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GENERALIZED TRANSVERSELY PROJECTIVE STRUCTURE ON A TRANSVERSELY HOLOMORPHIC FOLIATION

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The results of Biswas (2000) are extended to the situation of transversely projective foliations. In particular, it is shown that a transversely holomorphic foliation defined using everywhere locally nondegenerate maps to a projective space $\mathbb{CP}^n$, and whose transition functions are given by automorphisms of the projective space, has a canonical transversely projective structure. Such a foliation is also associated with a transversely holomorphic section of $N^{\otimes -k}$ for each $k \in [3, n+1]$, where N is the normal bundle to the foliation. These transversely holomorphic sections are also flat with respect to the Bott partial connection.

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1. Introduction. A projective structure on a Riemann surface X is defined by giving a covering of X by holomorphic coordinate charts such that all the transition functions are restrictions of Möbius transformations. It is well known that the notion of a projective structure can be extended to the situation of foliations (cf. [10]). To define this generalization, let $\mathcal{F}$ be a foliation of codimension two on a real manifold M. Let $\{U_i\}_{i\in I}$ be an open covering of M, and let $\phi_i : U_i \to \mathbb{C}$ be submersions onto the image such that the fibers of ϕ_i are leaves for $\mathcal{F}$. A transversely projective structure on $\mathcal{F}$ is defined by imposing the condition that, for every $i, j \in I$, there is a commutative diagram

$$\begin{array}{ccc} U_i \cap U_j & = & U_i \cap U_j \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_j} \\ \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j) \end{array} \tag{1.1}$$

such that $f_{i,j}$ is a restriction of some Möbius transformation [10].

A holomorphic immersion $\gamma : X \to \mathbb{CP}^n$ of a Riemann surface X is called everywhere locally nondegenerate if for every $x \in X$, the order of contact of the image $\gamma(U)$ at $\gamma(x)$, where U is a neighborhood of x in X, with any hyperplane in $\mathbb{CP}^n$ passing through $\gamma(x)$ is at most $n-1$ (see [3, 9]). Two such immersions are called equivalent if they differ by an automorphism of $\mathbb{CP}^n$. A $\mathbb{CP}^n$-structure on X is an equivalence class of an everywhere locally nondegenerate equivariant map of the universal cover of X into $\mathbb{CP}^n$. A $\mathbb{CP}^1$-structure on X is the same as a projective structure on X.

If $f : X \to \mathbb{CP}^n$ is a holomorphic map such that the image of f is not contained in any hyperplane of $\mathbb{CP}^n$, then there is a finite subset $S \subset X$ such that the restriction of

f to the complement $X\setminus S$ defines a $\mathbb{CP}^n$-structure on $X\setminus S$. Any Riemann surface has many $\mathbb{CP}^n$-structures. In [3], it has been shown that the space of $\mathbb{CP}^n$-structures on X, where $n \geq 2$, is canonically identified with the Cartesian product of the space of all projective structures on X with the direct sum $\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i})$.

The notion of a $\mathbb{CP}^n$-structure can be extended to the situation of foliations which will be called a *transversely* $\mathbb{CP}^n$*-structure*; see Definition 2.3 for the definition of a transversely $\mathbb{CP}^n$-structure.

Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension one. So the normal bundle N is a transversely holomorphic line bundle. The normal bundle N is equipped with the Bott partial connection obtained from the Lie bracket operation of vector fields. The transversely holomorphic structure of N is compatible with the Bott partial connection.

We prove that, giving a transversely $\mathbb{CP}^n$-structure on $\mathcal{F}$ is equivalent to giving a transversely projective structure on $\mathcal{F}$ together with a transversely holomorphic section ω_k of $N^{\otimes -k}$, for each $k \in [3, n+1]$, such that ω_k is flat with respect to the Bott partial connection (see Theorem 2.4). In particular, setting all ω_k to be zero we conclude that, for any transversely $\mathbb{CP}^n$-structure on $\mathcal{F}$ there is a canonically associated transversely projective structure on $\mathcal{F}$. When the foliation is trivial, that is, $\mathcal{F} = 0$, then Theorem 2.4 is the main result of [3] (see [3, Theorem 5.5]).

It is not easy to directly construct a transversely $\mathbb{CP}^n$-structure on a holomorphic foliation. In fact, when the foliation is trivial, namely we have a Riemann surface X, it is not easy to construct a map of the universal cover of X to $\mathbb{CP}^n$, which is everywhere locally nondegenerate. However, using Theorem 2.4 we can indirectly construct many examples of transversely $\mathbb{CP}^n$-structures, just as using [3, Theorem 5.5], we can indirectly construct examples of everywhere locally nondegenerate maps of the universal cover of a Riemann surface to $\mathbb{CP}^n$.

2. Transversely projective foliations defined by maps to a projective space.

Let M be a connected smooth real manifold of dimension $d+2$. Let $\mathcal{F}$ be a C^∞-subbundle of rank d of the tangent bundle TM.

