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MAPPINGS OF TERMINAL CONTINUA

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The concept of a terminal continuum introduced in 1973 by G. R. Gordh Jr., for hereditarily unicoherent continua is extended to arbitrary continua. Mapping properties of these two concepts are investigated. Especially the invariance of terminality under mappings satisfying some special conditions is studied. In particular, we conclude that the invariance holds for atomic mappings.

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Various kinds of nonseparating subcontinua were studied by a number of authors, see, for example, the expository paper [2], where a large amount of information on this subject is given. In the topological literature, or in continuum theory (to be more precise), the term "terminal," when applied either to subcontinua of a given continuum or to points, and the same name "terminal" was assigned to several concepts defined in quite different ways, see, for example, definitions of terminal points or terminal subcontinua of a given continuum in [2, Definition 1.1, page 7], [10, page 461], [12, page 458], [14, page 17], [15, page 190], and [16, Definition 1.54, page 107]. See [2, page 35] for a discussion on relations to some other concepts for which the name "terminal" (or a similar one) is used.

In the present paper we deal with terminal continua as defined by Gordh Jr. in [12]. To avoid any confusion or misunderstanding in the terminology, we have to use another name for the considered concept. Since Gordh restricts his considerations to subcontinua of hereditarily unicoherent continua only, we rename this concept, following [5, Section 3, page 380], as HU-terminal. To formulate the concept and to prove its properties, we have to recall some needed definitions and auxiliary results.

A *continuum* means a compact, connected Hausdorff space. A subcontinuum I of a continuum X is said to be *irreducible about a subset* $S \subset X$ provided that $S \subset I$ and no proper subcontinuum of I contains S. A continuum I is said to be *irreducible* provided that there are two points a and b in I such that I is irreducible about $\{a,b\}$. Then I is said to be irreducible *between* a and b or *from* a *to* b. Each continuum, containing some two points, contains a continuum which is irreducible between them. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two of its subcontinua is connected. A continuum X is hereditarily unicoherent if and only if for each subset S of X there is in X exactly one subcontinuum $I(S)$ irreducible about S (this was shown in [4, T1, page 187] for metric continua only, but the proof works in the nonmetric case as well; compare [12, Remark, page 458]).

Definition 1. A subcontinuum K of a hereditarily unicoherent continuum X is called an *HU-terminal continuum of* X if

(1.1) K is contained in an irreducible subcontinuum of X,

(1.2) for each irreducible subcontinuum I of X containing K, there is a point $x \in X$ such that I is irreducible about the union $K \cup \{x\}$.

For example, each subcontinuum K of the limit segment of the $\sin 1/x$-curve is an HU-terminal continuum of X. Note that if a continuum X is irreducible, and hereditarily unicoherent, then it is an HU-terminal subcontinuum of itself.

Some structural properties of HU-terminal continua, in particular their relations to other similarly defined concepts, can be found in [2, 12]. The present paper is devoted to their mapping properties. A leading problem is to find mappings between hereditarily unicoherent continua which map HU-terminal continua of the domain onto HU-terminal continua of the range, see [5, Question 22, page 382]. Thus the paper can be considered as a continuation of studies from [5], where some mapping properties of other nonseparating subcontinua were examined. However, as it was said in [5, page 381], if we are looking for a suitable class of mappings which preserve the concept of HU-terminality, then no condition expressed in terms of confluence is good enough for this goal. The reader is referred to [5, page 381] for examples of spaces and mappings justifying this statement. Therefore some other conditions concerning the mappings have to be considered.

Investigating conditions (1.1) and (1.2) of Definition 1, one can see that the conditions can be studied independently of the hereditary unicoherence of the continuum X in which the continuum K is located. This leads to the following concept.

Definition 2. A subcontinuum K of a continuum X is called a *G-terminal continuum of X* if conditions (1.1) and (1.2) are satisfied.

