IJMMS 30:10 (2002) 637–643
PII. S0161171202005331
http://ijmms.hindawi.com

RATE PRESERVATION OF DOUBLE SEQUENCES UNDER l-l TYPE TRANSFORMATION

RICHARD F. PATTERSON

Received 30 May 2000

Following the concepts of divergent rate preservation for ordinary sequences, we present a notion of rates preservation of divergent double sequences under l-l type transformations. Definitions for Pringsheim limit inferior and superior are also presented. These definitions and the notion of asymptotically equivalent double sequences, are used to present necessary and sufficient conditions on the entries of a four-dimensional matrix such that, the rate of divergence is preserved for a given double sequences under l-l type mapping where $l =: \{x_{k,l} : \sum_{k,l=1,1}^{\infty,\infty} |x_{k,l}| < \infty\}$.

2000 Mathematics Subject Classification: 42B15, 40C05.

1. Introduction. The concepts of rates preservation for P-convergence and P-divergence of asymptotically equivalent of double sequences under four-dimensional matrix transformation is presented in [2]. This paper presents necessary and sufficient conditions on the entries of a four-dimensional matrix such that the asymptotic properties of the given sequences are preserved under l-l type mapping where $l =: \{x_{k,l} : \sum_{k,l=1,1}^{\infty,\infty} |x_{k,l}| < \infty\}$.

2. Definitions, notation, and preliminary results

Definition 2.1 (see [5]). A double sequence $[x] = \{x_{k,l}\}$ has *Pringsheim limit* L (denoted by $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k > N$ and $l > N$. We will describe such an $[x]$ more briefly as "P-convergent."

A double sequence $[x]$ is *bounded* if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Note that a convergent double sequence need not be bounded. In 1900, Pringsheim gave the following definition: a double sequence $[x]$ is called *definite divergent* if for every (arbitrarily large) $G > 0$, there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1$, $k \geq n_2$. This definition is clearly equivalent to $P\text{-}\lim[|x|] = \infty$.

Definition 2.2 (see [4]). A number β is called a *Pringsheim limit point* of the double sequence $[x] = \{x_{n,k}\}$ provided that there exists a sequence $[y] = \{y_{n,k}\}$ of $\{x_{n,k}\}$ that has Pringsheim limit $\beta : P\text{-}\lim y_{n,k} = \beta$.

Definition 2.3 (see [4]). The double sequence $[y]$ is a double *subsequence* of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$

and $\{k_j\}$ such that if $\{z_j\} = \{x_{n_j,k_j}\}$, then $[y]$ is formed by

$$\begin{matrix} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & — \\ z_9 & z_8 & z_7 & — \\ — & — & — & —. \end{matrix}$$

Remark 2.4. The definition of a Pringsheim limit point can also be stated as follows: β is a Pringsheim limit point of x provided that there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_i x_{n_i,k_i} = \beta$. In addition to this reformulation of subsequence definition it should be noted that a finite number of unbounded rows and/or columns does not affect the P-convergence or P-divergence of $[x]$ and its subsequences.

Definition 2.5 (see [4]). A double sequence $[x]$ is *divergent in the Pringsheim sense* (P-divergent) provided that $[x]$ does not converge in the Pringsheim sense (P-convergent).

We consider the following notation: $l = \{x_{k,l} : \sum_{k,l=1,1}^{\infty,\infty} |x_{k,l}| < \infty\}$ denoted by l'', $d_A = \{x_{k,l}$: the $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ exists$\}$, P_δ is the set of all real double number sequences such that $x_{k,l} \geq \delta > 0$ for all k, and $l\}$, and P_0 is the set of all nonnegative sequences which have at most a finite number of columns and/or rows with zero entries.

Definition 2.6 (see [2]). Two nonnegative double sequences $[x]$ and $[y]$ are said to be *asymptotically equivalent* if

$$P\text{-}\lim_{k,l} \frac{x_{k,l}}{y_{k,l}} = 1 \tag{2.1}$$

(denoted by $x \overset{P}{\sim} y$).

