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ON A SUBCLASS OF α-CONVEX λ-SPIRAL FUNCTIONS

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Let H denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the unit disc $\Delta = \{z : |z| < 1\}$. In this paper, we introduce the class $M_{\alpha}^{\lambda}[A,B]$ of functions $f \in H$ with $f(z)f'(z)/z \neq 0$, satisfying for $z \in \Delta : \{(e^{i\lambda} - \alpha\cos\lambda)(zf'(z)/f(z)) + \alpha\cos\lambda(1 + zf''(z)/f'(z))\} \prec \cos\lambda((1+Az)/(1+Bz)) + i\sin\lambda$, where $\prec$ denotes subordination, α and λ are real numbers, $|\lambda| < \pi/2$ and $-1 \leq B < A \leq 1$. Functions in $M_{\alpha}^{\lambda}[A,B]$ are shown to be λ-spiral-like and hence univalent. Integral representation, coefficients bounds, and other results are given.

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1. Introduction. Let H denote the class of functions f analytic in the unit disc $\Delta = \{z : |z| < 1\}$ and be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Let $f \in H$ with $f(z)f'(z)/z \neq 0$ in Δ and α be a real number. Then f is said to be α-convex λ-spiral function, if and only if it satisfies the inequality $\operatorname{Re}\{(e^{i\lambda} - \alpha\cos\lambda)(zf'(z)/f(z)) + \alpha\cos\lambda(1 + zf''(z)/f'(z))\} > 0$ in Δ, for some λ, $|\lambda| < \pi/2$. The class of these functions, which is denoted by $\mathrm{SC}(\alpha,\lambda)$ was defined and studied by Umarany [8].

In this paper, we introduce and study a subclass of $\mathrm{SC}(\alpha,\lambda)$ defined by using subordination to convex functions.

Definition 1.1. Let $f \in H$ with $f(z)f'(z)/z \neq 0$ in Δ. Then f is said to belong to the class $MS_{\alpha}^{\lambda}[A,B]$ if and only if for $z \in \Delta$,

$$K(\alpha,\lambda,f(z)) = \left\{(e^{i\lambda} - \alpha\cos\lambda)\frac{zf'(z)}{f(z)} + \alpha\cos\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \prec \cos\lambda\frac{1+Az}{1+Bz} + i\sin\lambda, \tag{1.1}$$

where $\prec$ denotes subordination, α and λ are real numbers, $|\lambda| < \pi/2$ and A and B are arbitrary fixed numbers such that $-1 \leq B < A \leq 1$.

It is clear from Definition 1.1 that a function $f \in MS_{\alpha}^{\lambda}[A,B]$ if and only if there exists a function $w(z)$ analytic in Δ and satisfying $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$, such that

$$K(\alpha,\lambda,f(z)) = \cos\lambda\frac{1+Aw(z)}{1+Bw(z)} + i\sin\lambda. \tag{1.2}$$

It is also clear from Definition 1.1 that $M_{\alpha}^{0}[A,B] \equiv M_{\alpha}[A,B]$, the subclass of α-convex functions introduced by Kim and Jung [5] and $M_{0}^{\lambda}[A,B] \equiv \mathrm{Sp}^{\lambda}[A,B]$, the subclass of spiral-like functions introduced by Dashrath and Shukla [2].

In this paper, we show that functions in $M_\alpha^\lambda[A,B]$ are spiral-like and hence univalent in Δ. Integral representation, coefficient bounds, and other results are given.

2. Spiral-likeness.

To derive our main result, we prove the following lemma.

Lemma 2.1. *Let $f \in H$, then $f \in MS_\alpha^\lambda[A,B]$ if and only if*

$$|K(\alpha,\lambda,f(z))-m| < M, \quad z\in\Delta, \tag{2.1}$$

where

$$m=\cos\lambda\frac{1-AB}{1-B^2}+i\sin\lambda, \qquad M=\frac{(A-B)}{1-B^2}\cos\lambda. \tag{2.2}$$

Proof. Suppose that $f \in MS_\alpha^\lambda[A,B]$. Then from (1.2) we obtain

$$\begin{aligned} K(\alpha,\lambda,f(z))-m &= \frac{e^{i\lambda}-m+[(A-B)\cos\lambda+Be^{i\lambda}-mB]w(z)}{1+Bw(z)} \\ &= M\frac{B+w(z)}{1+Bw(z)}, \quad B\neq -1, \end{aligned} \tag{2.3}$$

using (2.2), hence

$$K(\alpha,\lambda,f(z))-m = Mq(z). \tag{2.4}$$

It is clear that the function q satisfies $|q(z)|<1$. Hence (2.1) follows from (2.4).

