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ON THE IRREGULARITY OF THE DISTRIBUTION OF THE SUMS OF PAIRS OF ODD PRIMES

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Let $P_2(n)$ denote the number of ways of writing n as a sum of two odd primes. We support a conjecture of Hardy and Littlewood concerning $P_2(n)$ by showing that it holds in a certain "average" sense. Thereby showing the irregularity of $P_2(n)$.

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1. Introduction. Let $P_2(n)$ be the number of ways of writing n as a sum of two odd primes. Goldbach conjectured that $P_2(n) \geq 1$ for all even positive integers n; Landau [2] proved an average result for even $n \geq 2$

$$\sum_{n\leq x} P_2(n) \sim \frac{x^2}{2\log^2(x)}. \tag{1.1}$$

Hardy and Littlewood [1] conjectured that, asymptotically, for even $n \geq 2$,

$$P_2(n) \sim C \prod_{\substack{p\mid n \\ p\neq 2}} \left(\frac{p-1}{p-2}\right)\frac{n}{\log^2 n}, \tag{1.2}$$

where

$$C = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right). \tag{1.3}$$

Thus, rather than $P_2(n)$ increasing monotonically with n (as one might guess at first glance), their conjecture implies that the value of $P_2(n)$ depends heavily on the small primes dividing n. So, for instance, if n is the product of the first k primes, then the above formula suggests that

$$P_2(n) \sim C' \frac{n}{\log^2 n}\log\log n \tag{1.4}$$

for some absolute constant $C' > 0$.

Whereas, if n is twice a prime then

$$P_2(n) \sim C\frac{n}{\log^2 n}. \tag{1.5}$$

Although we have no idea how one might prove (1.2), we attach, in this note, the question of proving that the behavior of $P_2(n)$ depends heavily on its small prime factors, as suggested by (1.2). In particular, we show that (1.2) holds in a certain "average" sense, as follows.

Theorem 1.1. *For fixed even integers $2 \le a \le m$, and as $x \to \infty$*

$$\frac{1}{x/m} \sum_{\substack{n \le x \\ n \equiv a \pmod{m}}} \frac{P_2(n)}{C_m n/\log^2 n} \sim \prod_{\substack{p \mid (a,m) \\ p>2}} \left(\frac{p-1}{p-2}\right), \tag{1.6}$$

where

$$C_m = \prod_{\substack{p \mid m \\ p>2}} \left(1 - \frac{1}{(p-1)^2}\right). \tag{1.7}$$

Taking the example $m = 6$, the theorem tells us that "on average," $P_2(n) \sim 2C_m n/\log^2 n$ if 6 divides n, but $P_2(n) \sim C_m n/\log^2 n$ if 6 does not divide n. In other words, if 3 divides n then we have proved, "on average," that $P_2(n)$ is twice as large as if 3 does not divide n, which captures the spirit of our earlier deduction from (1.2).

The sum in the theorem is estimated by making use of the Prime Number Theorem for arithmetic progressions in the form

$$\sum_{\substack{n \le x \\ n \equiv a \pmod{m}}} \Lambda(n) = \frac{x}{\phi(m)} + o(x) \quad \text{as } x \longrightarrow \infty, \tag{1.8}$$

where $\Lambda(n)$ is the von Mangoldt's function. By using (1.8) and using partial summation, we have

$$\sum_{p \le x} p\Lambda(p) = \frac{x^2}{2\phi(m)} + o(x^2). \tag{1.9}$$

Also we note that

$$\begin{aligned} \prod_{\substack{p \mid m \\ p \nmid a \\ p>2}} \left(1 - \frac{2}{p}\right) \prod_{\substack{p \mid m \\ p \mid a \\ p>2}} \left(1 - \frac{1}{p}\right) &= \prod_{\substack{p \mid m \\ p>2}} \left(1 - \frac{2}{p}\right) \frac{1}{\prod_{\substack{p \mid m \\ p \mid a \\ p>2}} (1-2/p)} \prod_{\substack{p \mid m \\ p \mid a \\ p>2}} \left(1 - \frac{1}{p}\right) \\ &= \prod_{\substack{p \mid m \\ p>2}} \left(1 - \frac{2}{p}\right) \prod_{\substack{p \mid m \\ p \mid a \\ p>2}} \left(\frac{p-1}{p-2}\right), \end{aligned} \tag{1.10}$$

