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DUALITY OF MEASURE AND CATEGORY IN INFINITE-DIMENSIONAL SEPARABLE HILBERT SPACE ℓ_2

GOGI PANTSULAIA

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We prove that an analogy of the Oxtoby duality principle is not valid for the concrete nontrivial σ-finite Borel invariant measure and the Baire category in the classical Hilbert space ℓ_2.

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As usual, we equip an infinite-dimensional separable Hilbert space ℓ_2 by such nonzero σ-finite Borel measures which are invariant with respect to everywhere dense vector subspaces and study duality between such measures and Baire category.

Section 1 contains constructions of nontrivial σ-finite Borel measures, which are defined in the infinite-dimensional separable Hilbert space ℓ_2 and are invariant with respect to some everywhere dense vector subspaces. The duality between invariant Borel measures and Baire category in the classical Hilbert space ℓ_2 is studied in Section 2. An idea applied in the process of proving of the main assertions allows us to obtain more general results for sufficiently large class of infinite-dimensional topological vector spaces.

1. Invariant Borel measures in classical Hilbert space ℓ_2. Let $\mathbb{R}^N$ be the space of all sequences of real numbers equipped with the Tychonoff topology. Denote by $B(\mathbb{R}^N)$ the σ-algebra of all Borel subsets in $\mathbb{R}^N$.

Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be sequences of real numbers such that

$$(\forall i)\quad (i\in\mathbb{N}\longrightarrow a_i<b_i). \tag{1.1}$$

We put

$$A_n=R_0\times\cdots\times R_n\times\left(\prod_{i>n}\Delta_i\right)\quad (n\in\mathbb{N}), \tag{1.2}$$

where

$$(\forall i)\quad (i\in\mathbb{N}\longrightarrow R_i=R,\ \Delta_i=[a_i,b_i[). \tag{1.3}$$

For an arbitrary natural number $i\in\mathbb{N}$, consider the Lebesgue measure μ_i defined on the space R_i and satisfying the condition $\mu_i(\Delta_i)=1$. Denote by λ_i the normed Lebesgue measure defined on the interval Δ_i.

For an arbitrary $n \in \mathbb{N}$, we denote by ν_n the measure defined by

$$\nu_n = \prod_{1 \le i \le n} \mu_i \times \prod_{i>n} \lambda_i, \tag{1.4}$$

and by $\bar{\nu}_n$ the Borel measure in the space $\mathbb{R}^N$ defined by

$$(\forall X) \quad (X \in B(\mathbb{R}^N) \longrightarrow \bar{\nu}_n(X) = \nu_n(X \cap A_n)). \tag{1.5}$$

The following assertion is valid.

Lemma 1.1. *For an arbitrary Borel set $X \subseteq \mathbb{R}^N$, there exists a limit*

$$\nu_\Delta(X) = \lim_{n \to \infty} \bar{\nu}_n(X). \tag{1.6}$$

Moreover, the functional ν_Δ is a nontrivial σ-finite measure defined on the Borel σ-algebra $B(\mathbb{R}^N)$.

Proof. First, observe that, for an arbitrary natural number n, the condition $A_n \subset A_{n+1}$ is valid. By the property of σ-additivity of the measure ν_{n+1}, we obtain

$$\begin{aligned} \bar{\nu}_{n+1}(X) &= \nu_{n+1}(X \cap A_{n+1}) = \nu_{n+1}(X \cap [A_{n+1} \setminus A_n] \cup A_n) \\ &= \nu_{n+1}[X \cap (A_{n+1} \setminus A_n)] + \nu_{n+1}(X \cap A_n). \end{aligned} \tag{1.7}$$

Note that the restriction $\nu_{n+1}|_{A_n}$ of the measure ν_{n+1} to the set A_n coincides with the measure ν_n.