Definition 2.1. A transversely holomorphic structure on $\mathcal{F}$ is defined by giving the following data (see [5]):

(1) a covering of M by open subsets U_i, where i runs over an index set I. So we have $\bigcup_{i\in I} U_i = M$;

(2) for each $i \in I$, a submersion ϕ_i of U_i to an open subset D_i of $\mathbb{C}$. The restriction $\mathcal{F}|_{U_i}$ is the kernel of the differential map $d\phi_i : TU_i \to \phi_i^* TD_i$;

(3) for every pair $i, j \in I$, there is a commutative diagram of maps

$$\begin{array}{ccc} U_i \cap U_j & \xrightarrow{Id} & U_i \cap U_j \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_j} \\ \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j), \end{array} \tag{2.1}$$

where $f_{i,j}$ is a holomorphic map.

Two such data $\{U_i, \phi_i\}_{i\in I}$ and $\{U_i, \phi_i\}_{i\in J}$ are called *equivalent* if their union, namely

$$\{U_i, \phi_i\}_{i\in I\cup J}, \tag{2.2}$$

also satisfies the above conditions. A *transversely holomorphic* structure on $\mathcal{F}$ will mean an equivalence class of data of the above type satisfying the three conditions.

Next we recall the definition of a transversely projective foliation.

Definition 2.2. A transversely projective structure on $\mathcal{F}$ is defined by giving a data $\{U_i, \phi_i\}_{i\in I}$ exactly as in Definition 2.1, but satisfying the extra condition (apart from the three conditions) that the holomorphic maps $f_{i,j}$ in condition (3) are of the form $z \mapsto (az+b)/(cz+d)$, where $a,b,c,d \in \mathbb{C}$ are constant scalars and $ad-bc = 1$, that is, each $f_{i,j}$ is the restriction of some Möbius transformation; the scalars a, b, c, d may depend on the index i. As before, two such data $\{U_i, \phi_i\}_{i\in I}$ and $\{U_i, \phi_i\}_{i\in J}$ are called *equivalent* if their union $\{U_i, \phi_i\}_{i\in I\cup J}$ is also a data for a transversely projective structure. A *transversely projective* structure on $\mathcal{F}$ will mean an equivalence class of such data.

Clearly, a transversely projective structure on $\mathcal{F}$ defines a transversely holomorphic structure on $\mathcal{F}$. If $\bar{\mathcal{F}}$ is a transversely holomorphic structure on $\mathcal{F}$, then a transversely projective structure on $\bar{\mathcal{F}}$ is a transversely projective structure on $\mathcal{F}$ such that, the transversely holomorphic structure defined by it coincides with $\bar{\mathcal{F}}$.

We now recall the notion of a locally nondegenerate immersion of a Riemann surface into a projective space (see [3, 9]).

Let X be a Riemann surface, that is, a complex manifold of complex dimension one. Let $\mathbb{CP}^n$, $n \geq 1$, denote the n-dimensional projective space consisting of all lines in $\mathbb{C}^{n+1}$. A holomorphic immersion

$$\gamma : X \longrightarrow \mathbb{CP}^n \tag{2.3}$$

is called *everywhere locally nondegenerate* if for every $x \in X$, the order of contact of the image $\gamma(U)$, where U is a neighborhood of x in X, at $\gamma(x)$ with any hyperplane in $\mathbb{CP}^n$ passing through $\gamma(x)$ is at most $n-1$. We need to consider a neighborhood in the definition since γ may not be injective.

An alternative description of the above nondegeneracy condition following [9] is given below.

Let

$$0 \longrightarrow S \longrightarrow V \xrightarrow{q} Q \longrightarrow 0 \tag{2.4}$$

be the universal exact sequence over $\mathbb{CP}^n$. The vector bundle V is the trivial vector bundle with $\mathbb{C}^{n+1}$ as fiber and S is the tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$. Consider the differential

$$d\gamma : T_X \longrightarrow \gamma^* T_{\mathbb{CP}^n} = \gamma^* \operatorname{Hom}(S,Q) \tag{2.5}$$

of the immersion γ; here T_X is the holomorphic tangent bundle of X. Since γ is an immersion, the homomorphism $d\gamma$ is injective.

Now, the homomorphism $d\gamma$ gives a homomorphism

$$\overline{d\gamma} : T_X^* \otimes \gamma^* S \longrightarrow \gamma^* Q, \tag{2.6}$$

where T_X^* is the holomorphic cotangent bundle of X. Let S_1 denote the inverse image $q^{-1}(\text{image}(\overline{d\gamma}))$, where the homomorphism q is defined in (2.4). The subbundle S_1 of $\gamma^* V$ defines a map

$$\gamma_1 : X \longrightarrow G(n+1, 2) \tag{2.7}$$

of X into the Grassmannian of two planes in $\mathbb{C}^{n+1}$.

Now assume that γ_1 is an immersion. Then repeating the above argument we get a map

$$\gamma_2 : X \longrightarrow G(n+1, 3) \tag{2.8}$$

of X into the Grassmannian of three planes in $\mathbb{C}^{n+1}$.

More generally, inductively we have a map

$$\gamma_i : X \longrightarrow G(n+1, i+1), \tag{2.9}$$

where $i \in [1, n-1]$, by assuming that γ_{i-1} is an immersion. (See also [9, Section 1] for the details of the construction of the maps γ_i described above.)

The condition that the map γ, together with each map γ_i, where $i \in [1, n-1]$, is an immersion, is equivalent to the condition that the map γ is everywhere locally nondegenerate.