$$\begin{aligned} \frac{1}{\prod_{\substack{p \mid m \\ p>2}} (1-1/p)^2} \prod_{\substack{p \mid m \\ p>2}} \left(1 - \frac{2}{p}\right) &= \prod_{\substack{p \mid m \\ p>2}} \left(\frac{(p-2)/p}{((p-1)/p)^2}\right) \\ &= \prod_{\substack{p \mid m \\ p>2}} \frac{p(p-2)}{(p-1)^2} \\ &= \prod_{\substack{p \mid m \\ p>2}} \left(1 - \frac{1}{(p-1)^2}\right) = C_m. \end{aligned} \tag{1.11}$$

We also need the following lemma.

Lemma 1.2. *Given integers* $1 \le a \le m$,

$$\#\{i, 1 \le i \le m : (i,m) = (a-i,m) = 1\} = m \prod_{\substack{p \mid m \\ p \nmid a}} \left(1 - \frac{2}{p}\right) \prod_{\substack{p \mid m \\ p \mid a}} \left(1 - \frac{1}{p}\right). \tag{1.12}$$

2. Proofs

Proof of Lemma 1.2. By the Chinese Remainder Theorem, we see that

$$(i,m) = (a-i,m) = 1 \tag{2.1}$$

if and only if

$$(i,p) = (a-i,p) = 1 \quad \text{for every prime } p \mid m \tag{2.2}$$

if and only if

$$i \not\equiv 0 \text{ or } a \pmod p \quad \text{for every prime } p \mid m. \tag{2.3}$$

Thus if p^l divides m (but p^{l+1} does not) then the number of i_p, $i \le i_p \le p^l$, such that $i_p \not\equiv 0$ or $a \pmod p$ is $p^l(1-2/p)$ if $p \nmid a$, and $p^l(1-1/p)$ if $p \mid a$.

Knowing the number of possibilities $(\bmod\, p^l)$ for each p dividing m, we apply the Chinese Remainder Theorem to find that the number of solutions $(\bmod\, m)$ is the product. This gives the desired formula, and the lemma is proved. □

Proof of Theorem 1.1. Define von Mangoldt's function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^a \text{ is a prime power,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Let

$$\begin{aligned} \Lambda_2(n) &= \sum_{p+q=n} \Lambda(p)\Lambda(q), \\ \Psi_2(x) &= \sum_{n \le x} \Lambda_2(n), \qquad \Psi_2(x;m,a) = \sum_{\substack{n \le x \\ n \equiv a \pmod m}} \Lambda_2(n). \end{aligned} \tag{2.5}$$

Just as in the proof of the Prime Number Theorem, where it is easier to prove that

$$\sum_{n \le x} \Lambda(n) \sim x \tag{2.6}$$

rather than

$$\sum_{\text{prime } p \le x} 1 \simeq \frac{x}{\log x} \tag{2.7}$$

and then show that these statements are equivalent, herein we will prove that

$$\Psi_2(x;m,a) \sim C_m \prod_{\substack{p \mid (a,m) \\ p > 2}} \left(\frac{p-1}{p-2}\right) \frac{x^2}{2m}, \tag{2.8}$$

and then note that this is equivalent to our theorem (which may similarly be deduced through partial summation). So

$$\begin{aligned}
\Psi_2(x;m,a) &= \sum_{\substack{p+q\le x\\ p+q\equiv a(\operatorname{mod} m)}} \Lambda(p)\Lambda(q)\\
&= \sum_{i=0}^{m} \sum_{\substack{p\le x\\ p\equiv i(\operatorname{mod} m)}} \Lambda(p) \sum_{\substack{q\le x-p\\ q\equiv a-i(\operatorname{mod} m)}} \Lambda(q)\\
&= \sum_{\substack{i=0\\ (i,m)=(a-i,m)=1}}^{m} \sum_{\substack{p\le x\\ p\equiv i(\operatorname{mod} m)}} \Lambda(p)\Psi(x-p;m,a-i)+o(x),
\end{aligned} \tag{2.9}$$

we now incorporate (1.8) into (2.9) and we establish

$$\Psi_2(x;m,a) = \sum_{(i,m)=(a-i,m)=1} \sum_{\substack{p\le x\\ p\equiv i(\operatorname{mod} m)}} \Lambda(p)\left\{\frac{x-p}{\phi(m)}+o(x)\right\}+o(x). \tag{2.10}$$

