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Research Article

The Dirichlet Problem for the Equation $\Delta u - k^2 u = 0$ in the Exterior of Nonclosed Lipschitz Surfaces

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We study the Dirichlet problem for the equation $\Delta u - k^2 u = 0$ in the exterior of nonclosed Lipschitz surfaces in R^3. The Dirichlet problem for the Laplace equation is a particular case of our problem. Theorems on existence and uniqueness of a weak solution of the problem are proved. The integral representation for a solution is obtained in the form of single-layer potential. The density in the potential is defined as a solution of the operator (integral) equation, which is uniquely solvable.

Weak solvability of elliptic boundary value problems with Dirichlet, Neumann, and mixed Dirichlet-Neumann boundary conditions in Lipschitz domains has been studied in [1–6]. It is pointed out in the book [1, page 91] that domains with cracks (cuts) are not Lipschitz domains. So, solvability of elliptic boundary value problems in domains with cracks does not follow from general results on solvability of elliptic boundary value problems in Lipschitz domains. In the present paper, the weak solvability of the Dirichlet problem for the equation $\Delta u - k^2 u = 0$ in the exterior of nonclosed Lipschitz surfaces (cracks) in R^3 is studied. The Dirichlet problem for the Laplace equation is a particular case of our problem. Theorems on existence and uniqueness of a weak solution are proved, integral representation for a solution in the form of single-layer potential is obtained, the problem is reduced to the uniquely solvable operator equation.

The weak solvability of the Neumann problem for the Laplace equation in the exterior of several smooth nonclosed surfaces in R^3 has been studied in [7]. Boundary value problems for the Helmholtz equation in the exterior of smooth nonclosed screens have been studied in [8, 9].

In Cartesian coordinates $x = (x_1, x_2, x_3)$ in R^3 consider bounded Lipschitz domain G with the boundary S, that is, S is closed Lipschitz surface. Let γ be nonempty subset of the boundary S and $\gamma \neq S$. Assume that γ is a nonclosed Lipschitz surface with Lipschitz boundary $\partial\gamma$ in the space R^3, and assume that γ includes its limiting points, or, alternatively, assume that γ is a union of finite number of such nonclosed surfaces, which do not have common points, in particular, they do not have common boundary points. In the latter case, γ is not a connected set. Notice that γ is a closed set. Let us introduce Sobolev spaces on γ as follows:

$$\begin{gathered} H^{1/2}(\gamma) = \left\{ v : \quad v = V|_{\gamma\backslash\partial\gamma}, \quad V \in H^{1/2}(S) \right\}, \\ \widetilde{H}^{-1/2}(\gamma) = \left\{ v : \quad v \in H^{-1/2}(S), \quad \operatorname{supp} v \subset \gamma \right\}. \end{gathered} \tag{1}$$

Spaces $H^{1/2}(\gamma)$ and $\widetilde{H}^{-1/2}(\gamma)$ are dual spaces in the sense of scalar product in $L_2(\gamma)$ [1, pages 91-92]. Furthermore, one can set $\|v\|_{\widetilde{H}^{-1/2}(\gamma)} = \|v\|_{H^{-1/2}(S)}$ for $v \in \widetilde{H}^{-1/2}(\gamma)$ (see [1, page 79]), and $\|v\|_{H^{1/2}(\gamma)} = \min_{V|_{\gamma\backslash\partial\gamma}=v, V\in H^{1/2}(S)} \|V\|_{H^{1/2}(S)}$ (see [1, pages 77, 99]). Spaces $H^{1/2}(S)$ and $H^{-1/2}(S)$ on closed Lipschitz surface S and their norms are defined, for example, in [1, page 98].

