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BOUNDEDNESS AND MONOTONICITY OF PRINCIPAL EIGENVALUES FOR BOUNDARY VALUE PROBLEMS WITH INDEFINITE WEIGHT FUNCTIONS

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We study the principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem: $-\Delta u(x) = \lambda g(x)u(x)$, $x \in D$; $(\partial u/\partial n)(x) + \alpha u(x) = 0$, $x \in \partial D$, where Δ is the standard Laplace operator, D is a bounded domain with smooth boundary, $g : D \to \mathbb{R}$ is a smooth function which changes sign on D and $\alpha \in \mathbb{R}$. We discuss the relation between α and the principal eigenvalues.

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1. Introduction. We investigate the property of principal eigenvalues for the boundary value problem

$$\begin{aligned} -\Delta u(x) &= \lambda g(x)u(x), \quad x \in D, \\ \frac{\partial u}{\partial n}(x) + \alpha u(x) &= 0, \quad x \in \partial D, \end{aligned} \tag{1.1}$$

where D is a bounded region in $\mathbb{R}^N$ with smooth boundary, $g : D \to \mathbb{R}$ is a smooth function which changes sign on D and $\alpha \in \mathbb{R}$.

Such problems have been studied in recent years since Fleming [4] studied the following associated nonlinear problems arising in the study of population genetics:

$$u_t(x,t) = \Delta u + \lambda g(x) f(u), \quad x \in D, \tag{1.2}$$

where f is some function of class C^1 such that $f(0) = 0 = f(1)$.

Fleming's results suggested that nontrivial steady-state solutions were bifurcating the trivial solutions $u \equiv 0$ and $u \equiv 1$. In order to investigate these bifurcation phenomena, it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

$$-\Delta u(x) = \lambda g(x)u(x), \quad x \in D. \tag{1.3}$$

The study of the linear ordinary differential equation case, however, goes back to Bocher [3]. Attention has been confined mainly to the cases of Dirichlet ($\alpha = \infty$) and Neumann boundary conditions.

In the case of Dirichlet boundary conditions, it is well known (see [5]) that there exists a double sequence of eigenvalues for (1.1)

$$\cdots < \lambda_2^- < \lambda_1^+ < 0 < \lambda_1^+ < \lambda_2^+ < \cdots, \tag{1.4}$$

$\lambda_1^+(\lambda_1^-)$ being the unique positive (negative) principal eigenvalue, that is, (1.1) has solution $u(v)$ which is positive in D. It is also well known that the case where $0<\alpha<\infty$ is similar to the Dirichlet case.

In the case of Neumann boundary conditions, 0 is clearly a principal eigenvalue and there is a positive (negative) principal eigenvalue if and only if $\int_D g(x)dx<0$ (>0); in the case where $\int_D g(x)dx=0$ there are no positive and no negative principal eigenvalues.

We show that the set of λ's such that λ is a principal eigenvalue of (1.1) is a bounded set and its bounds are independent of α, and also the positive principal eigenvalue λ of (1.1) is strictly an increasing function of α.

Our analysis is based on a method used by Hess and Kato [5]. Consider, for fixed λ, the eigenvalue problem

$$\begin{aligned} -\Delta u(x)-\lambda g(x)u(x)&=\mu u(x), \quad x\in D,\\ \frac{\partial u}{\partial n}(x)+\alpha u(x)&=0, \quad x\in\partial D. \end{aligned} \tag{1.5}$$

We denote the lowest eigenvalue of (1.5) by $\mu(\alpha,\lambda)$. Let

$$A_{\alpha,\lambda}=\left\{\int_D|\nabla\phi|^2dx+\alpha\int_{\partial D}\phi^2ds_x-\lambda\int_D g\phi^2dx:\phi\in W^{1,2}(D),\ \int_D\phi^2dx=1\right\} \tag{1.6}$$

when $\alpha\ge 0$, it is clear that $A_{\alpha,\lambda}$ is bounded below. It is shown in [6], by using variational arguments, that $\mu(\alpha,\lambda)=\inf A_{\alpha,\lambda}$ and that an eigenfunction corresponding to $\mu(\alpha,\lambda)$ does not change sign on D. Thus, clearly, λ is a principal eigenvalue of (1.1) if and only if $\mu(\alpha,\lambda)=0$.

