So $g_0 R = G$. Let $p, q \in R$ and $x = g_0(p + q) - (g_0 p + g_0 q)$. $xs = (g_0(p + q) - (g_0 p + g_0 q))s = (g_0(p + q))s - ((g_0 p + g_0 q))s = g_0(ps + qs) - (g_0 ps + g_0 qs) = (g_0(ps) + g_0(qs)) - (g_0(ps) + g_0(qs)) = 0$ for all $s \in S$. Therefore, $x = 0$, and hence g_0 is a generator of the right R-group G. It can be easily verified that the action of R on G is an extension of the action of S on G. So an ideal of the right R-group G is also an ideal of the right S-group G. Since the right S-group G has no nontrivial ideals, the right R-group G also has no nontrivial ideals. Therefore, G is also a right R-group of type 0. □

We show now that the Hoehnke radical J_0^r is idempotent in the variety $\mathcal{F}$.

Theorem 3.11. *Let $R \in \mathcal{F}$. Then $J_0^r(J_0^r(R)) = J_0^r(R)$.*

Proof. Let $I := J_0^r(R)$. I is the largest right quasiregular ideal of R. Since R_c is a right quasiregular ideal of R, $R_c \subseteq I$. So I is an invariant ideal of R. Suppose that $J_0^r(I) \neq I$. So there is a right I-group G of type 0. By Theorem 3.10, G is an R-group of type 0. Now, by Corollary 3.5, $GI = GJ_0^r(R) = \{0\}$. This is a contradiction to the fact that G is an I-group of type 0. Therefore, $J_0^r(I) = I$, that is, $J_0^r(J_0^r(R)) = J_0^r(R)$. □

From Theorems 3.7 and 3.11, we have the following.

Theorem 3.12. *J_0^r is a Kurosh-Amitsur radical in the variety $\mathcal{F}$.*

Theorem 3.13. *J_0^r is a Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

Theorem 3.14. *J_0^r is not s-hereditary in the class of all zero-symmetric near-rings.*

Proof. Consider $G := Z_8$, the group of integers under addition modulo 8. Now $T: G \to G$, defined by $T(g) = 5g$, for all $g \in G$, is an automorphism of G. T fixes 0, 2, 4, and 6, and maps 1 to 5 and 3 to 7. $A := \{I, T\}$ is an automorphism group of G. $\{0\}$, $\{2\}$, $\{4\}$, $\{6\}$, $\{1,5\}$, and $\{3,7\}$ are the orbits. Let R be the centralizer near-ring $M_A(G)$, the near-ring of all self maps of G which fix 0 and commute with T. An element of R is completely determined by its action on $\{1,2,3,4,6\}$. An element $f \in R$ maps $2G$ into $2G$ and $f(1)$ and $f(3)$ are arbitrary in G. This example was considered in [9] and showed that $P := (0 : 2G) = \{f \in R \mid f(h) = 0, \text{ for all } h \in 2G\}$ is the only nontrivial ideal of R. Let f_0 be the element of P which fixes all the elements in $G - 2G$. Clearly $f - f_0 f \in (2G : G) = \{t \in R \mid t(G) \subseteq 2G\}$ for all $f \in R$. Since $(2G : G)$ is a proper right ideal of R, f_0 is not right quasiregular in R. So P is not a right quasiregular ideal of R. Since R is a near-ring with identity, it is not right quasiregular. Therefore, $\{0\}$ is the largest right quasiregular ideal of R, and hence $J_0^r(R) = \{0\}$. So R is J_0^r-semisimple. It is shown in [9] that $K := (4G : G)_P$ is a nonzero ideal of P and $K^2 = \{0\}$. Since a nil ideal is right quasiregular, K is a right quasiregular ideal of P. Therefore, $\{0\} \neq K \subseteq J_0^r(P)$ and hence P is not J_0^r-semisimple. So J_0^r is not s-hereditary in the class of all zero-symmetric near-rings. □

Corollary 3.15. *J_0^r is not s-hereditary in the class of all near-rings.*

Theorem 3.16. *J_0^r is not an ideal-hereditary radical in the class of all zero-symmetric near-rings.*

It is not known to the authors whether J_0^r is a KA-radical in the class of all near-rings. J_0^r may fail to be idempotent and thus Kurosh-Amitsur in the class of all near-rings.

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