Clearly, the concepts of G-terminal and HU-terminal are the same for hereditarily unicoherent continua. If X is irreducible, then it is a G-terminal subcontinuum of itself.

We begin our study of mapping properties of terminal continua with the following result.

Theorem 3. *Let a subcontinuum K of a continuum X be G-terminal, and let a surjective mapping $f : X \to Y$ satisfy the following two conditions:*

(3.1) *for each subcontinuum I of X which is irreducible from a point $p \in K$ to some other point of X and which contains K, its image $f(I)$ is irreducible from $f(p)$ to some point of Y;*

(3.2) *for each irreducible subcontinuum J of Y such that $f(K) \subset J$, the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Then $f(K)$ is a G-terminal subcontinuum of Y.

Proof. Since K is a G-terminal subcontinuum of X, there is, according to (1.1), an irreducible subcontinuum I_0 of X with $K \subset I_0$. Thus $f(K) \subset f(I_0)$. By (1.2) there is a point $x_0 \in X$ such that I_0 is irreducible about $K \cup \{x_0\}$, that is,

$$I_0 = I(K \cup \{x_0\}). \tag{1}$$

Consider two cases. If $x_0 \in K$, then $I_0 = I(K) = K$, thus K is irreducible, and therefore by (3.1) its image $f(K)$ is an irreducible subcontinuum of Y, whence (1.1) holds for

$f(K)$ and Y. If $x_0 \notin K$, then x_0 is a point of irreducibility of I_0. Let $P = \{q \in I_0 : I_0$ is irreducible from x_0 to $q\}$. Then, by (1), $P \cap K \neq \emptyset$, so there is a point $q \in K$ such that I_0 is irreducible from x_0 to q. Thus the continuum $f(I_0)$ is irreducible from $f(x_0)$ to $f(q)$, whence again condition (1.1) holds for $f(K)$ and Y.

To show (1.2) (for $f(K)$ and Y) let J be an irreducible continuum in Y with $f(K) \subset J$. By (3.2) its inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X, and $K \subset f^{-1}(f(K)) \subset f^{-1}(J)$. By (1.2) there is a point $x \in X$ such that $f^{-1}(J) = I(K \cup \{x\})$. Therefore $f^{-1}(J)$ is irreducible from a point $p \in K$ to x. Applying (3.1), we infer that $J = f(f^{-1}(J))$ is irreducible from $f(p)$ to $f(x)$. Thus (1.2) holds as needed, and the proof is complete. □

The next two examples show that the assumptions of Theorem 3 are essential.

Example 4. Condition (3.1) is indispensable in Theorem 3.

Proof. In the plane equipped with the polar coordinate system (ρ,φ), let $C_k = \{(k,\varphi) : \varphi \in [0,2\pi]\}$ for $k \in \{1,2\}$ be two concentric circles, and let

$$S = \left\{(\rho,\varphi) : \rho = \frac{2+e^{\varphi}}{1+e^{\varphi}},\ \varphi \in (-\infty,+\infty)\right\} \tag{2}$$

be the spiral line approximating both C_1 and C_2. Put $X = C_1 \cup S \cup C_2$ and $K = C_1$. Thus K is a G-terminal subcontinuum of X.

Let $f : X \to C_1 = Y$ be the central projection determined by $f((\rho,\varphi)) = (1,\varphi)$ for each $(\rho,\varphi) \in X$. Then condition (3.1) is not satisfied, because for the irreducible continuum $I = X$ containing K its image $f(I) = Y$ is not irreducible. However, condition (3.2) is satisfied vacuously, since there is no irreducible subcontinuum J of Y containing $f(K)$. And $f(K) = Y$ is not a G-terminal subcontinuum of Y since it is contained in no irreducible subcontinuum of Y. □

Example 5. Condition (3.2) is indispensable in Theorem 3.