Now, we extend the above notion of everywhere locally nondegenerate map to the context of foliations, which we call transversely $\mathbb{CP}^n$-structure.

Definition 2.3. A transversely $\mathbb{CP}^n$-structure on $\mathcal{F}$ is defined by giving a data $\{U_i, \phi_i\}_{i \in I}$ exactly as in Definition 2.1 satisfying conditions (1) and (2) and the following stronger version of (3): for every $i \in I$, there is an everywhere locally nondegenerate map

$$\gamma_i : D_i := \text{image}(\phi_i) \longrightarrow \mathbb{CP}^n \tag{2.10}$$

such that, for every pair $i, j \in I$, there is a commutative diagram of maps

$$\begin{array}{ccc} U_i \cap U_j & \xrightarrow{\text{Id}} & U_i \cap U_j \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_j} \\ \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j) \\ \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \gamma_j} \\ \mathbb{CP}^n & \xrightarrow{T} & \mathbb{CP}^n, \end{array} \tag{2.11}$$

where T is an automorphism of $\mathbb{CP}^n$, that is, $T \in \text{GL}(n+1, \mathbb{C})$. As before, two such data $\{U_i, \phi_i, \gamma_i\}_{i \in I}$ and $\{U_i, \phi_i, \gamma_i\}_{i \in J}$ are called *equivalent* if their union $\{U_i, \phi_i, \gamma_i\}_{i \in I \cup J}$ is also a data for a transversely $\mathbb{CP}^n$-structure. A *transversely* $\mathbb{CP}^n$*-structure* on $\mathcal{F}$ will mean an equivalence class of such data.

The above condition forces the map $f_{i,j}$ to be holomorphic. So, a transversely $\mathbb{CP}^n$-structure on $\mathcal{F}$ defines a transversely holomorphic structure on $\mathcal{F}$. If $\bar{\mathcal{F}}$ is a transversely holomorphic structure on $\mathcal{F}$, then a transversely $\mathbb{CP}^n$-structure on $\bar{\mathcal{F}}$ is a transversely $\mathbb{CP}^n$-structure on $\mathcal{F}$ such that the underlying transversely holomorphic structure coincides with $\bar{\mathcal{F}}$.

Note that, a transversely $\mathbb{CP}^1$-structure on $\mathcal{F}$ is by definition a transversely projective structure on $\mathcal{F}$.

We fix a transversely holomorphic structure $\bar{\mathcal{F}}$ on $\mathcal{F}$.

The normal bundle

$$N := \frac{TM}{\mathcal{F}} \tag{2.12}$$

is a complex line bundle. Therefore, for every integer $k \in \mathbb{Z}$, we have a complex line bundle $N^{\otimes k}$ obtained by taking the kth tensor power of the complex line bundle N. By $N^{\otimes -1}$ we mean the dual line bundle N^*.

Any such line bundle $N^{\otimes k}$ has a natural transversely holomorphic structure. This means that, there is a Dolbeault operator

$$\bar{\partial}_{N^{\otimes k}} : N^{\otimes k} \longrightarrow N^* \otimes N^{\otimes k} = N^{\otimes k-1} \tag{2.13}$$

satisfying the Leibniz identity. The operator $\bar{\partial}_{N^{\otimes k}}$ is simply the Dolbeault operator on the holomorphic tangent bundle $T_{\mathbb{C}}^{\otimes k}$ of the complex line $\mathbb{C}$ transported to M using the projections ϕ_i. It may be noted that, the condition in Definition 2.1(3) that every $f_{i,j}$ is holomorphic ensures that these locally defined operators patch compatibly to define the global differential operator $\bar{\partial}_{N^{\otimes k}}$.

Also, the line bundle N, and hence any $N^{\otimes k}$, has the Bott partial connection (see [8]). Recall that, the Lie bracket operation on the sheaf of sections of the tangent bundle TM defines the Bott partial connection

$$N \longrightarrow \mathcal{F}^* \otimes N \tag{2.14}$$

along the foliation $\mathcal{F}$. The Jacobi identity for Lie bracket ensures that this partial connection is flat.

It is easy to see that both the complex structure of N and the transversely holomorphic structure of N are compatible with respect to the Bott partial connection. In other words, both the complex vector space structure of the fibers of N and the Dolbeault operator $\bar{\partial}_N$ defined in (2.13) commute with the differential operator in (2.14) defining the Bott connection. Equivalently, parallel translation (for the Bott connection) along the leaves of the foliation $\bar{\mathcal{F}}$ of holomorphic sections of N remain holomorphic. Also, parallel translations for the Bott connection commute with multiplication by $\sqrt{-1}$ of the fibers of N.

The Bott partial connection on N induces a flat partial connection on any $N^{\otimes k}$. All the above compatibility properties of the Bott connection on N evidently remain valid for any $N^{\otimes k}$.

Let $\mathcal{V}_{\bar{\mathcal{F}}}(k)$ denote the space of all globally defined smooth sections s of the complex line bundle $N^{\otimes k}$ such that s is transversely holomorphic for the transversely holomorphic foliation $\bar{\mathcal{F}}$ and it is flat with respect to the Bott partial connection for $\bar{\mathcal{F}}$. So $\mathcal{V}_{\bar{\mathcal{F}}}(k)$ is a complex vector space; it need not be of finite dimension. However, in the situation where M is compact, it was proved by Duchamp and Kalka [4, Theorem 1.27, page 323], and also independently by Gómez-Mont [6, Theorem 1, page 169], that the dimension of $\mathcal{V}_{\bar{\mathcal{F}}}(k)$ is finite.