Indeed, we have

$$\begin{aligned} \nu_{n+1}(A_n \cap X) &= \left(\prod_{1 \le i \le n+1} \mu_i \times \prod_{i>n+1} \lambda_i \right)(A_n \cap X) \\ &= \left\{ \prod_{1 \le i \le n} \mu_i \times [\mu_{n+1} | \Delta_{n+1} + \mu_{n+1} | \{R \setminus \Delta_{n+1}\}] \times \prod_{i>n+1} \lambda_i \right\}(A_n \cap X) \\ &= \left(\prod_{1 \le i \le n} \mu_i \times \prod_{i>n} \lambda_i \right)(A_n \cap X) \\ &\quad + \left(\prod_{1 \le i \le n} \mu_i \times (\mu_{n+1} \mid \{R \setminus \Delta_{n+1}\}) \times \prod_{i>n+1} \lambda_i \right)(A_n \cap X) \\ &= \nu_n(A_n \cap X). \end{aligned} \tag{1.8}$$

Since for an arbitrary $n \in \mathbb{N}$, the inclusion $A_n \subset A_{n+1}$ holds, we have

$$(\forall X) \quad (X \in B(\mathbb{R}^N) \longrightarrow \nu_n(A_n \cap X) \le \nu_{n+1}(A_n \cap X)). \tag{1.9}$$

Hence there exists a limit $\lim_{n \to \infty} \bar{\nu}_n(X)$ which we denote by $\nu_\Delta(X)$.

The proof of the fact that the measure ν_Δ is countably additive is trivial.

Establish the following properties of ν_Δ.

(I) The measure ν_Δ is nontrivial, since

$$\nu_\Delta\left(\prod_{i\in\mathbb{N}}\Delta_i\right) = 1. \tag{1.10}$$

(II) The measure ν_Δ is σ-finite. Indeed, we have

$$\mathbb{R}^N = \left(\mathbb{R}^N \setminus \bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{n\in\mathbb{N}} A_n\right). \tag{1.11}$$

Since $\mathbb{R}^N \setminus \bigcup_{n\in\mathbb{N}} A_n \in B(\mathbb{R}^N)$, by the definition of the measure ν_Δ we have

$$\nu_n\left(\mathbb{R}^N \setminus \bigcup_{k\in\mathbb{N}} A_k\right) = \nu_n\left(\left(\mathbb{R}^N \setminus \bigcup_{k\in\mathbb{N}} A_k\right) \cap A_n\right) = \nu_n(\varnothing) = 0. \tag{1.12}$$

Since, for an arbitrary natural number $n \in \mathbb{N}$, the measure $\bar{\nu}_n$ is σ-finite, there exists a countable family $(B_k^{(n)})_{k\in\mathbb{N}}$ of Borel measurable subsets of the space $\mathbb{R}^N$ such that

$$\begin{aligned} &(\forall k) \quad \left(k \in \mathbb{N} \longrightarrow \bar{\nu}_n\left(B_k^{(n)}\right) < +\infty\right); \\ &(\forall n) \quad \left(n \in \mathbb{N} \longrightarrow A_n = \bigcup_{k\in\mathbb{N}} B_k^{(n)}\right). \end{aligned} \tag{1.13}$$

Consider the family $(B_k^{(n)})_{k\in\mathbb{N},n\in\mathbb{N}}$.

It is clear that

$$(\forall k)\ (\forall n) \quad \left(k \in \mathbb{N},\ n \in \mathbb{N} \longrightarrow \nu_\Delta(B_k^{(n)}) = \bar{\nu}_n(B_k^{(n)}) < +\infty\right). \tag{1.14}$$

On the other hand, we have

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}} B_k^{(n)}, \tag{1.15}$$

that is,

$$\mathbb{R}^N = \left(\mathbb{R}^N \setminus \bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{n\in\mathbb{N},k\in\mathbb{N}} B_k^{(n)}\right). \tag{1.16}$$

The proof is completed. □

Remark 1.2. The measure ν_Δ described in Lemma 1.1 can be regarded as an inductive limit of the family $(\bar{\nu})_{n\in\mathbb{N}}$ of invariant measures.

Recall that an element $h \in \mathbb{R}^N$ is called an admissible translation (in the sense of invariance) of the measure ν_Δ if

$$(\forall X) \quad \left(X \in B(\mathbb{R}^N) \longrightarrow \nu_\Delta(X+h) = \nu_\Delta(X)\right). \tag{1.17}$$

We define

$$G_\Delta = \{h : h \in \mathbb{R}^N,\ h \text{ is an admissible translation for } \nu_\Delta\}. \tag{1.18}$$

It is easy to show that G_Δ is a vector subspace of the space $\mathbb{R}^N$.