Proof. Let X be a simple triod with center v and end points a, b, and c, that is, $X = va \cup vb \cup vc$. Choose a point $a' \in av \setminus \{a,v\}$ and let $K = a'v \cup vb$. Thus conditions (1.1) and (1.2) are satisfied, so K is G-terminal, (even HU-terminal), subcontinuum of X.

Define $f : X \to f(X) = Y$ as identification of points b and c only, that is, $f(b) = f(c)$ and the partial mapping $f|(X \setminus \{b,c\})$ is a homeomorphism. Note that each subcontinuum I of X which is irreducible from a point $p \in K$ to some other point of X and which contains K is an arc $xv \cup vb$ where $x \in aa'$, so its image $f(I)$ is an arc $f(xv) \cup f(vb)$ in Y, and thus condition (3.1) is satisfied. Condition (3.2) does not hold, for if $J = f(K)$, then $f^{-1}(J)$ has two components: K and $\{c\}$. Finally, $f(K)$ is not a G-terminal subcontinuum of Y since condition (1.2) is not satisfied: for a point $c' \in vc \setminus \{v,c\} \subset X$ the arc $f(av) \cup f(vb) \cup f(cc') \subset Y$ contains $f(K)$ but it is not irreducible about the union $f(K) \cup \{y\}$ for any point $y \in Y$. □

Since the concepts of G-terminal and HU-terminal are the same for hereditarily unicoherent continua, we have the following corollary to Theorem 3.

Corollary 6. *Let continua X and Y be hereditarily unicoherent and a subcontinuum K of X be HU-terminal. If a surjective mapping $f : X \to Y$ satisfy conditions (3.1) and (3.2), then $f(K)$ is an HU-terminal subcontinuum of Y.*

Remark 7. The converse implications to the ones in Theorem 3 and Corollary 6 are not true. Indeed, in the plane with Cartesian coordinate system put

$$v = (0,0), \qquad a = (-1,0), \qquad b = (0,1), \qquad c = (1,0), \tag{3}$$

let va, vb, and vc be the straight line segments, and let $T = va \cup vb \cup vc$. Define $f : [0,4] \to T$ as a piecewise linear mapping determined by $f(0) = a$, $f(1) = f(3) = v$, $f(2) = b$, and $f(4) = c$. Then $\{0\}$ and $\{4\}$ are degenerate HU-terminal subcontinua of $[0,4]$, their images $\{a\}$ and $\{c\}$ are degenerate HU-terminal subcontinua of T, while none of conditions (3.1) and (3.2) is satisfied.

As another consequence of Theorem 3 we get the following result.

Corollary 8. *Let a surjective mapping $f : X \to Y$ between continua X and Y satisfy the following two conditions:*

(8.1) *for each subcontinuum I of X which is irreducible from a point p of a G-terminal subcontinuum of X to some other point of X, its image $f(I)$ is irreducible from $f(p)$ to some point of Y;*

(8.2) *for each irreducible subcontinuum J of Y containing the image under f of a G-terminal subcontinuum of X, the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Then

(8.3) *for each G-terminal subcontinuum of X, its image is a G-terminal subcontinuum of Y.*

Moreover, if the continua X and Y are assumed to be hereditarily unicoherent, then "G-terminal" can be replaced by "HU-terminal" in (8.1), (8.2), and (8.3).

Since conditions (8.1) and (8.2) are implied by (9.1) and (9.2) below, respectively, we have the next corollary.

Corollary 9. *Let a surjective mapping $f : X \to Y$ between continua X and Y satisfy the following two conditions:*

(9.1) *for each subcontinuum I of X which is irreducible from a point p of X to some other point of X, its image $f(I)$ is irreducible from $f(p)$ to some point of Y;*

(9.2) *for each irreducible subcontinuum J of Y, the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Then

(8.3) *for each G-terminal subcontinuum of X its image is a G-terminal subcontinuum of Y.*

Moreover, if continua X and Y are assumed to be hereditarily unicoherent, then "G-terminal" can be replaced by "HU-terminal" in (8.3).