Let $\mathcal{P}(\bar{\mathcal{F}})$ denote the space of all equivalence classes of transversely projective structures on the transversely holomorphic foliation $\bar{\mathcal{F}}$. Transversely projective structures were defined in Definition 2.2 and transversely projective structures on $\bar{\mathcal{F}}$ were defined in the paragraph following Definition 2.2. The space $\mathcal{P}(\bar{\mathcal{F}})$ may be empty.

The following theorem is the main result of this section.

Theorem 2.4. *There is a canonical bijective map from the space of all transversely $\mathbb{CP}^n$-structures on $\bar{\mathcal{F}}$ and the Cartesian product*

$$\mathcal{P}(\bar{\mathcal{F}}) \times \left(\bigoplus_{k=3}^{n+1} \mathcal{V}_{\bar{\mathcal{F}}}(-k) \right). \tag{2.15}$$

In particular, a transversely $\mathbb{CP}^n$-structure gives a transversely projective structure on $\bar{\mathcal{F}}$ by simply taking the zero section in $\mathcal{V}_{\bar{\mathcal{F}}}(-k)$ for all $k \in [3, n+1]$.

The theorem will be proved after establishing a few lemmas. We start with the definition of jet bundles and differential operators.

Let E be a holomorphic vector bundle on a Riemann surface X, and let n be a positive integer. The nth-order *jet bundle* of E, denoted by $J^n(E)$, is defined to be the following direct image on X:

$$J^n(E) := p_{1*}\left(\frac{p_2^* E}{p_2^* E \otimes \mathcal{O}_{X\times X}(-(n+1)\Delta)} \right), \tag{2.16}$$

where $p_i : X \times X \to X$, $i = 1, 2$, is the projection onto the ith factor, and Δ is the diagonal divisor on $X \times X$. Therefore, for any $x \in X$, the fiber $J^n(E)_x$ is the space of all sections of E over the nth-order infinitesimal neighborhood of x.

Let K_X denote the holomorphic cotangent bundle of X. There is a natural exact sequence

$$0 \longrightarrow K_X^{\otimes n} \otimes E \longrightarrow J^n(E) \longrightarrow J^{n-1}(E) \longrightarrow 0 \tag{2.17}$$

constructed using the obvious inclusion of $\mathcal{O}_{X\times X}(-(n+1)\Delta)$ in $\mathcal{O}_{X\times X}(-n\Delta)$. The inclusion map $K_X^{\otimes n} \bigotimes E \to J^n(E)$ is constructed by using the homomorphism

$$K_X^{\otimes n} \longrightarrow J^n(\mathcal{O}_X), \tag{2.18}$$

which is defined at any $x \in X$ by sending $(df)^{\otimes n}$, where f is any holomorphic function with $f(x) = 0$, to the jet of the function $f^n/n!$ at x.

The sheaf of *differential operators* $\mathrm{Diff}_X^n(E,F)$ is defined to be $\mathrm{Hom}(J^n(E),F)$. The homomorphism

$$\sigma : \mathrm{Diff}_X^n(E,F) \longrightarrow \mathrm{Hom}\left(K_X^{\otimes n} \otimes E, F\right), \tag{2.19}$$

obtained by restricting a homomorphism from $J^n(E)$ to F to the subsheaf $K_X^{\otimes n} \bigotimes E$ in (2.17), is known as the *symbol map*.

Let X denote a simply connected open subset of $\mathbb{CP}^1$. Take a holomorphic map $\gamma : X \to \mathbb{CP}^n$. Let ζ denote the line bundle $\gamma^*\mathbb{O}_{\mathbb{CP}^n}(1)$ over X. In the notation of the exact sequence (2.4), the line bundle $\mathbb{O}_{\mathbb{CP}^n}(1)$ is S^*. Pulling back the universal exact sequence (2.4) to X and then taking the dual, we have

$$0 \longrightarrow \gamma^* Q^* \longrightarrow W \xrightarrow{p} \zeta \longrightarrow 0, \tag{2.20}$$

where W is the trivial vector bundle of rank $n+1$ over X with fiber $(\mathbb{C}^{n+1})^*$. Of course, $(\mathbb{C}^{n+1})^* = \mathbb{C}^{n+1}$.

The trivialization of W induces a homomorphism

$$\bar{p} : W \longrightarrow J^n(\zeta) \tag{2.21}$$

which can be defined as follows: for any point $x \in X$ and vector $w \in W_x$ in the fiber, let $\bar{w}$ denote the unique flat section of W such that $\bar{w}(x) = w$. Now, $\bar{p}(w)$ is the restriction of the section $p(\bar{w})$ of ζ to the nth-order infinitesimal neighborhood of x. Recall that, the fiber $J^n(\zeta)_x$ is the space of sections of ζ over the nth-order infinitesimal neighborhood of x.