Let Δ be Laplacian in R^3, then for the equation

$$\Delta u(x) - k^2 u(x) = 0, \quad k = \text{const} \geq 0, \tag{2}$$

consider the single-layer potential

$$U[h](x) = \frac{1}{4\pi} \int_S h(y) \frac{\exp(-k|x-y|)}{|x-y|} ds_y, \tag{3}$$

with the density $h \in H^{-1/2}(S)$. The function (3) is defined for $x \in R^3 \setminus S$. According to Theorem 6.11 in [1], the potential

$U[h](x)$ belongs to $H^1_{\text{loc}}(R^3)$ and does not have jump on S, when approaching S from G and from $R^3 \setminus \overline{G}$ it has the same trace $U[h]|_S \in H^{1/2}(S)$. The overline means closure. Moreover, potential $U[h](x)$ belongs to $C^\infty(R^3 \setminus S)$ (see [1, page 202]), obeys (2) in $R^3 \setminus S$, and satisfies conditions at infinity

$$u = O\left(|x|^{-1}\right), \quad |\nabla u| = o\left(|x|^{-1}\right), \quad |x| \longrightarrow \infty. \tag{4}$$

Lemma 1. *Let $h \in H^{-1/2}(S)$, $k > 0$, and S is a boundary of an open bounded Lipschitz domain G. Then there is such a constant $c > 0$, that inequality*

$$\left(U[h]|_S, h\right)_{L_2(S)} \geq c\|h\|^2_{H^{-1/2}(S)} \tag{5}$$

holds.

Proof. Note that normal vector exists on the Lipschitz surface almost everywhere [1, page 96]. Let B_r be an open ball of the radius r with the center in the origin and $\overline{G} \subset B_r$. By n denote outward (with respect to G) unit normal vector on S where exists and outward unit normal vector on ∂B_r. Writing down Green's formula [1, page 118] for the function $U[h](x)$ in $B_r \setminus \overline{G}$ and in G, we obtain

$$\begin{aligned} \|\nabla U[h]\|^2_{L_2(B_r\setminus\overline{G})} &+ k^2\|U[h]\|^2_{L_2(B_r\setminus\overline{G})} \\ &= -\left(U[h]|_S, \left(\frac{\partial U[h]}{\partial n}\right)^+\right)_{L_2(S)} \\ &\quad + \left(U[h], \frac{\partial U[h]}{\partial n}\right)_{L_2(\partial B_r)}, \end{aligned} \tag{6}$$

$$\|\nabla U[h]\|^2_{L_2(G)} + k^2\|U[h]\|^2_{L_2(G)} = \left(U[h]|_S, \left(\frac{\partial U[h]}{\partial n}\right)^-\right)_{L_2(S)}. \tag{7}$$

By $(\partial U[h]/\partial n)^-$ and $(\partial U[h]/\partial n)^+$, we mean traces of normal derivative of the function $U[h](x)$ on S when approaching S from G and from $R^3 \setminus \overline{G}$, respectively. According to Theorem 6.11 in [1], traces $(\partial U[h]/\partial n)^+$ and $(\partial U[h]/\partial n)^-$ of the normal derivative of the function $U[h](x)$ exist and belong to $H^{-1/2}(S)$. Remind that under conditions of the lemma, the function $U[h](x)$ has the same trace $U[h]|_S \in H^{1/2}(S)$ when approaching S both from G and from $R^3 \setminus \overline{G}$. Since spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ are dual, the scalar products are defined in $L_2(S)$ in right sides of (6) and (7). Tending $r \to \infty$ in (6) and taking into account that the potential $U[h](x)$ satisfies conditions (4), we obtain

$$\begin{aligned} \|\nabla U[h]\|^2_{L_2(R^3\setminus\overline{G})} &+ k^2\|U[h]\|^2_{L_2(R^3\setminus\overline{G})} \\ &= -\left(U[h]|_S, \left(\frac{\partial U[h]}{\partial n}\right)^+\right)_{L_2(S)}. \end{aligned} \tag{8}$$

By Theorem 6.11 in [1], the jump of the normal derivative of potential $U[h]$ on S is given by the following formula:

$$\left(\frac{\partial U[h]}{\partial n}\right)^- - \left(\frac{\partial U[h]}{\partial n}\right)^+ = h. \tag{9}$$

Adding (7) and (8), we obtain

$$\|\nabla U[h]\|^2_{L_2(R^3\setminus S)} + k^2\|U[h]\|^2_{L_2(R^3\setminus S)} = \left(U[h]|_S, h\right)_{L_2(S)}. \tag{10}$$