Conversely, suppose that (2.1) holds. Then

$$\left|\frac{K(\alpha,\lambda,f(z))}{M}-\frac{m}{M}\right|<1, \quad B\neq -1. \tag{2.5}$$

Let

$$g(z)=\frac{K(\alpha,\lambda,f(z))}{M}-\frac{m}{M}, \quad B\neq -1, \tag{2.6}$$

$$w(z)=\frac{g(z)-g(0)}{1-g(z)g(0)}=\frac{K(\alpha,\lambda,f(z))-e^{i\lambda}}{(A-B)\cos\lambda+Be^{i\lambda}-BK(\alpha,\lambda,f(z))}. \tag{2.7}$$

Clearly $w(0)=0$ and $|w(z)|<1$. Rearranging (2.7) we get (1.2), hence $f\in MS_\alpha^\lambda[A,B]$. We note that condition (2.1) can be written as

$$\left|\frac{K(\alpha,\lambda,f(z))-i\sin\lambda-(1-A)/(1-B)\cos\lambda}{\cos\lambda-\cos\lambda(1-A)/(1-B)}-\frac{1}{1+B}\right|<\frac{1}{1+B}, \quad z\in\Delta. \tag{2.8}$$

As $B\to -1$ and $A=1$, the above condition reduces to the necessary and sufficient condition for f to belong to $MS_\alpha^\lambda[1,-1]$ (see [8]). □

The following lemma is due to Jack [3].

Lemma 2.2. *Let w be a nonconstant and analytic function in Δ, $w(0)=0$. Then if $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at z_0 we can write*

$$z_0 w'(z_0) = \phi w(z_0), \tag{2.9}$$

where ϕ is a real number such that $\phi \geq 1$.

Remark 2.3. Throughout, $-1 \leq B < A \leq 1$, unless otherwise indicated, $|\lambda| < \pi/2$.

Theorem 2.4. *If $f \in MS_\alpha^\lambda[A,B]$, then $f \in \mathrm{Sp}^\lambda[A,B]$ and hence univalent.*

Proof. Let

$$\frac{zf'(z)}{f(z)} = \frac{1+[(A-B)\cos\lambda e^{-i\lambda}+B]w(z)}{1+Bw(z)} = \frac{1+\eta w(z)}{1+Bw(z)}, \tag{2.10}$$

where $\eta = (A-B)\cos\lambda e^{-i\lambda}+B$. Differentiating (2.10) logarithmically, we get

$$1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{\eta z w'(z)}{1+\eta w(z)} - \frac{Bzw'(z)}{1+Bw(z)}. \tag{2.11}$$

Multiplying both sides of (2.11) by $\alpha\cos\lambda$ and adding $e^{i\lambda}(zf'(z)/f(z))$ to both sides, we obtain

$$\begin{aligned}\alpha\cos\lambda&\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+e^{i\lambda}\frac{zf'(z)}{f(z)}\\&=\left(\frac{\eta z w'(z)}{1+\eta w(z)}-\frac{Bzw'(z)}{1+Bw(z)}\right)\alpha\cos\lambda+e^{i\lambda}\frac{1+\eta w(z)}{1+Bw(z)}.\end{aligned} \tag{2.12}$$