Lemma 2.5. *The map γ is everywhere locally nondegenerate if and only if the homomorphism $\bar{p}$ in (2.21) is an isomorphism.*

Proof. This is a straightforward consequence of the condition of everywhere locally nondegeneracy. For some point $x \in X$, if $\bar{p}_x : W_x \to J^n(\zeta)_x$ is not an isomorphism, then take a nonzero vector w in the kernel of $\bar{p}_x$, since $W_x = (\mathbb{C}^{n+1})^*$, the vector w defines a hyperplane H in $\mathbb{CP}^n$. Clearly, H contains $\gamma(x)$. The given condition $\bar{p}_x(w) = 0$ can be seen to be equivalent to the condition that the order of contact of H with $\gamma(X)$ at $\gamma(x)$ is at least n. In other words, γ is degenerate at x.

Conversely, if γ is degenerate at a point $x \in X$, take a hyperplane H in $\mathbb{CP}^n$ containing $\gamma(x)$ such that the order of contact between $\gamma(X)$ and H at $\gamma(x)$ is at least n. Let $w \in (\mathbb{C}^{n+1})^*$ be a functional defining the hyperplane H. It is easy to see that $\bar{p}_x(w) = 0$. This completes the proof. □

Assume that γ is everywhere locally nondegenerate. So the homomorphism $\bar{p}$ in (2.21) gives a trivialization of the jet bundle $J^n(\zeta)$. Now, from (2.17) it follows that $\bigwedge^{n+1} J^n(\zeta)$ is canonically isomorphic to $K_X^{n(n+1)/2} \bigotimes \zeta^{n+1}$. The trivialization of $J^n(\zeta)$ induces a trivialization of $K_X^{n(n+1)/2} \bigotimes \zeta^{n+1}$. Fix a square-root ξ of the holomorphic tangent bundle T_X. In other words, ξ is a holomorphic line bundle and an isomorphism between T_X and $\xi^{\otimes 2}$ is chosen. The above trivialization of $K_X^{n(n+1)/2} \bigotimes \zeta^{n+1}$ induces an isomorphism

$$J^i(\zeta^j) = J^i(\xi^{nj}) \otimes (\xi^{nj})^* \otimes \zeta^j \tag{2.22}$$

for every i and j. Indeed, this is an immediate consequence of the fact that ζ and ξ^n differ by tensoring with a finite-order line bundle. By a finite-order line bundle we mean a line bundle some tensor power of which has a canonical trivialization.

Consider the homomorphism

$$\hat{p} : W \longrightarrow J^{n+1}(\zeta) \tag{2.23}$$

which sends any $w \in W_x$ to the restriction of the section $p(\bar{w})$ of ζ to the $(n+1)$th-order infinitesimal neighborhood of x. Here p as in (2.20) and $\bar{w}$ as in the definition of the map $\bar{p}$ in (2.21). From its definition it is immediate that the composition $f_n \circ \hat{p} \circ \bar{p}^{-1}$ is the identity map of $J^n(\zeta)$, where f_n is the projection $J^{n+1}(\zeta) \to J^n(\zeta)$ defined in (2.17). In other words, $\hat{p} \circ \bar{p}^{-1}$ is a splitting of the jet sequence

$$0 \longrightarrow K_X^{n+1} \otimes \zeta \longrightarrow J^{n+1}(\zeta) \longrightarrow J^n(\zeta) \longrightarrow 0 \tag{2.24}$$

defined in (2.17).

There is a unique homomorphism $J^{n+1}(\zeta) \to K_X^{n+1} \otimes \zeta$ satisfying the two conditions that its kernel is the image of $\hat{p} \circ \bar{p}^{-1}$ and the composition of the natural inclusion of $K_X^{n+1} \otimes \zeta$ in $J^{n+1}(\zeta)$ (as in (2.17)) with it is the identity map of $K_X^{n+1} \otimes \zeta$. By the earlier definition of differential operators given in terms of jet bundles, this homomorphism defines a differential operator

$$D_\gamma \in H^0\Big(X, \mathrm{Diff}_X^{n+1}\left(\zeta, K_X^{n+1} \otimes \zeta\right)\Big). \tag{2.25}$$

Since D_γ is defined by a splitting of a jet sequence, its symbol is the constant function 1 (the symbol of a differential operator is defined in (2.19)). Now, using (2.22), the differential operator D_γ gives a differential operator

$$D(\gamma) \in H^0\Big(X, \mathrm{Diff}_X^{n+1}\left(\xi^n, \xi^{-n-2}\right)\Big) \tag{2.26}$$

of symbol 1.

It can be deduced from the definition of jet bundles that, for any holomorphic vector bundle E, there is a natural injective homomorphism $J^{i+j}(E) \to J^i(J^j(E))$ for any $i, j \geq 0$. Therefore, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \xi^{\otimes(-n-2)} & \longrightarrow & J^{n+1}(\xi^{\otimes n}) & \longrightarrow & J^n(\xi^{\otimes n}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \tau & & \| & & \\ 0 & \longrightarrow & K_X \otimes J^n(\xi^{\otimes n}) & \longrightarrow & J^1(J^n(\xi^{\otimes n})) & \longrightarrow & J^n(\xi^{\otimes n}) & \longrightarrow & 0, \end{array} \tag{2.27}$$

where the injective homomorphism τ is obtained from the above remark.