Remark 1.3. The construction of the measure ν_Δ belongs to Kharazishvili [1].

Our next theorem gives a representation of the algebraic structure of the vector subspace G_Δ of all admissible translations for ν_Δ.

Theorem 1.4. *The following conditions are equivalent:*

$$g = (g_1, g_2, \dots) \in G_\Delta, \tag{1.19}$$

$$(\exists n_g) \quad \left(n_g \in \mathbb{N} \longrightarrow \text{ the series } \sum_{i=n_g}^{\infty} \ln\left(1 - \frac{|g_i|}{b_i - a_i}\right) \text{ is convergent}\right). \tag{1.20}$$

Proof. Assume that for an element $g = (g_1, g_2, \dots) \in \mathbb{R}^N$, the condition (1.19) is satisfied. Then we have

$$\nu_\Delta(\Delta + g) = \nu_\Delta(\Delta) = 1. \tag{1.21}$$

On the other hand, we have

$$\begin{aligned}
\nu_\Delta(\Delta + g) &= \nu_\Delta(\Delta + g) \\
&= \nu_\Delta\left(\prod_{i\in\mathbb{N}} [a_i + g_i, b_i + g_i[\right) \\
&= \lim_{n\to\infty} \bar{\nu}_n(A_n \cap (\Delta + g)) \\
&= \lim_{n\to\infty} \left(\prod_{1\le i\le n} \mu_i \times \prod_{i>n} \lambda_i\right)\left(\left(\prod_{1\le i\le n} R_i \times \prod_{i>n} [a_i, b_i[\right) \cap \prod_{i\in\mathbb{N}} [a_i + g_i, b_i + g_i[\right) \\
&= \lim_{n\to\infty} \left(\prod_{1\le i\le n} \mu_i \left(\prod_{1\le i\le n} [a_i + g_i, b_i + g_i[\right)\right) \times \left(\prod_{i>n} \lambda_i([a_i + g_i, b_i + g_i[)\right) \\
&= \lim_{n\to\infty} \prod_{i>n} \lambda_i([a_i, b_i[\cap [a_i + g_i, b_i + g_i[) = 1.
\end{aligned} \tag{1.22}$$

We show that

$$(\forall g) \quad \left(g = (g_1, g_2, \dots) \in G_\Delta \longrightarrow \lim_{i\to\infty} \frac{|g_i|}{|b_i - a_i|} = 0\right). \tag{1.23}$$

Indeed, if we assume the contrary, then there exist a countable subset $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ and a positive real number $\epsilon > 0$, such that

$$(\forall k) \quad \left(k \in \mathbb{N} \longrightarrow \frac{|g_{n_k}|}{b_{n_k} - a_{n_k}} > \epsilon\right). \tag{1.24}$$

Choose a number $m > 0$ such that $\epsilon \cdot m > 1$. Since $g \in G_\Delta$, we have

$$m \cdot g = (m \cdot g_1, m \cdot g_2, \dots) \in G_\Delta. \tag{1.25}$$

In view of the property of σ-additivity of the measure ν_Δ, we obtain

$$\nu_\Delta(\Delta) = \nu_\Delta(\Delta + m \cdot g) = 1. \tag{1.26}$$

But note that

$$(\Delta + m \cdot g) \cap \left(\bigcup_{n \in \mathbb{N}} A_n \right) = \varnothing. \tag{1.27}$$

Indeed, assume the contrary and take

$$(x_i)_{i \in \mathbb{N}} \in (\Delta + m \cdot g) \cap \left(\bigcup_{n \in \mathbb{N}} A_n \right). \tag{1.28}$$

Then it is clear that, for the n_kth coordinate, we have

$$\begin{aligned}(\exists k_0) \quad (k_0 \in \mathbb{N} \text{ and } (\forall k) \quad (k \geq k_0 \longrightarrow (a_{n_k} + m \cdot g_{n_k} \leq x_{n_k} < b_{n_k} + m \cdot g_{n_k}), \\ (a_{n_k} \leq x_{n_k} < b_{n_k}))).\end{aligned} \tag{1.29}$$

On the other hand, the validity of the condition

$$(\forall k) \quad \left(k \in \mathbb{N} \longrightarrow \frac{|g_{n_k}|}{b_{n_k} - a_{n_k}} > \epsilon \right) \tag{1.30}$$

implies the validity of the relation

$$(\forall k) \quad (k \in \mathbb{N} \longrightarrow m \cdot |g_{n_k}| > b_{n_k} - a_{n_k}), \tag{1.31}$$

which shows that the intervals $[a_{n_k}, b_{n_k}[$ and $[a_{n_k} + m \cdot g_{n_k}, b_{n_k} + m \cdot g_{n_k}[$ have an empty intersection. Hence the condition $\lim_{i\to\infty}(|g_i|/(b_i - a_i)) = 0$ holds.