When $\alpha<0$, the boundedness below of $A_{\alpha,\lambda}$ is no longer obvious a priori, and it is shown by Afrouzi and Brown [2].

2. Boundedness and monotonicity of principal eigenvalues. The following theorem is proved in [1, Theorem 1.8].

Theorem 2.1. *If*

$$\lambda_1=\inf\left\{\int_D\left[|\nabla\phi|^2+q\phi^2\right]dx+\alpha\int_{\partial D}\phi^2ds_x:\phi\in W^{1,2}(D),\ \int_D\phi^2dx=1\right\}, \tag{2.1}$$

where $q\in L^\infty(D)$, then there exists $\phi_1\in W^{1,2}(D)$, $\int_D\phi_1^2dx=1$, such that

$$\lambda_1=\int_D\left[|\nabla\phi_1|^2+q\phi_1^2\right]dx+\alpha\int_{\partial D}\phi_1^2ds_x. \tag{2.2}$$

Moreover, λ_1 is the principal eigenvalue and $\phi_1>0$ is a principal eigenfunction of

$$\begin{aligned} -\Delta u(x)+q(x)u(x)&=\lambda u(x), \quad x\in D,\\ \frac{\partial u}{\partial n}(x)+\alpha u(x)&=0, \quad x\in\partial D. \end{aligned} \tag{2.3}$$

It is obvious that λ_1 *is the principal eigenvalue of (1.1) if and only if* 0 *is the principal eigenvalue of*

$$\begin{aligned} -\Delta u(x) - \lambda_1 g(x)u(x) &= \mu u(x), \quad x \in D, \\ \frac{\partial u}{\partial n}(x) + \alpha u(x) &= 0, \quad x \in \partial D. \end{aligned} \tag{2.4}$$

Here we are ready to prove one of the main results of this section about the uniformly boundedness of principal eigenvalues of (1.1) with respect to α.

Theorem 2.2. *There exist* $m < 0$ *and* $M > 0$ *such that if* λ *is a principal eigenvalue of (1.1), then* $\lambda \in [m, M]$ *and also* m, M *are independent of* α.

Proof. Suppose that λ_1 is a principal eigenvalue of (1.1). Then 0 is a principal eigenvalue of (2.4) and so by Theorem 2.1, we have

$$0 = \inf\left\{\int_D |\nabla\phi|^2 + \alpha\int_{\partial D}\phi^2 ds_x - \lambda_1\int_D g\phi^2 dx : \phi \in W^{1,2}(D),\ \int_D \phi^2 dx = 1\right\}. \tag{2.5}$$

Now, by considering test functions $\phi_1, \phi_2 \in C_0^\infty(D)$ such that $\int_D \phi_1^2 dx = 1$ and $\int_D g\phi_1^2 dx > 0$ also $\int_D \phi_2^2 dx = 1$ and $\int_D g\phi_2^2 dx < 0$ we have

$$0 \le \int_D |\nabla\phi_1|^2 + \alpha\int_{\partial D}\phi_1^2 ds_x - \lambda_1\int_D g\phi_1^2 dx \tag{2.6}$$

and also

$$0 \le \int_D |\nabla\phi_2|^2 + \alpha\int_{\partial D}\phi_2^2 ds_x - \lambda_1\int_D g\phi_2^2 dx. \tag{2.7}$$

Hence from (2.6) and (2.7) we obtain, respectively,

$$\lambda_1 \le \frac{\int_D |\nabla\phi_1|^2 dx}{\int_D g\phi_1^2 dx}, \qquad \frac{\int_D |\nabla\phi_2|^2 dx}{\int_D g\phi_2^2 dx} \le \lambda_1. \tag{2.8}$$

So by assuming $M = \int_D |\nabla\phi_1|^2 dx / \int_D g\phi_1^2 dx$ and $m = \int_D |\nabla\phi_2|^2 dx / \int_D g\phi_2^2 dx$, we have obtained $\lambda \in [m, M]$, and also we see that m, M are independent of α. □

In the case $0 < \alpha < \infty$, it is known [1, Lemmas 1.18 and 1.19] that problem (1.1) has the unique positive (negative) principal eigenvalue, that is, $\lambda_1^+ (\lambda_1^-)$, such that if u and v are being eigenfunctions corresponding to λ_1^+ and λ_1^-, respectively, then $\int_D gu^2 dx > 0$ and $\int_D gv^2 dx < 0$. Also in the case $\alpha < 0$, the following theorem [2, Theorem 5] is proved.