If

$$f : J^n(\xi^{\otimes n}) \longrightarrow J^{n+1}(\xi^{\otimes n}) \tag{2.28}$$

is a splitting of the top exact sequence in (2.27), then the composition $\tau \circ f$ defines a splitting of the bottom exact sequence in (2.27). But a splitting of the exact sequence

$$0 \longrightarrow K_X \otimes E \longrightarrow J^1(E) \longrightarrow E \longrightarrow 0 \tag{2.29}$$

is a holomorphic connection on E (see [1]). Furthermore, any holomorphic connection on a Riemann surface is flat. Therefore, $\tau \circ f$ defines a flat connection on $J^n(\xi^{\otimes n})$. Let ∇^f denote this flat connection on $J^n(\xi^{\otimes n})$ obtained from a splitting f.

Since X is simply connected, ∇^f gives a trivialization of $J^n(\xi^{\otimes n})$. In other words, if we choose a point $z \in X$, using parallel translations, $J^n(\xi^{\otimes n})$ gets identified with the trivial vector bundle over X with $J^n(\xi^{\otimes n})_z$ as the fiber.

Fix an isomorphism of the fiber $J^n(\xi^{\otimes n})_z$ with $\mathbb{C}^{n+1}$. As before, let W denote the trivial vector bundle over X with $\mathbb{C}^{n+1}$ as the fiber. So we have $J^n(\xi^{\otimes n}) = W$.

For any point $y \in X$, consider the one-dimensional subspace $(\xi^{\otimes n} \otimes K_X^n)_y$ of the fiber $J^n(\xi^{\otimes n})_y$ given in (2.17). Let

$$\gamma : X \longrightarrow \mathbb{CP}^n \tag{2.30}$$

denote the map that sends any point $y \in X$ to the line in $\mathbb{C}^{n+1}$ that corresponds to the line $(\xi^{\otimes n} \otimes K_X^n)_y$ by the isomorphism between the fibers $J^n(\xi^{\otimes n})_y$ and W_y.

If we change the isomorphism between $J^n(\xi^{\otimes n})_z$ and $\mathbb{C}^{n+1}$ by an automorphism $A \in \mathrm{GL}(n+1,\mathbb{C})$, then the map γ is altered by the automorphism A of $\mathbb{CP}^n$.

Lemma 2.6. *Let $f : J^n(\xi^{\otimes n}) \to J^{n+1}(\xi^{\otimes n})$ be a splitting of the top exact sequence in (2.27). Then the map γ constructed in (2.30) from f is everywhere locally nondegenerate.*

Proof. The lemma follows from Lemma 2.5 and the fact that the connection ∇^f, from which γ is constructed, is given by a splitting f (as in (2.28)). In [3], a different but equivalent formulation of the lemma can be found. □

Two everywhere locally nondegenerate maps f_1 and f_2 of X into $\mathbb{CP}^n$ are called equivalent if there is an automorphism $A \in \mathrm{Aut}(\mathbb{CP}^n) = \mathrm{PGL}(n+1,\mathbb{C})$ such that $A \circ f_1 = f_2$.

Let $\mathcal{A}$ denote the space of all equivalence classes of everywhere locally nondegenerate maps of X into $\mathbb{CP}^n$.

Take a differential operator $D \in H^0(X, \mathrm{Diff}_X^{n+1}(\xi^n, \xi^{-n-2}))$ of symbol 1. Since the symbol of D is 1, it gives a splitting of the top exact sequence in (2.27). Denoting this splitting $J^n(\xi^{\otimes n}) \to J^{n+1}(\xi^{\otimes n})$ by $\bar{D}$, consider $\tau \circ \bar{D}$, which, as we already noted, is a flat connection on $J^n(\xi^{\otimes n})$. It may be noted that since $\xi^{\otimes 2} = T_X$, the line bundle $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ is canonically trivialized.

Let $\mathcal{B}$ denote the space of global differential operators

$$D \in H^0(X, \mathrm{Diff}_X^{n+1}(\xi^n, \xi^{-n-2})) \tag{2.31}$$

of symbol 1 *and satisfying the condition that the connection on $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ induced by the connection $\tau \circ \bar{D}$ on $J^n(\xi^{\otimes n})$ preserves the trivialization of $\bigwedge^{n+1} J^n(\xi^{\otimes n})$.*

From the construction of the differential operator $D(\gamma)$ in (2.26) it follows that $D(\gamma) \in \mathcal{B}$.

Let

$$F : \mathcal{A} \longrightarrow \mathcal{B} \tag{2.32}$$

be the map that sends any everywhere locally nondegenerate map γ to the differential operator $D(\gamma)$ constructed in (2.26).

As above, for a differential operator $D \in \mathcal{B}$, the corresponding splitting is denoted by $\bar{D}$. Let

$$G : \mathcal{B} \longrightarrow \mathcal{A} \tag{2.33}$$

be the map that sends any operator D to the map γ constructed in (2.30) using the splitting $f = \bar{D}$ as in (2.28).