From the validity of the condition $\lim_{i\to\infty}(|g_i|/(b_i - a_i)) = 0$, we conclude that there exists a natural number n_g such that

$$(\forall i) \quad \left(i > n_g \longrightarrow \frac{|g_i|}{b_i - a_i} < 1 \right), \tag{1.32}$$

since

$$(\forall i) \quad \left(i > n_g \longrightarrow \lambda_i([a_i, b_i[\cap [a_i + g_i, b_i + g_i[) = \frac{b_i - a_i - |g_i|}{b_i - a_i} = 1 - \frac{|g_i|}{b_i - a_i} \right). \tag{1.33}$$

Keeping in mind that

$$\lim_{p\to\infty} \prod_{i \geq n_g + p} \left(1 - \frac{|g_i|}{b_i - a_i} \right) = 1 \tag{1.34}$$

and considering the logarithms of both sides, we have

$$\lim_{p\to\infty} \sum_{i \geq n_g + p} \ln \left(1 - \frac{|g_i|}{b_i - a_i} \right) = 0. \tag{1.35}$$

This means that the series $\sum_{i \geq n_g} \ln(1 - |g_i|/(b_i - a_i))$ is convergent.

The validity of the implication (1.19)→(1.20) is proved.

Now we prove (1.20)→(1.19). Let n_g be a natural number such that the series $\sum_{i\geq n_g}\ln(1-|g_i|/(b_i-a_i))$ is convergent.

Consider an arbitrary element X having the form

$$X = B\times\prod_{i>n}\Delta_i, \tag{1.36}$$

where $B\in B(\mathbb{R}^N)$ $(n\in\mathbb{N})$.

The sets of these forms generate the σ-algebra $B(A_n)$ of the space A_n, and the condition $B(A_n) = B(\mathbb{R}^N)\cap A_n$ holds. To prove the implication (1.20)→(1.19), it is sufficient to show the validity of the condition

$$\begin{aligned}
\nu_\Delta(X+g) &= \nu_\Delta\left\{\left[\left(B\times\prod_{n+1\leq i\leq n_g+n}\Delta_i\right)+(g_1,\dots,g_{n_g})\right]\times\prod_{i>n_g+n}[a_i+g_i,b_i+g_i[\right\}\\
&= \lim_{n\to\infty}\prod_{i=1}^{n_g+n}\mu_i\left(B\times\prod_{n+1\leq i\leq n_g+n}\Delta_i\right)\\
&\quad\times\prod_{i>n_g+n}\lambda_i([a_i+g_i,b_i+g_i[\cap[a_i,b_i[) = \nu_\Delta\left(B\times\prod_{i>n}\Delta_i\right)\\
&\quad\times\lim_{n\to\infty}\prod_{i>n_g+n}\left(1-\frac{|g_i|}{b_i-a_i}\right) = \nu_\Delta\left(B\times\prod_{i>n}\Delta_i\right) = \nu_\Delta(X).
\end{aligned}\tag{1.37}$$

We have used the well-known result from mathematical analysis

$$\begin{aligned}
&\left(\text{the series }\sum_{i\geq n_g}\ln\left(1-\frac{|g_i|}{b_i-a_i}\right)\text{ is convergent}\right)\\
&\quad\Longleftrightarrow\lim_{n\to\infty}\prod_{i\geq n_g+n}\left(1-\frac{|g_i|}{b_i-a_i}\right)=\ln 1\\
&\quad\Longleftrightarrow\lim_{n\to\infty}\prod_{i>n_g+n}\left(1-\frac{|g_i|}{b_i-a_i}\right)=1.
\end{aligned}\tag{1.38}$$