Definition 2.7. For any double sequence $[x]$, let $[Sx]$ denote the sequence of *partial double sums* whose m and nth term is given by

$$S_{m,n}x := \sum_{k,l \leq m,n} |x_{k,l}|. \tag{2.2}$$

Definition 2.8 (see [3]). Let $[x] = \{x_{k,l}\}$ be a double sequence of real numbers and for each n, let $\alpha_n = \sup_n\{x_{k,l} : k > n \geq$ and $l \geq n\}$. The *Pringsheim limit superior* of $[x]$ is defined as follows:

(1) if $\alpha = +\infty$ for each n, then $P\text{-}\limsup[x] := +\infty$;

(2) if $\alpha < \infty$ for some n, then $P\text{-}\limsup[x] := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{k,l} : k \geq n$ and $l \geq n\}$ then the *Pringsheim limit inferior* of $[x]$ is defined as follows:

(1) if $\beta_n = -\infty$ for each n, then $P\text{-}\liminf[x] := -\infty$;

(2) if $\beta_n > -\infty$ for some n, then $P\text{-}\liminf[x] := \sup_n\{\beta_n\}$.

3. Main results. For sequences x and y not in $l = \{x_k : \sum_{k=1}^{\infty} |x_k| < \infty\}$, Marouf in [1] used partial sum to characterize divergence rate preserving for such sequences under l-l transformation. The following theorem is a multiple-dimensional analog of Marouf theorem.

Theorem 3.1. *If A is a nonnegative real four-dimensional matrix, then the following statements are equivalent:*

(1) *if $[x]$ and $[y]$ are bounded double sequences such that $x \overset{P}{\sim} y$, $[x] \in P_0$, and $[y] \in P_\delta$ for some $\delta > 0$, then*
   - (a) *$[Ax]$ and $[Ay]$ are not in l'' and*
   - (b) $S_{m,n}(Ax) \overset{P}{\sim} S_{m,n}(Ay)$

(2) (a)

$$\left\{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\right\}_{m,n} \notin l''; \tag{3.1}$$

   (b) *for some p and q*

$$P\text{-}\lim_{\alpha,\beta}\left\{\frac{\sum_{m,n=1,1}^{\alpha,\beta} a_{m,n,p,q}}{\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}\right\} = 0. \tag{3.2}$$

Proof. We will begin by proving that (2) implies (1). Suppose (2) holds and $[x]$ and $[y]$ are such that $x \overset{P}{\sim} y$, $[x] \in P_0$, and $[y] \in P_\delta$ for some $\delta > 0$. First we will show that $[Ax]$ and $[Ay]$ are not in l''. Since $[y]$ is in P_δ and 2(a) grant us that

$$\sum_{m,n=1,1}^{\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = +\infty, \tag{3.3}$$

we obtain the following inequality

$$\sum_{m,n=1,1}^{\infty,\infty}(Ay)_{m,n} = \sum_{m,n=1,1}^{\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}y_{k,l} \geq \delta\sum_{m,n=1,1}^{\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}. \tag{3.4}$$

Thus $[Ay]$ is not in l''. Also since $x \overset{P}{\sim} y$, given $\epsilon > 0$ there exist $\bar{K}$ and $\bar{L}$ such that $|x_{k,l}/y_{k,l} - 1| < \epsilon$ whenever $k > \bar{K}$ and $l > \bar{L}$. This implies that

$$(1-\epsilon)y_{k,l} \leq x_{k,l} \leq (1+\epsilon)y_{k,l} \quad \text{for } k > \bar{K},\ l > \bar{L}. \tag{3.5}$$

As a consequence of (3.5), the following inequality holds:

$$\begin{aligned}\sum_{m,n=1,1}^{\infty,\infty}(Ax)_{m,n} &= \sum_{m,n=1,1}^{\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}\\ &\geq (1-\epsilon)\sum_{m,n=1,1}^{\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}y_{k,l}\\ &\geq (1-\epsilon)\delta\sum_{m,n=1,1}^{\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\\ &= +\infty.\end{aligned} \tag{3.6}$$