Since $L_2(R^3 \setminus S) = L_2(R^3)$ and $U[h] \in H^1_{\text{loc}}(R^3)$, then taking into account the theorem on equivalence of Sobolev spaces [1, Theorem 3.16], we observe, that there is such a constant $c_1 > 0$, for which inequality holds

$$\begin{aligned} c_1\|U[h]\|^2_{H^1(R^3)} &\leq \min\left\{k^2, 1\right\}\left(\|\nabla U[h]\|^2_{L_2(R^3\setminus S)} + \|U[h]\|^2_{L_2(R^3\setminus S)}\right) \\ &\leq \left(U[h]|_S, h\right)_{L_2(S)}. \end{aligned} \tag{11}$$

Using inequality for single-layer potential from [1, page 227] (it follows from [1, Lemma 4.3]), for some constant $c_2 > 0$, we obtain

$$\begin{aligned} \|h\|^2_{H^{-1/2}(S)} &= \left\|\left(\frac{\partial U[h]}{\partial n}\right)^+ - \left(\frac{\partial U[h]}{\partial n}\right)^-\right\|^2_{H^{-1/2}(S)} \\ &\leq c_2\left\|U_0[h]\right\|^2_{H^1(R^3)}. \end{aligned} \tag{12}$$

Here $U_0[h](x) = \delta(x)U[h](x)$, where $\delta(x) \in C^\infty(R^3)$ is a cutoff function, such that $\delta(x) \leq 1$ for all $x \in R^3$, $\delta(x) \equiv 1$ in an open bounded domain containing $\overline{G}$, and $\delta(x) \equiv 0$ in the exterior of some ball with the center in the origin. Clearly,

$$\left\|U_0[h]\right\|^2_{H^1(R^3)} \leq c_3\|U[h]\|^2_{H^1(R^3)} \tag{13}$$

for some constant $c_3 > 0$, so

$$\|h\|^2_{H^{-1/2}(S)} \leq c_2c_3\|U[h]\|^2_{H^1(R^3)}. \tag{14}$$

Using (11), we obtain

$$c\|h\|^2_{H^{-1/2}(S)} \leq \left(U[h]|_S, h\right)_{L_2(S)}, \quad c = \frac{c_1}{c_2c_3}. \tag{15}$$

Lemma is proved. □

Let us formulate the Dirichlet problem for (2) in the exterior of nonclosed Lipschitz surfaces γ.

Problem $\mathscr{D}$. Find a function $u(x) \in H^1_{\text{loc}}(R^3) \cap C^2(R^3 \setminus \gamma)$, that obeys (2) in $R^3 \setminus \gamma$, satisfies the boundary condition

$$u|_\gamma = f \in H^{1/2}(\gamma) \tag{16}$$

and conditions at infinity (4).

Note that Lapalce equation $\Delta u = 0$ is a particular case of (2) as $k = 0$. So, the Dirichlet problem for Laplace equation is included in the Problem $\mathscr{D}$.

Boundary condition (16) implies that the function $u(x)$ has the same trace $u|_\gamma$, when approaching γ from G and from $R^3 \setminus \overline{G}$, and this trace has to satisfy condition (16).

Let us construct the solution of the problem. We look for a solution in the form of a single-layer potential

$$u(x) = U[g](x) = \int_\gamma g(y) \frac{\exp(-k|x-y|)}{4\pi|x-y|} ds_y = \int_S g(y) \frac{\exp(-k|x-y|)}{4\pi|x-y|} ds_y \tag{17}$$

with the density $g \in \widetilde{H}^{-1/2}(\gamma) \subset H^{-1/2}(S)$. The function (17) is defined as $x \in R^3 \setminus \gamma$.

It follows from aforementioned properties of a single-layer potential (3) that the potential $U[g](x)$ belongs to $H^1_{\text{loc}}(R^3)$, has a trace on $S : U[g]|_S \in H^{1/2}(S)$, and has a trace on $\gamma : U[g]|_\gamma \in H^{1/2}(\gamma)$. Furthermore, the potential $U[g](x)$ belongs to $C^\infty(R^3 \setminus \gamma)$ (see [1, page 202]), satisfies (2) in $R^3 \setminus \gamma$, and conditions at infinity (4). Therefore, for any function g from the space $\widetilde{H}^{-1/2}(\gamma)$, the potential $U[g](x)$ satisfies all conditions of the Problem $\mathscr{D}$, except for the boundary condition (16). We have to find the function $g \in \widetilde{H}^{-1/2}(\gamma)$ to satisfy the boundary condition (16). Substituting (17) into the boundary condition (16), we arrive at the operator equation

$$U[g]|_\gamma = f \in H^{1/2}(\gamma). \tag{18}$$

Here by $U[g]|_\gamma$, we mean the trace of the function (17) on γ, this trace belongs to $H^{1/2}(\gamma)$. To prove solvability of (18), we have to study properties of the operator in the left side of the equation.