Lemma 2.7. *The map F defined in (2.32) is one-to-one and onto.*

Proof. In fact, unraveling the definitions of the maps F and G, defined in (2.32) and (2.33), respectively, yields that they are inverses of each other. We omit the details; it can be found in [3]. □

Let $\mathcal{P}(X)$ denote the space of all projective structures on the Riemann surface X. It is known that $\mathcal{P}(X)$ is an affine space for the space of quadratic differentials, namely, $H^0(X, K_X^2)$ (see [7]).

Lemma 2.8. *There is a natural bijective map between $\mathcal{B}$ and the Cartesian product*

$$\mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right) \tag{2.34}$$

if $n \geq 2$. If $n = 1$, then $\mathcal{B}$ is in bijective correspondence with $\mathcal{P}(X)$.

Proof. The key input in the proof is [2, Theorem 6.3, page 19]. Now we recall its statement.

Let Y be a Riemann surface equipped with a projective structure. Let $k, l \in \mathbb{Z}$ and let $n \in \mathbb{N}$ be such that $k \notin [-n+1, 0]$ and $l - k - j \notin \{0, 1\}$ for any integer $j \in [1, n]$. Then,

$$H^0(Y, \mathrm{Diff}_Y^n(\mathcal{L}^k, \mathcal{L}^l)) = \bigoplus_{i=0}^{n} H^0(Y, \mathcal{L}^{l-k-2n+2i}), \tag{2.35}$$

where $\mathcal{L}$ is the square-root of the canonical bundle defined by the projective structure.

A clarification of the above statement is needed. In [2], a projective structure means an $\mathrm{SL}(2,\mathbb{C})$ structure. But here projective structure means a $\mathrm{PGL}(2,\mathbb{C})$ structure. But we know that a $\mathrm{PGL}(2,\mathbb{C})$ structure on a Riemann surface always lifts to an $\mathrm{SL}(2,\mathbb{C})$ structure [7]. Furthermore, the space of such lifts is in bijective correspondence with the space of theta-characteristics (square-root of the holomorphic cotangent bundle) of Y.

Therefore, given a $\mathrm{PGL}(2,\mathbb{C})$ structure P on X, the pair (P, ξ) determines a unique $\mathrm{SL}(2,\mathbb{C})$ structure.

Now, set $k = -n$ and $l = n+2$ in (2.35). This yields an isomorphism

$$F : H^0(X, \mathrm{Diff}_X^{n+1}(\xi^n, \xi^{-n-2})) \longrightarrow \bigoplus_{i=0}^{n} H^0(X, K_X^{\otimes i}). \tag{2.36}$$

For any $D \in H^0(X, \mathrm{Diff}_X^{n+1}(\xi^n, \xi^{-n-2}))$, the component of $F(D)$ in

$$H^0(X, K_X^{\otimes 0}) = H^0(X, \mathcal{O}_X) \tag{2.37}$$

is the symbol of D. Furthermore, the condition in the definition of $\mathcal{B}$ that, the connection on $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ induced by the connection $\tau \circ \bar{D}$ on $J^n(\xi^{\otimes n})$ preserves the trivialization of $\bigwedge^{n+1} J^n(\xi^{\otimes n})$, is actually equivalent to the condition that the component of $F(D)$ in $H^0(X, K_X)$ vanishes (see [3]). Therefore, using F, the space $\mathcal{B}$ gets identified with the direct sum

$$\bigoplus_{i=2}^{n+1} H^0(X, K_X^{\otimes i}), \tag{2.38}$$

if X is equipped with a projective structure.

Using the fact that the space of projective structures on X, namely $\mathcal{P}(X)$, is an affine space for $H^0(X, K_X^2)$, it is easy to deduce that given any

$$D \in H^0(X, \mathrm{Diff}_X^{n+1}(\xi^n, \xi^{-n-2})), \tag{2.39}$$

there is a unique projective structure $P \in \mathcal{P}(X)$ such that, for the map F in (2.36) corresponding to P, the component of $F(D)$ in $H^0(X, K_X^{\otimes 2})$ vanishes identically. Let $\overline{F(D)}$ denote the projection of $F(D)$ in $\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i})$; F corresponds to this unique projective structure. Now, we have a bijective map

$$\bar{F} : \mathcal{B} \longrightarrow \mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right), \tag{2.40}$$

that sends any D to the pair $(P, \overline{F(D)})$ constructed above. (See [3, Section 4] for the details.)

If $n = 1$, then using [2, Theorem 6.3] and the fact that $\mathcal{P}(X)$ is an affine space for $H^0(X, K_X^{\otimes 2})$, it follows immediately that $\mathcal{B} = \mathcal{P}(X)$. This completes the proof of the lemma. □

For the first part of the proof of Lemma 2.8, we should have directly used [2, Corollary 6.6] instead of deriving it using [2, Theorem 6.3]. Unfortunately, in the statement of [2, Corollary 6.6], the word "compact" is used which technically makes it useless for our purpose. But, of course, compactness is not used in the proof of [2, Corollary 6.6]. When [2, 3] were written, we had primarily compact Riemann surfaces in mind.

Combining Lemmas 2.7 and 2.8, we have the following corollary.

Corollary 2.9. *There is a natural bijective map*

$$\Gamma : \mathcal{A} \longrightarrow \mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right) \tag{2.41}$$

for $n \geq 2$. If $n = 1$ then $\mathcal{A}$ is in bijective correspondence with $\mathcal{P}(X)$.