Theorem 2.3. *There exists* $\alpha_0 \le 0$ *such that*

(i) *if* $\alpha < \alpha_0$*, then (1.1) does not have a principal eigenvalue;*

(ii) *if* $\alpha = \alpha_0$*, then (1.1) has a unique principal eigenvalue with the corresponding eigenfunction* u_0 *such that* $\int_D g(x)u_0^2(x)dx = 0$*;*

(iii) *if* $\alpha > \alpha_0$*, then (1.1) has exactly two principal eigenvalues* λ *and* $\mu, \lambda < \mu$*; if* u_0 *and* v_0 *are eigenfunctions corresponding to* $\lambda < \mu$*, respectively, then* $\int_D g(x)u_0^2(x)dx < 0$ *and* $\int_D g(x)v_0^2(x)dx > 0$*;*

(iv) $\alpha_0 = 0$ *if and only if* $\int_D g(x)dx = 0$.

Now we prove the monotonicity of principal eigenvalues of (1.1) with respect to α.

Theorem 2.4. *Suppose that λ_1 is a principal eigenvalue of*

$$\begin{aligned} -\Delta u(x) &= \lambda g(x)u(x), \quad x \in D, \\ \frac{\partial u}{\partial n}(x) + \alpha_1 u(x) &= 0, \quad x \in \partial D \end{aligned} \tag{2.9}$$

such that the corresponding principal eigenvalue, say u_1, satisfies $\int_D g u_1^2 dx > 0$. If $\alpha_2 > \alpha_1$ and λ_2, u_2 are, respectively, principal eigenvalue and eigenfunction of

$$\begin{aligned} -\Delta u(x) &= \lambda g(x)u(x), \quad x \in D, \\ \frac{\partial u}{\partial n}(x) + \alpha_2 u(x) &= 0, \quad x \in \partial D \end{aligned} \tag{2.10}$$

such that $\int_D g u_2^2 dx > 0$, then $\lambda_2 < \lambda_1$.

Proof. Since λ_1 is a principal eigenvalue of (2.9), so 0 is a principal eigenvalue of

$$\begin{aligned} -\Delta u(x) - \lambda_1 g(x)u(x) &= \mu u(x), \quad x \in D, \\ \frac{\partial u}{\partial n}(x) + \alpha_1 u(x) &= 0, \quad x \in \partial D, \end{aligned} \tag{2.11}$$

and so we have

$$0 = \int_D |\nabla u_1|^2 dx + \alpha_1 \int_{\partial D} u_1^2 ds_x - \lambda_1 \int_D g u_1^2 dx \tag{2.12}$$

and also

$$0 = \inf\left\{ \int_D |\nabla u|^2 dx + \alpha_2 \int_{\partial D} u^2 ds_x - \lambda_2 \int_D g u^2 dx : u \in W^{1,2}(D),\ \int_D u^2 dx = 1 \right\}. \tag{2.13}$$

If $\lambda_2 \geq \lambda_1$, then

$$\begin{aligned} 0 &= \int_D |\nabla u_1|^2 dx + \alpha_1 \int_{\partial D} u_1^2 ds_x - \lambda_1 \int_D g u_1^2 dx \\ &> \int_D |\nabla u_1|^2 dx + \alpha_2 \int_{\partial D} u_1^2 ds_x - \lambda_2 \int_D g u_1^2 dx \\ &\geq 0 \end{aligned} \tag{2.14}$$

which is impossible. Hence $\lambda_2 < \lambda_1$ and the proof is complete. □

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