Let r^* be the distance from the origin to the pole of w nearest to the origin. Then w is analytic in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.2 for $|z| \leq r$ $(r < r_0)$, there exists a point z_0 such that

$$z_0 w'(z) = \phi w(z_0), \quad \phi \geq 1. \tag{2.13}$$

From (2.11) and (2.12), we obtain

$$K(\alpha,\lambda,f(z)) - m = \frac{N(z_0)}{R(z_0)}, \quad B \neq -1, \tag{2.14}$$

where

$$\begin{aligned}N(z_0) &= e^{i\lambda}-m+[(\eta e^{i\lambda}-Bm)+(e^{i\lambda}-m)\eta+\alpha\cos\lambda(\eta-B)\phi]w(z_0)\\&\quad+(\eta e^{i\lambda}-Bm)\eta w^2(z_0),\end{aligned} \tag{2.15}$$

$$R(z_0) = 1+(B+\eta)w(z_0)+B\eta w^2(z_0). \tag{2.16}$$

Now suppose that it was possible to have $\max_{|z|=r}|w(z)| = 1$ for some $r, r < r_0 \le 1$. At the point z_0 where this occurred, we would have $|w(z_0)| = 1$. Then, by using the identities

$$e^{i\lambda} - m = BM, \quad \eta e^{i\lambda} - Bm = M, \quad B \neq -1 \tag{2.17}$$

in (2.15) we have

$$N(z_0) = BM + [\alpha\cos\lambda(\eta - B)\phi + \eta BM + M]w(z_0) + \eta M w^2(z_0). \tag{2.18}$$

From (2.16) and (2.18), we get

$$|N(z_0)|^2 - M^2|R(z_0)|^2 = \tilde{a} + 2\tilde{b}\operatorname{Re}\{w(z_0)\}, \tag{2.19}$$

where

$$\begin{aligned} \tilde{a} &= \alpha\cos\lambda(\eta - B)\phi\{\alpha\cos\lambda(\eta - B)\phi + 2M(1 + B\eta)\}, \\ \tilde{b} &= \alpha\cos\lambda(\eta - B)\phi M(B + \eta). \end{aligned} \tag{2.20}$$

From (2.19), we have

$$|N(z_0)|^2 - M^2|R(z_0)|^2 > 0, \quad \text{provided } \tilde{a} \pm 2\tilde{b} > 0. \tag{2.21}$$

Now

$$\begin{aligned} \tilde{a} + \tilde{b} &= \alpha\cos\lambda(\eta - B)\phi\{\alpha\cos\lambda(\eta - B)\phi + M(2 + 2B\eta + B + \eta)\} > 0, \\ \tilde{a} - \tilde{b} &= \alpha\cos\lambda(\eta - B)\phi\{\alpha\cos\lambda(\eta - B)\phi + M(2 + 2B\eta - B - \eta)\} > 0. \end{aligned} \tag{2.22}$$

Thus it follows from (2.14) and (2.21) that

$$|K(\alpha, \lambda, f(z)) - m| > M. \tag{2.23}$$

But in view of Lemma 2.1, this is contrary to our assumption $f \in MS_\alpha^\lambda[A, B]$. So we cannot have $|w(z_0)| = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $w(0) = 0$, $|w(z)|$ is continuous and $|w(z_0)| \neq 1$ in $|z| < r_0, w$ cannot have a pole at $|z| = r_0$. Therefore, w is analytic in Δ and satisfies $|w(z)| < 1$ for $z \in \Delta$. Hence $f \in \mathrm{Sp}^\lambda[A, B]$. □

Remark 2.5. When $A = 1$ and $B = -1$, a result of Umarani [8] follows from Theorem 2.4.

3. Integral representation

Theorem 3.1. *A necessary and sufficient condition for the function f to be in $MS_\alpha^\lambda[A, B]$, $\alpha > 0$, is that f has the integral representation*

$$f(z) = \left[\frac{e^{i\lambda}}{\alpha\cos\lambda}\int_0^z (g(t))^{e^{i\lambda}/\alpha\cos\lambda} t^{-1} dt\right]^{\alpha e^{-i\lambda}\cos\lambda}, \tag{3.1}$$

for some $g \in \mathrm{Sp}^\lambda[A, B]$, where the powers are assumed to be principal values.