Thus $[Ax] \notin l''$. Second we show that $S_{m,n}(Ax) \overset{P}{\sim} S_{m,n}(Ay)$. Consider the following:

$$\begin{aligned}
S_{\alpha,\beta}(Ax) &= \sum_{m,n=1,1}^{\alpha,\beta} (Ax)_{m,n} = \sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \\
&= \sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} a_{m,n,k,l} x_{k,l} + \sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=\bar{K},\bar{L}}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \\
&\quad + \sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=\bar{K},l}^{\infty,\bar{L}-1} a_{m,n,k,l} x_{k,l} + \sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} a_{m,n,k,l} x_{k,l} \\
&\le \sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} x_{k,l} \sum_{m,n=1,1}^{\alpha,\beta} \max_{1\le k<\bar{K};1\le l<\bar{L}} \{a_{m,n,k,l}\} \\
&\quad + (1+\epsilon) \sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=\bar{K},\bar{L}}^{\infty,\infty} a_{m,n,k,l} y_{k,l} \\
&\quad + \sum_{k,l=\bar{K},1}^{\infty,\bar{L}-1} x_{k,l} \sum_{m,n=1,1}^{\alpha,\beta} \sup_{\bar{K}\le k<\infty;1\le l\infty} \{a_{m,n,k,l}\} \\
&\quad + \sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} x_{k,l} \sum_{m,n=1,1}^{\alpha,\beta} \sup_{1\le K<\bar{k};\bar{L}\le l\infty} \{a_{m,n,k,l}\}.
\end{aligned} \tag{3.7}$$

As a consequence of the last inequality, we obtain the following:

$$\begin{aligned}
\frac{S_{\alpha,\beta}(Ax)}{S_{\alpha,\beta}(Ay)} &\le \left(\frac{\sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} x_{k,l}}{\delta}\right)\left(\frac{\sum_{m,n=1,1}^{\alpha,\beta} \max_{1\le k<\bar{K};1\le l<\bar{L}} \{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}\right) \\
&\quad + (1+\epsilon) + \frac{\sum_{k,l=\bar{K},1}^{\infty,\bar{L}-1} x_{k,l} \sum_{m,n=1,1}^{\alpha,\beta} \sup_{\bar{K}\le k<\infty;1\le l\infty} \{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\
&\quad + \frac{\sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} x_{k,l} \sum_{m,n=1,1}^{\alpha,\beta} \sup_{1\le K<\bar{k};\bar{L}\le l\infty} \{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}.
\end{aligned} \tag{3.8}$$

Observe that condition 2(b) implies the following:

$$\begin{aligned}
P\text{-}\lim_{\alpha,\beta} \frac{\sum_{m,n=1,1}^{\alpha,\beta} \max_{1\le k<\bar{K};1\le l<\bar{L}} \{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} &= 0, \\
P\text{-}\lim_{\alpha,\beta} \frac{\sum_{k,l=\bar{K},1}^{\infty,\bar{L}-1} \sum_{m,n=1,1}^{\alpha,\beta} \sup_{\bar{K}\le k<\infty;1\le l\infty} \{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} &= 0, \\
P\text{-}\lim_{\alpha,\beta} \frac{\sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} x_{k,l} \sum_{m,n=1,1}^{\alpha,\beta} \sup_{1\le K<\bar{k};\bar{L}\le l\infty} \{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} &= 0.
\end{aligned} \tag{3.9}$$

Therefore,

$$P\text{-}\lim_{\alpha,\beta}\frac{S_{\alpha,\beta}(Ax)}{S_{\alpha,\beta}(Ay)} \leq (1+\epsilon). \tag{3.10}$$

Similar to the above inequalities we obtain

$$\begin{aligned}
S_{\alpha,\beta}(Ax) &= \sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l} + \sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} a_{m,n,k,l}x_{k,l} + \sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=\bar{K},\bar{L}}^{\infty,\infty} a_{m,n,k,l}x_{k,l} \\
&\quad + \sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=\bar{K},l}^{\infty,\bar{L}-1} a_{m,n,k,l}x_{k,l} + \sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} a_{m,n,k,l}x_{k,l} \\
&\geq \sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\min_{1\leq k<\bar{K};1\leq l<\bar{L}}\{a_{m,n,k,l}\} + (1-\epsilon)\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=\bar{K},\bar{L}}^{\infty,\infty} a_{m,n,k,l}y_{k,l} \\
&\quad + \sum_{k,l=\bar{K},l}^{\infty,\bar{L}-1} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\inf_{\bar{K}\leq k<\infty;1\leq l\infty}\{a_{m,n,k,l}\} + \sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\inf_{1\leq K<\bar{k};\bar{L}\leq l\infty}\{a_{m,n,k,l}\} \\
&\geq \sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\min_{1\leq k<\bar{K};1\leq l<\bar{L}}\{a_{m,n,k,l}\} \\
&\quad + (1-\epsilon)\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}y_{k,l} - (1-\epsilon)\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} a_{m,n,k,l}y_{k,l} \\
&\quad - (1-\epsilon)\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=\bar{K},1}^{\infty,\bar{L}-1} a_{m,n,k,l}y_{k,l} - (1-\epsilon)\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} a_{m,n,k,l}y_{k,l} \\
&\quad + \sum_{k,l=\bar{K},1}^{\infty,\bar{L}-1} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\inf_{\bar{K}\leq k<\infty;1\leq l\infty}\{a_{m,n,k,l}\} + \sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\inf_{1\leq K<\bar{k};\bar{L}\leq l\infty}\{a_{m,n,k,l}\},
\end{aligned}$$