The proof is completed. □

Remark 1.5. Let $\mathbb{R}^{(N)}$ be the space of all finite sequences, that is,

$$\mathbb{R}^{(N)} = \{(g_i)_{i\in\mathbb{N}}\mid(g_i)_{i\in\mathbb{N}}\in\mathbb{R}^N,\ \operatorname{card}\{i\mid g_i\neq 0\}<\aleph_0\}. \tag{1.39}$$

It is clear that, on the one hand, for an arbitrary compact infinite-dimensional parallelepiped $\Delta=\prod_{k\in\mathbb{N}}[a_k,b_k[$, we have

$$\mathbb{R}^{(N)}\subset G_\Delta. \tag{1.40}$$

On the other hand, $G_\Delta \setminus \mathbb{R}^{(N)} \neq \emptyset$, since the element $(g_i)_{i\in\mathbb{N}}$ defined by

$$(\forall i) \quad \left(i \in \mathbb{N} \longrightarrow g_i = \left(1 - \exp\left\{-\frac{b_i - a_i}{2^i}\right\} \times (b_i - a_i)\right)\right) \tag{1.41}$$

belongs to the difference $G_\Delta \setminus \mathbb{R}^{(N)}$.

It is easy to show that the vector space G_Δ is everywhere dense in $\mathbb{R}^N$ with respect to the Tychonoff topology, since $\mathbb{R}^{(N)} \subset G_\Delta$.

In the sequel, we will need the following result.

Theorem 1.6. *In the separable Hilbert space ℓ_2, there exists a σ-finite Borel measure λ such that*

(1) $\lambda(\Delta_0) = 1$;

(2) *a group G_{Δ_0} of all admissible translations of the measure λ has the form*

$$G_{\Delta_0} = \Big\{ (c_k)_{k\in\mathbb{N}} \mid (c_k)_{k\in\mathbb{N}} \in \ell_2, \\ (\exists n_p) \left(n_p \in \mathbb{N} \longrightarrow \text{ the series } \sum_{n=n_p}^{\infty} \ln\left(1 - |c_k|(i+1)\right) \text{ is convergent}\right)\Big\}, \tag{1.42}$$

where $\Delta_0 = \prod_{i\in\mathbb{N}} [0;1/(i+1)[$.

Proof. According to Suslin's theorem we have $B(\ell_2) \subseteq B(\mathbb{R}^N)$. Now the proof of Theorem 1.6 can be obtained easily if we put

$$(\forall X) \quad \left(X \in B(\ell_2) \Longrightarrow \lambda(X) = \nu_{\Delta_0}(\ell_2 \cap X)\right). \tag{1.43}$$

□

2. Duality of measure and category in the infinite-dimensional separable Hilbert space ℓ_2. In this section, we continue our discussion of some properties of invariant measures in the infinite-dimensional separable Hilbert space ℓ_2 and study the question of the duality between the Baire category and the above-constructed measure λ.

The following definitions are important for our investigation.

Let (E,T) be a nonempty topological vector space. Denote by $B(E)$ the Borel σ-algebra of subsets of the space E, generated by the topology T. Consider a nontrivial Borel measure μ defined on the σ-algebra $B(E)$. A subset $X \subseteq E$ is called small in the sense of measure if $\mu^*(X) = 0$. Analogously, a subset $Y \subseteq E$ is called small in the sense of category if it is the first category set in the topological space (E,T). Further, let P be a such sentence in formulation of which the notions of measure zero and of the first category are used. We say that the duality between the measure μ and the Baire category is valid with respect to the sentence P if the sentence P is equivalent to the sentence P^* obtained from the sentence P by interchanging the notions of the above small sets. We also say that strict duality between the measure μ and Baire category is valid if the duality between the measure μ and the Baire category is valid for all the above P sentences formulated only by using the notions of measure zero, of first category and of purely set-theoretical notions.

The following result is known as the Erdős-Sierpiński duality principle.

Theorem 2.1 (duality principle)**.** *If the continuum hypothesis is true, then the strict duality between a linear Lebesgue measure and the Baire category of the real axis $\mathbb{R}$ is valid.*

The proof of Theorem 2.1 can be found, for example, in [4].

Using the same argument applied in the process of the proving of Theorem 2.1 (see [4, pages 129–131]), it is easy to conclude that if the continuum hypothesis is true, then the strict duality between the measure λ and Baire category of ℓ_2 is valid also.