Proof. Let $f, g \in H$ and f be given as in (3.1). Differentiating both sides and simplifying we get

$$g(z) = f(z)\left(\frac{zf'(z)}{f(z)}\right)^{\alpha e^{-i\lambda}\cos\lambda}. \tag{3.2}$$

Differentiating (3.2) logarithmically and multiplying both sides by $ze^{i\lambda}$ we obtain

$$e^{i\lambda}\frac{zg'(z)}{g(z)} = (e^{i\lambda} - \alpha\cos\lambda)\frac{zf'(z)}{f(z)} + \alpha\cos\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{3.3}$$

Hence $f \in MS_\alpha^\lambda[A,B]$ if and only if $g \in \mathrm{Sp}^\lambda[A,B]$. □

Remark 3.2. Using the integral representation and the external function of the class $\mathrm{Sp}^\lambda[A,B]$ (see [2]) we get the external function of the class $MS_\alpha^\lambda[A,B]$ as

$$f(z) = \begin{cases} \left(\dfrac{e^{i\lambda}}{\alpha\cos\lambda}\displaystyle\int_0^z t^{e^{i\lambda}/\alpha\cos\lambda - 1}(1+Bt)^{((A-B)/\alpha B)}dt\right)^{\alpha e^{-i\lambda}\cos\lambda} & \text{if } B \neq 0, \\ \left(\dfrac{e^{i\lambda}}{\alpha\cos\lambda}\displaystyle\int_0^z t^{e^{i\lambda}/\alpha\cos\lambda}\left(\exp\left(Ate^{-i\lambda}\cos\lambda\right)\right)^{e^{i\lambda}/\alpha\cos\lambda}dt\right)^{\alpha e^{-i\lambda}\cos\lambda} & \text{if } B = 0. \end{cases} \tag{3.4}$$

4. Coefficients bounds. To derive our next result, we need the following lemma (see [4]).

Lemma 4.1. *Let $w(z) = c_1z + c_2z^2 + \cdots$ be an analytic function with $|w(z)| < 1$ in Δ. If ν is any complex number, then*

$$\left|c_2 - \nu c_1^2\right| \leq \max\{1, |\nu|\}. \tag{4.1}$$

The equality may be attained with the functions $w(z) = z^2$ and $w(z) = z$.

Theorem 4.2. *Let $f \in MS_\alpha^\lambda[A,B]$ given by $f(z) = z + \sum_{k=2}^\infty a_kz^k$, and let μ be any complex number. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)\cos\lambda}{2\left|e^{i\lambda} + 2\alpha\cos\lambda\right|}\max\{1, |\nu|\}, \tag{4.2}$$

where

$$\nu = \frac{2\mu(e^{i\lambda} + 2\alpha\cos\lambda)(A-B)\cos\lambda - (A-B)\cos\lambda(e^{i\lambda} + 3\alpha\cos\lambda)}{(e^{i\lambda} + \alpha\cos\lambda)^2} + \frac{(e^{i\lambda} + \alpha\cos\lambda)^2B}{(e^{i\lambda} + \alpha\cos\lambda)^2}. \tag{4.3}$$

This result is sharp.

Proof. Let $f \in MS_\alpha^\lambda[A,B]$ and let $w(z) = c_1z + c_2z^2 + \cdots$ be an analytic function with $|w(z)| < 1$ in Δ. Then

$$(e^{i\lambda} - \alpha\cos\lambda)\frac{zf'(z)}{f(z)} + \alpha\cos\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{e^{i\lambda} + e^{i\lambda}\eta w(z)}{1 + Bw(z)}, \tag{4.4}$$

where $\eta = (A-B)\cos\lambda e^{-i\lambda} + B$, from (1.2). Equating the coefficients in both sides of (4.4) we get

$$c_1 = \frac{e^{i\lambda} + \alpha\cos\lambda}{(A-B)\cos\lambda} a_2, \tag{4.5}$$

$$c_2 = \frac{2(e^{i\lambda} + 2\alpha\cos\lambda)}{(A-B)\cos\lambda} a_3 - \frac{(A-B)\cos\lambda(e^{i\lambda} + 3\alpha\cos\lambda) - (e^{i\lambda} + \alpha\cos\lambda)^2 B}{(A-B)^2\cos^2\lambda} a_2^2. \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$c_2 - \nu c_1^2 = \frac{2(e^{i\lambda} + 2\alpha\cos\lambda)}{(A-B)\cos\lambda}\{a_3 - \mu a_2^2\}, \tag{4.7}$$

where

$$\mu = \frac{(A-B)\cos\lambda}{2(e^{i\lambda} + 2\alpha\cos\lambda)}\left\{\frac{(e^{i\lambda} + 3\alpha\cos\lambda)(A-B)\cos\lambda - (e^{i\lambda} + \alpha\cos\lambda)^2 B}{(A-B)^2\cos^2\lambda} + \nu\frac{(e^{i\lambda} + \alpha\cos\lambda)^2}{(A-B)^2\cos^2\lambda}\right\}. \tag{4.8}$$