Operator U is bounded when acting from $H^{-1/2}(S)$ into $H^{1/2}(S)$ by Theorem 6.11 in [1], so when acting from $\widetilde{H}^{-1/2}(\gamma) \subset H^{-1/2}(S)$ into $H^{1/2}(S)$ it is bounded as well. If a set of functions is bounded (in norm) in $H^{1/2}(S)$, by a constant, then set of restrictions of these functions to γ is bounded (in norm) in $H^{1/2}(\gamma)$ also and by the same constant. Therefore, the operator U is bounded when acting from $\widetilde{H}^{-1/2}(\gamma)$ into $H^{1/2}(\gamma)$. Since $g \in \widetilde{H}^{-1/2}(\gamma) \subset H^{-1/2}(S)$, we have for $k \geq 0$

$$\left(U[g]|_S, g\right)_{L_2(S)} = \left(U[g]|_\gamma, g\right)_{L_2(\gamma)} \geq c\|g\|^2_{H^{-1/2}(S)} = c\|g\|^2_{\widetilde{H}^{-1/2}(\gamma)}. \tag{19}$$

If $k > 0$, then this estimate follows from Lemma 1, while if $k = 0$, then this estimate is proved in Corollary 8.13 in [1]. Therefore, for some constant $c > 0$, we have

$$\left(U[g]|_\gamma, g\right)_{L_2(\gamma)} \geq c\|g\|^2_{\widetilde{H}^{-1/2}(\gamma)}. \tag{20}$$

Note, that the operator U acts from $\widetilde{H}^{-1/2}(\gamma)$ into $H^{1/2}(\gamma)$ and is bounded, while spaces $\widetilde{H}^{-1/2}(\gamma)$, $H^{1/2}(\gamma)$ are dual in the sense of scalar product in $L_2(\gamma)$. Inequality (20) implies that the operator U is positive and bounded below. Consequently, from Lemma 2.32 in [1, page 43], it follows that the operator U is invertible (it has bounded inverse operator). Therefore, (18) has unique solution $g \in \widetilde{H}^{-1/2}(\gamma)$ for any function $f \in H^{1/2}(\gamma)$. The potential (17), constructed on this solution, satisfies all conditions of the Problem $\mathscr{D}$. From above considerations it follows the theorem.

Theorem 2. *The solution of the Problem $\mathscr{D}$ exists and is given by formula* (17), *where $g \in \widetilde{H}^{-1/2}(\gamma)$ is a solution of* (18), *which is uniquely solvable in $\widetilde{H}^{-1/2}(\gamma)$.*

Let us prove the uniqueness of a solution to the Problem $\mathscr{D}$.

Theorem 3. *The Problem $\mathscr{D}$ has at most one solution.*

Proof. Let $u(x)$ be a solution of the homogeneous Problem $\mathscr{D}$. Consider the ball B_r of enough large radius r with the center in the origine. Suppose that $\overline{G} \subset B_r$ and $\overline{G} \cap \partial B_r = \emptyset$. The overline means closure, while ∂B_r is a sphere, the boundary of the ball B_r. Since $u \in H^1_{\text{loc}}(R^3)$, the Green's formulae [1, Theorem 4.4, page 118]

$$\|\nabla u\|^2_{L_2(G)} + k^2\|u\|^2_{L_2(G)} = \left(u|_S, \left(\frac{\partial u}{\partial n}\right)^-\right)_{L_2(S)}, \tag{21}$$