Here we apply the well-known method to establish one important property of Baire second category subsets in the infinite-dimensional separable Hilbert space ℓ_2.

Theorem 2.2. *For an arbitrary second category Baire subset $X \subseteq \ell_2$, there exists a positive number $\delta > 0$ such that*

$$(\forall x) \quad (x \in \ell_2,\ \|x\| < \delta \longrightarrow (X + x) \cap X \neq \varnothing). \tag{2.1}$$

Proof. Since the set X has the Baire property, there exist an open subset $G \subseteq \ell_2$ and a first category subset $P \subseteq \ell_2$ such that the equality

$$X = G \Delta P \tag{2.2}$$

is fulfilled.

Evidently, there exists an open nonempty ball $B \subseteq G$.

Note that the inclusion

$$[(x + B) \cap B] \setminus [P \cup (x + P)] \subseteq (x + X) \cap X \tag{2.3}$$

holds for arbitrary $x \in \ell_2$. If $\|x\| < \operatorname{diam}(B)$, then the set, the left-hand side of (2.3), is a nonempty open set minus a first category set.

Using the well-known Baire theorem, we complete the proof of Theorem 2.2. □

Remark 2.3. The method considered in the proof of Theorem 2.2 was worked out and applied by many authors, for example, Oxtoby who establishes an analogous result for linear Baire second category subsets in $\mathbb{R}$ (cf. [4]).

The following simple result (which is however important from the viewpoint of applications) is also essentially due to Steinhaus.

Theorem 2.4. *Let X be an arbitrary linear Borel subset in $\mathbb{R}$ with a positive Lebesgue measure. Then there exists a positive number δ such that the condition*

$$(\forall x) \quad (x \in \mathbb{R},\ |x| < \delta \longrightarrow (x + X) \cap X \neq \varnothing) \tag{2.4}$$

holds.

The proof of Theorem 2.4 can be found in [4].

The next theorem plays the main role in our further consideration.

Theorem 2.5. *In the infinite-dimensional separable Hilbert space ℓ_2, there exists a Borel subset $Y \subset \ell_2$ with $\lambda(Y) > 0$ such that*

$$(\forall \delta) \quad (\delta > 0 \longrightarrow (\exists y)\ (\|y\| < \delta \longrightarrow Y \cap (Y + y) = \varnothing)). \tag{2.5}$$

Proof. Let

$$Y \equiv \Delta_0 = \prod_{i\in\mathbb{N}} \left[0, \frac{1}{i+1}\right[. \tag{2.6}$$

For an arbitrary positive real number $\delta > 0$, denote by n_δ a natural number such that

$$\sum_{i=n_\delta}^{\infty} \frac{1}{(i+1)^2} < \delta^2. \tag{2.7}$$

Assume that

$$\begin{aligned} &(\forall k) \quad (1 \le k < n_\delta \longrightarrow h_k = 0), \\ &(\forall k) \quad \left(k \ge n_\delta \longrightarrow h_k = \frac{1}{k+1}\right). \end{aligned} \tag{2.8}$$

It is clear that $h = (h_k)_{k\in\mathbb{N}} \notin G_{\Delta_0}$, $\|h\| < \delta$, and $\Delta_0 \cap (\Delta_0 + h) = \varnothing$. Theorem 2.5 is proved. □

Summarizing all the above results, we obtain the following statement.

Theorem 2.6. *The duality between the measure λ and the Baire category with respect to the sentence P_0, where*

$$\begin{aligned} P_0 = (\forall X) \quad &(X \subseteq \ell_2,\ X \text{ is a Baire subset of second category} \\ &\longrightarrow (\exists \delta)\ (\delta > 0 \longrightarrow (\forall x)\ (\|x\| < \delta \longrightarrow X \cap (X + x) \neq \varnothing))), \end{aligned} \tag{2.9}$$

is not valid.

Remark 2.7. By Remark 2.3 and Theorem 2.4, it is easy to obtain the validity of the duality between the linear Lebesgue measure and the Baire category with respect to the sentence P_0 in $\mathbb{R}$. This result is essentially due to Oxtoby and may be called Oxtoby duality principle in $\mathbb{R}$ (cf. [4]).