We expand (2.10)

$$\Psi_2(x;m,a) = \sum_{(i,m)=(a-i,m)=1} \left(\sum_{\substack{p\le x\\ p\equiv i(\operatorname{mod} m)}} \frac{\Lambda(p)x}{\phi(m)} - \sum_{\substack{p\le x\\ p\equiv i(\operatorname{mod} m)}} p\Lambda(p)+o(x) \sum_{\substack{p\le x\\ p\equiv i(\operatorname{mod} m)}} 1\right). \tag{2.11}$$

By incorporating (1.8) and (1.9) into (2.11), we have

$$\begin{aligned}
\Psi_2(x;m,a) &= \left(\frac{x^2}{2\phi(m)^2}+o(x^2)\right) \sum_{(i,m)=(a-i,m)=1} 1\\
&= \frac{x^2}{2\phi(m)^2}\left(\sum_{(i,m)=(a-i,m)=1} 1\right)+o(x^2).
\end{aligned} \tag{2.12}$$

From the lemma we see that (2.12) now becomes

$$\begin{aligned}
\Psi_2(x;m,a) &= \frac{x^2}{2m^2}\frac{1}{\prod_{p\mid m}(1-1/p)^2}\, m \prod_{\substack{p\mid m\\ p\nmid a}}\left(1-\frac{2}{p}\right)\prod_{\substack{p\mid m\\ p\mid a}}\left(1-\frac{1}{p}\right)+o(x^2)\\
&= \frac{x^2}{2m(1/4)}\frac{1}{\prod_{\substack{p\mid m\\ p>2}}(1-1/p)^2}\,\frac{1}{2}\prod_{\substack{p\mid m\\ p\nmid a\\ p>2}}\left(1-\frac{2}{p}\right)\frac{1}{2}\prod_{\substack{p\mid m\\ p\mid a\\ p>2}}\left(1-\frac{1}{p}\right)+o(x^2).
\end{aligned} \tag{2.13}$$

Using (1.10) we see that (2.11) becomes

$$\Psi_2(x;m,a) = \frac{x^2}{2m}\frac{1}{\prod_{\substack{p\mid m\\ p>2}}(1-1/p)^2}\prod_{\substack{p\mid m\\ p>2}}\left(1-\frac{2}{p}\right)\prod_{\substack{p\mid m\\ p\mid a\\ p>2}}\left(\frac{p-1}{p-2}\right)+o(x^2). \tag{2.14}$$

Now incorporate (1.11) into (2.14) we now establish

$$\Psi_2(x;m,a)=\frac{x^2}{2m}C_m\prod_{\substack{p\mid(a,m)\\p>2}}\left(\frac{p-1}{p-2}\right)+o(x^2). \tag{2.15}$$

Using the definition of left-hand side of (2.15), we have

$$\sum_{\substack{n\le x\\n\equiv a(\operatorname{mod} m)}}\left(\sum_{p+q=n}\Lambda(p)\Lambda(q)\right)=\frac{x^2}{2m}C_m\prod_{\substack{p\mid(a,m)\\p>2}}\left(\frac{p-1}{p-2}\right)+o(x^2). \tag{2.16}$$

Rearranging the inner sum of the left-hand side and applying partial sum, we get

$$\sum_{\substack{n\le x\\n\equiv a(\operatorname{mod} m)}}\left(\frac{1}{2}P_2(n)\log^2(n)\right)=\frac{x^2}{2m}C_m\prod_{\substack{p\mid(a,m)\\p>2}}\left(\frac{p-1}{p-2}\right)+o(x^2), \tag{2.17}$$

from which we can now establish the theorem using another partial summation. □

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