$$\begin{aligned}
\frac{S_{\alpha,\beta}(Ax)}{S_{\alpha,\beta}(Ay)} &\geq \left(\frac{\sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} x_{k,l}}{\sup_{k,l} y_{k,l}}\right)\left(\frac{\sum_{m,n=1,1}^{\alpha,\beta}\min_{1\leq k<\bar{K};1\leq l<\bar{L}}\{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}\right) \\
&\quad + (1+\epsilon) - \frac{(1-\epsilon)\sum_{k,l=1,1}^{\bar{K}-1,\bar{L}-1} y_{k,l}}{\delta}\frac{\sum_{m,n=1,1}^{\alpha,\beta}\max_{1\leq k<\bar{K};1\leq l<\bar{L}}\{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\
&\quad - \frac{(1-\epsilon)\sum_{k,l=\bar{K},1}^{\infty,\bar{L}-1} y_{k,l}}{\delta}\frac{\sum_{m,n=1,1}^{\alpha,\beta}\sup_{\bar{K}\leq k<\infty;1\leq l\infty}\{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\
&\quad - \frac{(1-\epsilon)\sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} y_{k,l}}{\delta}\frac{\sum_{m,n=1,1}^{\alpha,\beta}\sup_{1\leq K<\bar{k};\bar{L}\leq l\infty}\{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\
&\quad + \frac{\sum_{k,l=\bar{K},1}^{\infty,\bar{L}-1} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\inf_{\bar{K}\leq k<\infty;1\leq l\infty}\{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\
&\quad + \frac{\sum_{k,l=1,\bar{L}}^{\bar{K}-1,\infty} x_{k,l}\sum_{m,n=1,1}^{\alpha,\beta}\inf_{1\leq K<\bar{k};\bar{L}\leq l\infty}\{a_{m,n,k,l}\}}{\sum_{m,n=1,1}^{\alpha,\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}.
\end{aligned} \tag{3.11}$$

Condition 2(b) implies that each of the nonconstant terms of the last inequality has a Pringsheim limit of zero this yields

$$P\text{-}\lim_{\alpha,\beta}\frac{S_{\alpha,\beta}(Ax)}{S_{\alpha,\beta}(Ay)} \geq 1-\epsilon. \tag{3.12}$$

Therefore,

$$P\text{-}\lim_{\alpha,\beta}\frac{S_{\alpha,\beta}(Ax)}{S_{\alpha,\beta}(Ay)} = 1 \tag{3.13}$$

(that is, $S_{m,n}(Ax) \overset{P}{\sim} S_{m,n}(Ay)$). This completes the sufficiency part of this theorem. To establish the necessary part of this theorem (that is, (1) implies (2)). We form two double sequences $[x]$ and $[y]$ such that $x \overset{P}{\sim} y$, 1(a) and 1(b) holds but $[S_{m,n}(Ax)]$ is not asymptotically equivalent to $[S_{m,n}(Ay)]$. Let $y_{k,l} = 1$ for all k and l and

$$x_{k,l} := \begin{cases} 1, & \text{if } k \geq \alpha,\ l \geq \beta; \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