Hence applying Lemma 4.1, we get

$$|a_3 - \mu a_2^2| = \left|\frac{(A-B)\cos\lambda}{2(e^{i\lambda} + 2\alpha\cos\lambda)}\right| |c_2 - \nu c_1^2| \le \frac{(A-B)\cos\lambda}{2|e^{i\lambda} + 2\alpha\cos\lambda|}\max\{1, |\nu|\}. \tag{4.9}$$

The sharpness of (4.9) follows from the sharpness of inequality (4.1). □

5. Some radius problems. In this section, we discuss the covering theorem of the class $MS_\alpha^\lambda[A,B]$, that is, we find the radius of the largest disk covered by the image of the unit disk Δ under the mapping $f \in MS_\alpha^\lambda[A,B]$. We also find the α-convex β-spiral radius of functions in $MS_\alpha^\lambda[A,B]$.

Theorem 5.1. *Let $f \in MS_\alpha^\lambda[A,B]$. Then the disk Δ is mapped onto a domain that contains the disk*

$$|w| < \frac{|e^{i\lambda} + \alpha\cos\lambda|}{2|e^{i\lambda} + \alpha\cos\lambda| + (A-B)\cos\lambda}. \tag{5.1}$$

Proof. Let $w = f(z) \in MS_\alpha^\lambda[A,B]$ given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and let w_0 be any complex number such that $f(z) \neq w_0$ for $z \in \Delta$. Then

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \cdots \tag{5.2}$$

belongs to the class S of univalent functions. Hence (see [1])

$$\left|a_2 + \frac{1}{w_0}\right| \le 2. \tag{5.3}$$

Substituting $|c_1| \le 1$ in (4.5) (see [7]), we get

$$|a_2| \le \frac{(A-B)\cos\lambda}{|e^{i\lambda} + \alpha\cos\lambda|}. \tag{5.4}$$

From (5.3) and (5.4), we obtain

$$|w_0| \geq \frac{|e^{i\lambda} + \alpha\cos\lambda|}{2|e^{i\lambda} + \alpha\cos\lambda| + (A-B)\cos\lambda}, \tag{5.5}$$

which is the required result. □

To derive our next theorem we need the following lemma (see [6]).

Lemma 5.2. *Let $f \in \mathrm{Sp}^{\lambda}[A,B]$, then $f \in \mathrm{Sp}^{\beta}[A,B]$, $|\beta| < \pi/2$, on the disc $|z| < r^{\star\star}$ where $r^{\star\star}$ is the smallest positive root of the equation*

$$B[A\cos(\lambda+\beta) + B\sin\lambda\sin(\lambda-\beta)]r^2 + (A-B)r\cos\lambda - \cos\beta = 0. \tag{5.6}$$

This result is sharp.

Theorem 5.3. *Let $f \in MS_{\alpha}^{\lambda}[A,B]$. Then $f \in MS_{\alpha}^{\beta}[A,B]$, $|\beta| < \pi/2$ on the disk $|z| < r^{\star\star}$, where $r^{\star\star}$ is the smallest positive root of (5.6). This result is sharp.*

Proof. Let $f \in MS_{\alpha}^{\lambda}[A,B]$. Then from Theorem 3.1, there exists a function $g \in \mathrm{Sp}^{\lambda}[A,B]$ such that (3.1) is satisfied. From Lemma 5.2, $g \in \mathrm{Sp}^{\beta}[A,B]$, $|\beta| < \pi/2$, on $|z| < r^{\star\star}$. Applying Theorem 3.1 again, we find that $f \in MS_{\alpha}^{\beta}[A,B]$ on $|z| < r^{\star\star}$. The radius $r^{\star\star}$ is best possible as shown by the function f given by (3.4). □

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