Remark 2.8. Theorem 2.6 states that an analogy of the Oxtoby duality principle is not valid for the measure λ and the Baire category in the infinite-dimensional separable Hilbert space ℓ_2.

There are also several important works devoted to the solution of analogous problems in various topological vector spaces (cf. [2, 3] and others).

The following notion is frequently useful in studying various questions of measure theory.

We say that the measure μ defined in a topological vector space (E, T) satisfies the axiom of Steinhaus if the following condition:

$$\begin{aligned} (\forall X) \quad &\big(X \in \mathrm{dom}(\mu),\ \mu(X) < \infty \\ &\longrightarrow (\forall \epsilon)\ (\epsilon > 0 \longrightarrow ((\text{there exists a neighborhood } V_\epsilon \text{ of the zero vector } 0), \\ &\qquad ((\forall h)\ (h \in V_\epsilon \longrightarrow \mu((X + h) \triangle X) < \epsilon))))\big) \end{aligned} \tag{2.10}$$

holds.

Theorem 2.9. *The measure λ does not satisfy the axiom of Steinhaus.*

Proof. Assume the contrary. Then for the set Δ_0 and for the number $\epsilon = 1/2$, there exists a number $\delta > 0$ such that

$$(\forall x)\ \Big(\|x\| < \delta \longrightarrow \lambda((\Delta_0 + x)\triangle\Delta_0) < \frac{1}{2}\Big). \tag{2.11}$$

Consider the element $h = (h_k)_{k\in\mathbb{N}}$ constructed in Theorem 2.5. Since $\|h\| < \delta$, $(\Delta_0 + h) \cap (\bigcup_{n\in\mathbb{N}} A_n) = \emptyset$, and the measure λ is concentrated on the set $\bigcup_{n\in\mathbb{N}} A_n$, where A_n is defined in Section 1 for $\Delta = \Delta_0$, we have $\lambda((\Delta_0 + h)\triangle\Delta_0) = \lambda(\Delta_0) = 1$. This contradicts the condition

$$\lambda((\Delta_0 + h)\triangle\Delta_0) < \frac{1}{2}. \tag{2.12}$$

Thus, Theorem 2.9 is proved. □

Remark 2.10. We must say that the analogies of Theorems 2.6 and 2.9 are valid for an arbitrary nontrivial σ-finite Borel measure and Baire category defined in infinite-dimensional Polish topological vector space, but this question will not concern us here.

Example 2.11. Define the measure μ_0 by

$$(\forall B)\quad \big(B \in B(\ell_2) \longrightarrow \mu_0(B) = \begin{cases} \infty, & \text{if } \mathbf{B} \text{ is of second category}, \\ 0, & \text{if } \mathbf{B} \text{ is of first category}\big). \end{cases} \tag{2.13}$$

It is proved that, on the one hand, the measure μ_0 satisfies Suslin's property and is invariant with respect to the vector space ℓ_2 (see [3]). On the other hand, using Theorem 2.2, we conclude that the measure μ_0 (unlike the measure λ) satisfies

$$\begin{aligned} (\forall X)\quad & \big(X \in B(\ell_2),\ \mu(X) > 0 \\ & \longrightarrow (\exists\delta)\ (\delta > 0 \longrightarrow (\forall h)\ (\|h\| < \delta \longrightarrow (X + h) \cap X \neq \emptyset))\big). \end{aligned} \tag{2.14}$$

This means that the duality between the measure μ_0 (which is not σ-finite) and the Baire category, with respect to the property P_0, is valid in the separable Hilbert space ℓ_2. Also note that the measure μ_0 satisfies the axiom of Steinhaus.

Remark 2.12. Clearly, it is not possible to define, in the space ℓ_2, a translation-invariant nontrivial σ-finite Borel measure. But if we ignore the condition of σ-finiteness, then in some consistent system of axioms, the construction of such Borel measures is possible (cf. [5]). In connection with the above results, one can pose the problem of the validity of the duality between the translate-invariant Borel measure and the Baire category with respect to the property P_0 in the infinite-dimensional separable Hilbert space ℓ_2.

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Gogi Pantsulaia: Department of Higher Mathematics, Georgian Technical University, 77 Kostava Street, Tbilisi 380043, Georgia
E-mail address: gogi@gtu.edu.ge