$$\|\nabla u\|^2_{L_2(B_r\setminus\overline{G})} + k^2\|u\|^2_{L_2(B_r\setminus\overline{G})} = -\left(u|_S, \left(\frac{\partial u}{\partial n}\right)^+\right)_{L_2(S)} + \left(u, \frac{\partial u}{\partial n}\right)_{L_2(\partial B_r)} \tag{22}$$

hold for the function u. By n on ∂B_r, the outward (regarding to B_r) unite normal vector is understood, while by n on S, the outward (regarding to G) unite normal vector is understood (where exists). By $(\partial u/\partial n)^-$ and $(\partial u/\partial n)^+$, we denote the traces of the normal derivative of the function $u(x)$ on S when approaching to S from G and from $R^3 \setminus \overline{G}$, respectively. Since the function $u(x)$ belongs to $H^1_{\text{loc}}(R^3)$, the traces of this function exist on S when approaching both from G and from $R^3 \setminus \overline{G}$. According to the formulation of the Problem $\mathscr{D}$, these traces are the same, they are denoted by $u|_S$ and belong to $H^{1/2}(S)$ (see [1, Theorems 3.37, 3.38, page 102]). Since, in addition, the function $u(x)$ obeys (2) outside S, the traces $(\partial u/\partial n)^+$ and $(\partial u/\partial n)^-$ of the normal derivative of the function u exist and belong to $H^{-1/2}(S)$ by Lemma 4.3 in [1]. Since spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ are dual, the scalar product in $L_2(S)$ in the right sides of (21) and (22) is defined. Note that $u|_\gamma = 0 \in H^{1/2}(\gamma)$, since u is a solution of the homogeneous

Problem $\mathscr{D}$. Moreover, $(\partial u/\partial n)^+ = (\partial u/\partial n)^-$ on $S\setminus\gamma$. Adding (21) and (22), we obtain

$$\begin{aligned}\|\nabla u\|^2_{L_2(B_r\setminus S)} + k^2\|u\|^2_{L_2(B_r\setminus S)} &= \left(u, \frac{\partial u}{\partial n}\right)_{L_2(\partial B_r)} \\ &= \int_{\partial B_r} u\frac{\partial u}{\partial n}ds.\end{aligned} \tag{23}$$

Using conditions (4) at infinity, we obtain from (23) as $r \to \infty$

$$\begin{aligned}&\lim_{r\to\infty}\left(\|\nabla u\|^2_{L_2(B_r\setminus S)} + k^2\|u\|^2_{L_2(B_r\setminus S)}\right) \\ &\quad = \|\nabla u\|^2_{L_2(R^3\setminus S)} + k^2\|u\|^2_{L_2(R^3\setminus S)} = 0.\end{aligned} \tag{24}$$

Since $k \geq 0$, we have that $u \equiv c_1$ in G and $u \equiv c_2$ in $R^3\setminus G$, where c_1 and c_2 are some constants. Furthermore, since $u \in C^2(R^3 \setminus \gamma)$, we observe that $c_1 = c_2$ and $u \equiv$ const in $R^3 \setminus \gamma$. Taking into account conditions at infinity (4), we have const $= 0$, so $u \equiv 0$ in $R^3\setminus\gamma$. Thus, the homogeneous Problem $\mathscr{D}$ has only the trivial solution. In view of the linearity of the Problem $\mathscr{D}$, the inhomogeneous Problem $\mathscr{D}$ has at most one solution. The theorem is proved. □

In conclusion we note that the paper [10] treats the Dirichlet problem for the Laplace equation in planar domains with cracks without compatibility conditions at the tips of the cracks. The well-posed classical formulation of the problem is given. It is shown that classical solution exists and unique, while weak solution in H^1_{loc} space does not exist typically.

In addition, the Dirichlet problem for the Laplace equation in a planar domain with cracks with compatibility conditions at the tips of the cracks has been studied in [11] (bounded domain) and in [12] (unbounded domain). The Dirichlet problem for the Helmholtz equation in both bounded and unbounded planar domains with cracks with compatibility conditions at the tips of the cracks has been treated in [13, 14]. Furthermore, problems in [11–14] have been reduced to the uniquely solvable integral equations of the 2nd kind and index zero. Moreover, theorems on uniqueness and existence of a classical solution have been proved in [11–14], and integral representations for solutions in the form of potentials have been obtained.

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