Now, consequences of Corollary 9 will be presented. We start with some conditions that imply assumption (9.1), and next we discuss other ones that are related to (9.2).

Since condition (9.1) is connected with quasi-monotone mappings, some known results concerning these and other related mappings of continua will be needed. We recall definitions of these mappings first.

A surjective mapping $f : X \to Y$ between continua X and Y is said to be

(i) *monotone* if for each point $y \in Y$ the set $f^{-1}(y)$ is connected;

(ii) *quasi-monotone* if for each subcontinuum Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ has finitely many components, each of which is mapped by f onto Q;

(iii) *confluent* if for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped by f onto Q;

(iv) *hereditarily monotone* (*hereditarily confluent*) if for each subcontinuum C of X the partial mapping $f|C : C \to f(C)$ is monotone (confluent, resp.);

(v) a *local homeomorphism* if for each point of X there exists an open neighborhood U of this point such that $f(U)$ is open in Y and $f|U : U \to f(U)$ is a homeomorphism.

Obviously, each monotone mapping of a continuum is quasi-monotone. It is known that each local homeomorphism of a continuum is quasi-monotone (see [11, Theorems 5 and 7, pages 223 and 224] and [13, Table II, page 28]) and that each hereditarily confluent mapping is also quasi-monotone, (see [13, Corollary 4.45, page 26]). Thus we have the following known assertion (cf. [11, page 221]).

Assertion 10. If a surjective mapping of a continuum is either monotone, or hereditarily confluent, or a local homeomorphism, then it is quasi-monotone.

We need also a result concerning images of irreducible continua under quasi-monotone mappings. The result is a slightly stronger version of [11, Theorem 3, page 222] with the same proof (compare also [7, Theorem 4, page 71] and [13, (8.1) and (8.2), page 71]).

Proposition 11. *If a continuum X is irreducible between points a and b, and if a surjective mapping $f : X \to Y$ is quasi-monotone, then Y is irreducible between $f(a)$ and $f(b)$.*

The following theorem is an immediate consequence of Theorem 3 and Proposition 11.

Theorem 12. *Assume there are continua X and Y, a G-terminal subcontinuum K of X, and a surjective mapping $f : X \to Y$ that satisfies the following two conditions:*

(12.1) *for each subcontinuum I of X which is irreducible from a point $p \in K$ to some other point of X and which contains K the partial mapping $f|I : I \to f(I)$ is quasi-monotone;*

(3.2) *for each irreducible subcontinuum J of Y such that $f(K) \subset J$, the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Then $f(K)$ is a G-terminal subcontinuum of Y. Moreover, if continua X and Y are assumed to be hereditarily unicoherent, then "G-terminal" can be replaced by "HU-terminal" both in the assumption and in the conclusion of the theorem.

The next corollary is a consequence of Theorem 12 as well as Corollary 6 and Proposition 11.

Corollary 13. *Let a surjective mapping $f : X \to Y$ between continua X and Y satisfy the following two conditions:*

(13.1) *for each subcontinuum I of X which is irreducible from a point p of a G-terminal subcontinuum of X to some other point of X, the partial mapping $f|I : I \to f(I)$ is quasi-monotone;*

(13.2) *for each irreducible subcontinuum J of Y containing the image under f of a G-terminal subcontinuum of X the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Then

(8.3) *for each G-terminal subcontinuum of X its image is a G-terminal subcontinuum of Y.*

Moreover,

(13.3) *if continua X and Y are assumed to be hereditarily unicoherent then "G-terminal" can be replaced by "HU-terminal."*

The next corollary is a particular case of the previous one.

Corollary 14. *Let a surjective mapping $f : X \to Y$ between continua X and Y satisfy the following two conditions:*

(14.1) *for each irreducible subcontinuum I of X, the partial mapping $f|I : I \to f(I)$ is quasi-monotone;*

(14.2) *for each irreducible subcontinuum J of Y, the inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Then implications (8.3) and (13.3) hold.