Observe that $x \overset{P}{\sim} y$, $[x] \in P_0$, and $[y] \in P_1$. Assume that condition 2(a) does not hold. Also let β and α be two fixed positive integers and consider the following inequality

$$\begin{aligned} S_{p,q}(Ax) &= \sum_{m,n=1,1}^{p,q}(Ax)_{m,n} = \sum_{m,n=1,1}^{p,q}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l} \\ &= \sum_{m,n=1,1}^{p,q}\sum_{k,l=\alpha,\beta}^{\infty,\infty} a_{m,n,k,l} = \sum_{m,n=1,1}^{p,q}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \\ &\quad - \sum_{m,n=1,1}^{p,q}\sum_{k,l=\alpha,1}^{\infty,\beta-1} a_{m,n,k,l} - \sum_{m,n=1,1}^{p,q}\sum_{k,l=1,\beta}^{\alpha-1,\infty} a_{m,n,k,l} - \sum_{m,n=1,1}^{p,q}\sum_{k,l=1,1}^{\alpha-1,\beta-1} a_{m,n,k,l} \\ &< \sum_{m,n=1,1}^{p,q}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} - \sum_{m,n=1,1}^{p,q} a_{m,n,\alpha,\beta-1} \\ &\quad - \sum_{m,n=1,1}^{p,q} a_{m,n,\alpha-1,\beta} - \sum_{m,n=1,1}^{p,q} a_{m,n,\alpha-1,\beta-1}. \end{aligned} \tag{3.15}$$

This inequality implies

$$\begin{aligned} \frac{S_{p,q}(Ax)}{S_{p,q}(Ay)} < 1 &- \frac{\sum_{m,n=1,1}^{p,q} a_{m,n,\alpha,\beta-1}}{\sum_{m,n=1,1}^{p,q}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\ &- \frac{\sum_{m,n=1,1}^{p,q} a_{m,n,\alpha-1,\beta}}{\sum_{m,n=1,1}^{p,q}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\ &- \frac{\sum_{m,n=1,1}^{p,q} a_{m,n,\alpha-1,\beta-1}}{\sum_{m,n=1,1}^{p,q}\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}. \end{aligned} \tag{3.16}$$

Thus,

$$P\text{-}\liminf_{p,q} \frac{S_{p,q}(Ax)}{S_{p,q}(Ay)} < 1 - P\text{-}\limsup_{p,q} \frac{\sum_{m,n=1,1}^{p,q} a_{m,n,\alpha,\beta-1}}{\sum_{m,n=1,1}^{p,q} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} - P\text{-}\limsup_{p,q} \frac{\sum_{m,n=1,1}^{p,q} a_{m,n,\alpha-1,\beta}}{\sum_{m,n=1,1}^{p,q} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} - P\text{-}\limsup_{p,q} \frac{\sum_{m,n=1,1}^{p,q} a_{m,n,\alpha-1,\beta-1}}{\sum_{m,n=1,1}^{p,q} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}. \tag{3.17}$$

Therefore,

$$P\text{-}\liminf_{p,q} \frac{S_{p,q}(Ax)}{S_{p,q}(Ay)} < 1. \tag{3.18}$$

This implies that $[S_{m,n}(Ax)]$ is not asymptotically equivalent to $[S_{m,n}(Ay)]$. We need only to show that 2(a) holds. If we let $y_{k,l} = 1$ for all k and l, then

$$(Ay)_{m,n} = \sum_{k,l}^{\infty,\infty} a_{m,n,k,l}. \tag{3.19}$$

Since $[Ay] \notin l''$ then

$$\left\{ \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \right\}_{m,n} \notin l''; \tag{3.20}$$

thus (1) implies (2). This completes the proof of this theorem. □

References

[1] M. Marouf, *Summability matrices that preserve various types of sequential equivalence*, Ph.D. thesis, Kent State University, 1989.

[2] R. F. Patterson, *Some characterization of asymptotic equivalent double sequences*, to appear in Soochow J. Math.

[3] ______, *Double sequence core theorems*, Int. J. Math. Math. Sci. **22** (1999), no. 4, 785–793.

[4] ______, *Analogues of some fundamental theorems of summability theory*, Int. J. Math. Math. Sci. **23** (2000), no. 1, 1–9.

[5] A. Pringsheim, *Zür theorie der zweifach unendlichen zahlenfolgen*, Mathematische Annalen **53** (1900), 289–321.

Richard F. Patterson: Department of Mathematics and Statistics, University of North Florida, Building 11, Jacksonville, FL 32224, USA

E-mail address: rpatters@unf.edu