When X is a compact Riemann surface, the above corollary is [3, Theorem 5.5]. Again since "compactness" condition is thrown in [3] indiscriminately, a vast part of it is technically useless for our present purpose. Nevertheless, the ideas of [3] have been borrowed here.

Let $Y \subset X$ be a simply connected open subset. Let $\mathcal{A}_Y$ denote the space of all equivalence classes of everywhere locally nondegenerate maps of Y into $\mathbb{CP}^n$. In other words, $\mathcal{A}_Y$ is obtained by substituting Y in place of X in the definition of $\mathcal{A}$. The space of all projective structures on Y is denoted by $\mathcal{P}(Y)$.

The restriction of ξ to Y defines a square-root of the tangent bundle T_Y. There is a natural restriction map $\mathcal{P}(X) \to \mathcal{P}(Y)$ and also there are homomorphisms

$$H^0(X, K_X^{\otimes i}) \longrightarrow H^0(Y, K_Y^{\otimes i}) \tag{2.42}$$

for every $i \in \mathbb{Z}$ defined by restriction of sections. Similarly, we have a map $\mathcal{A} \longrightarrow \mathcal{A}_Y$, which sends a map γ of X to $\mathbb{CP}^n$ to the restriction of γ to Y.

Let

$$\Gamma_Y : \mathcal{A}_Y \longrightarrow \mathcal{P}(Y) \times \left(\bigoplus_{i=3}^{n+1} H^0(Y, K_Y^{\otimes i}) \right) \tag{2.43}$$

be the isomorphism for Y obtained in Corollary 2.9. The map Γ in Corollary 2.9 has the property that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right) \\ \downarrow & & \downarrow \\ \mathcal{A}_Y & \xrightarrow{\Gamma_Y} & \mathcal{P}(Y) \times \left(\bigoplus_{i=3}^{n+1} H^0(Y, K_Y^{\otimes i}) \right). \end{array} \tag{2.44}$$

The vertical maps are defined by restriction. The commutativity of this diagram is indeed easy to see from the construction of Γ.

Now that we have Corollary 2.9 and (2.44), we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Assume that $n \geq 2$, since the theorem is obvious in the case of $n = 1$.

Suppose we are given a transversely $\mathbb{CP}^n$-structure, as defined in Definition 2.3. We assume that all the subsets $D_i := \operatorname{image}(\phi_i)$ of $\mathbb{C}$ in Definition 2.1 are simply connected. Clearly, this is a harmless assumption.

Consider a triplet (U_i, ϕ_i, γ_i) as in Definition 2.3. Now, using the map Γ in Corollary 2.9, from the everywhere locally nondegenerate map γ_i we have a projective structure on $D_i = \operatorname{image}(\phi_i)$ together with a holomorphic section of $T_{D_i}^{\otimes -l}$ for all $l \in [3, n+1]$. This projective structure on D_i is denoted by $\mathcal{P}_i$, and the holomorphic section of $T_{D_i}^{\otimes -l}$ obtained above is denoted by ω_i^l. The projective structure $\mathcal{P}_i$ induces a transversely projective structure on the open subset U_i of M. We denote this transversely projective structure on U_i by $\bar{\mathcal{P}}_i$. The pullback, using the map ϕ_i, of the holomorphic section ω_i^l of $T_{D_i}^{\otimes -l}$ defines a section of $N^{\otimes -l}$ over U_i. This section of $N^{\otimes -l}$ over U_i is denoted by $\bar{\omega}_i^l$. Since ω_i^l is holomorphic, we have the section $\bar{\omega}_i^l$ over U_i to be transversely holomorphic. Furthermore, $\bar{\omega}_i^l$ is obviously flat with respect to the Bott partial connection. The proof of the theorem is completed by showing that all these locally defined transversely projective structures $\bar{\mathcal{P}}_i$ (resp., transversely holomorphic flat sections $\bar{\omega}_i^l$) patch compatibly to define globally on M a transversely projective structure (resp., transversely holomorphic flat section of $N^{\otimes -l}$).

If we take another triplet (U_j, ϕ_j, γ_j), $j \in I$, as in Definition 2.3, then the two projective structures on $D_i \cap D_j$, namely $\mathcal{P}_i$ and $\mathcal{P}_j$, coincide. This is an immediate consequence of the commutativity of the diagram (2.44). Therefore, we have a projective structure on the union $D_i \cup D_j$, and hence the two transversely projective structures, namely $\bar{\mathcal{P}}_i$ and $\bar{\mathcal{P}}_j$, coincide over $U_i \cap U_j$. Consequently, the transversely projective structures $\{\bar{\mathcal{P}}_i\}_{i\in I}$ patch together compatibly to define a transversely projective structure on $\bar{\mathcal{F}}$. Similarly, from the commutativity of the diagram (2.44), it follows that the two sections $\bar{\omega}_i^l$ and $\bar{\omega}_j^l$ coincide over $U_i \cap U_j$. In other words, these local sections $\bar{\omega}_i^l$ of $N^{\otimes -l}$ patch together to give an element of $\mathcal{V}_{\bar{\mathcal{F}}}(-l)$. This completes the proof of the theorem. □

Theorem 2.4 can be considered as a generalization of [10, Theorem 6.1].

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