Corollary 14 and Assertion 10 imply the next corollary.

Corollary 15. *Let a surjective mapping $f : X \to Y$ between continua X and Y satisfy condition (14.2) and be such that*

(15.1) *for each irreducible subcontinuum I of X the partial mapping $f|I : I \to f(I)$ is either monotone, or hereditarily confluent, or a local homeomorphism.*

Then implications (8.3) and (13.3) hold.

Note that each hereditarily monotone mapping $f : X \to Y$ satisfies condition (15.1) and that hereditary unicoherence of the domain continuum X implies one of the range if the mapping is monotone, see [13, (7.6), page 59]. Further, a continuum X is hereditarily unicoherent if and only if each monotone mapping defined on X is hereditarily monotone, see [13, (6.10), page 53]. Thus Corollary 15 implies the next one.

Corollary 16. *Let a hereditarily monotone surjective mapping $f : X \to Y$ between continua X and Y satisfy condition (14.2). Then implication (8.3) holds. Moreover, if X is hereditarily unicoherent, then the assumption of hereditary monotoneity of f can be relaxed to its monotoneity, the range Y is hereditarily unicoherent, and "G-terminal" can be replaced by "HU-terminal."*

A continuum X is said to be *arc-like* if every open cover of X can be refined by a finite open cover whose nerve is an arc; equivalently, for the metric case, if for each $\epsilon > 0$ there is an arc A and a surjective mapping $f : X \to A$ such that f is an *ϵ-mapping* (i.e., $\operatorname{diam} f^{-1}(y) < \epsilon$ for each $y \in T$). We mention that a continuum X is arc-like if and only if it is the inverse limit of an inverse sequence of arcs with surjective bonding mappings; see [17, page 24]; for the original definition using ϵ-chains, see Bing's paper [3, page 653]. It is well known that each arc-like continuum is hereditarily unicoherent and irreducible and that any subcontinuum of an arc-like continuum is arc-like (thus irreducible). Therefore if a continuum X is arc-like and a surjective mapping $f : X \to Y$ is monotone, then condition (14.2) is satisfied, and thus (8.3) holds with "HU-terminal" in place of "G-terminal" according to Corollary 16. Hence we have the following result.

Proposition 17. *If a surjective mapping of an arc-like continuum is monotone, then the image of each HU-terminal subcontinuum of the domain is an HU-terminal subcontinuum of the range.*

Recall that a surjective mapping $f : X \to Y$ between continua X and Y is said to be *atomic* if for each subcontinuum K of X such that the set $f(K)$ is nondegenerate, condition $K = f^{-1}(f(K))$ holds, see [1]. Any atomic mapping is known to be hereditarily monotone, see [9, Theorem 1, page 49] and [13, (4.14), page 17]. Further, each atomic mapping $f : X \to Y$ between continua satisfies condition (14.2), see [6, Theorem 2, page 132]. Therefore Corollary 16 implies the following one, which is an extended version of [8, Theorem 4.21], where a direct proof is presented for hereditarily unicoherent continua.

Principle Corollary 18. *Let a surjective mapping $f : X \to Y$ between continua X and Y be atomic. Then*

(8.3) *for each G-terminal subcontinuum of X its image is a G-terminal subcontinuum of Y.*

Moreover, if the continuum X is hereditarily unicoherent, then Y is hereditarily unicoherent too, and "G-terminal" can be replaced by "HU-terminal."

Remark 19. The above result cannot be extended from atomic to hereditarily monotone mappings because by shrinking the arm vb of the triod T of Remark 7 to the point v we have a hereditarily monotone mapping $f : T \to av \cup vc$ which maps an HU-terminal point b of the domain to an interior point $v = f(b)$ of the range.

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