IJMMS 30:6 (2002) 319–325
PII. S0161171202110350
http://ijmms.hindawi.com

SOME RESULTS ON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN COMPLETE METRIC SPACES

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Received 17 October 2001

Using the concept of w-distance, we improve some well-known fixed point theorems.

2000 Mathematics Subject Classification: 47H10.

1. Introduction. Recently, Ume [3] improved the fixed point theorems in a complete metric space using the concept of w-distance, introduced by Kada, Suzuki, and Takahashi [2], and more general contractive mappings than quasi-contractive mappings.

In this paper, using the concept of w-distance, we first prove common fixed point theorems for multivalued mappings in complete metric spaces, then these theorems are used to improve Ćirić's fixed point theorem [1], Kada-Suzuki-Takahashi's fixed point theorem [2], and Ume's fixed point theorem [3].

2. Preliminaries. Throughout, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Definition 2.1 (see [2]). Let (X,d) be a metric space, then a function $p : X\times X \to [0,\infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x,z) \le p(x,y)+p(y,z)$ for all $x,y,z \in X$;
(2) for any $x \in X$, $p(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \epsilon$.

Definition 2.2. Let (X,d) be a metric space with a w-distance p, then

(1) for any $x \in X$ and $A \subseteq X$, $d(x,A) := \inf\{d(x,y) : y \in A\}$ and $d(A,x) := \inf\{d(y,x) : y \in A\}$;
(2) for any $x \in X$ and $A \subseteq X$, $p(x,A) := \inf\{p(x,y) : y \in A\}$ and $p(A,x) := \inf\{p(y,x) : y \in A\}$;
(3) for any $A,B \subseteq X$, $p(A,B) := \inf\{p(x,y) : x \in A,\ y \in B\}$;
(4) $CB_p(X) = \{A \mid A$ is nonempty closed subset of X and $\sup_{x,y\in A} p(x,y) < \infty\}$.

The following lemmas are fundamental.

Lemma 2.3 (see [2]). *Let X be a metric space with a metric d, let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x,y,z \in X$. Then the following hold:*

(1) *if* $p(x_n, y) \le \alpha_n$ *and* $p(x_n, z) \le \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $p(x, y) = 0$ *and* $p(x, z) = 0$, *then* $y = z$;
(2) *if* $p(x_n, y_n) \le \alpha_n$ *and* $p(x_n, z) \le \beta_n$ *for any* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* z;
(3) *if* $p(x_n, x_m) \le \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence;*
(4) *if* $p(y, x_n) \le \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 2.4 (see [3]). *Let* X *be a metric space with a metric* d, *let* p *be a* w*-distance on* X, *and let* T *be a mapping of* X *into itself satisfying*

$$p(Tx, Ty) \le q \cdot \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\} \tag{2.1}$$

for all $x, y \in X$ *and some* $q \in [0, 1)$. *Then*
(1) *for each* $x \in X$, $n \in \mathbb{N}$, *and* $i, j \in \mathbb{N}$ *with* $i, j \le n$,

$$p(T^i x, T^j x) \le q \cdot \delta(O(x, n)); \tag{2.2}$$

(2) *for each* $x \in X$ *and* $n \in \mathbb{N}$, *there exist* $k, l \in \mathbb{N}$ *with* $k, l \le n$ *such that*

$$\delta(O(x, n)) = \max\{p(x, x), p(x, T^k x), p(T^l x, x)\}; \tag{2.3}$$

(3) *for each* $x \in X$,

$$\delta(O(x, \infty)) \le \frac{1}{1-q}\{p(x, x) + p(x, Tx) + p(Tx, x)\}; \tag{2.4}$$

(4) *for each* $x \in X$, $\{T^n x\}_{n=1}^{\infty}$ *is a Cauchy sequence.*

3. Main results

Theorem 3.1. *Let* X *be a complete metric space with a metric* d *and let* p *be a* w*-distance on* X. *Suppose that* S *and* T *are two mappings of* X *into* $CB_p(X)$ *and* $\varphi : X \times X \to [0, \infty)$ *is a mapping such that*

$$\max\{p(u_1, u_2), p(v_1, v_2)\} \le q \cdot \varphi(x, y) \tag{3.1}$$

for all nonempty subsets A, B *of* X, $u_1 \in SA$, $u_2 \in S^2 A$, $v_1 \in TB$, $v_2 \in T^2 B$, $x \in A$, $y \in B$, *and some* $q \in [0, 1)$,

$$\sup\left\{\sup\left(\frac{\varphi(x, y)}{\min[p(x, SA), p(y, TB)]} : x \in A,\ y \in B\right) : A, B \subseteq X\right\} < \frac{1}{q}, \tag{3.2}$$

$$\inf\{p(y, u) + p(x, Sx) + p(y, Ty) : x, y \in X\} > 0, \tag{3.3}$$

for every $u \in X$ *with* $u \notin Su$ *or* $u \notin Tu$, *where* SA *means* $\bigcup_{a \in A} Sa$. *Then* S *and* T *have a common fixed point in* X.

Proof. Let

$$\beta = \sup\left\{\sup\left(\frac{\varphi(x, y)}{\min[p(x, SA), p(y, TB)]} : x \in A,\ y \in B\right) : A,\ B \subseteq X\right\}, \tag{3.4}$$

and $k = \beta q$. Define $x_{n+1} \in Sx_n$ and $y_{n+1} \in Ty_n$ for all $n \in \mathbb{N}$. Then $x_n \in Sx_{n-1}$, $x_{n+1} \in S^2 x_{n-1}$, $y_n \in Ty_{n-1}$, and $y_{n+1} \in T^2 y_{n-1}$. From (3.1) and (3.2), we have

$$p(x_n, x_{n+1}) \le kp(x_{n-1}, x_n) \le \cdots \le k^{n-1} p(x_1, x_2), \tag{3.5}$$

$$p(y_n, y_{n+1}) \le kp(y_{n-1}, y_n) \le \cdots \le k^{n-1} p(y_1, y_2), \tag{3.6}$$

for all $n \in \mathbb{N}$ and some $k \in [0,1)$. Let n and m be any positive integers such that $n < m$. Then, from (3.6), we obtain

$$\begin{aligned} p(y_n, y_m) &\le p(y_n, y_{n+1}) + \cdots + p(y_{m-1}, y_m) \\ &= \sum_{i=0}^{m-n-1} p(y_{n+i}, y_{n+i+1}) \\ &\le \sum_{i=0}^{m-n-1} k^{n+i-1} p(y_1, y_2) \\ &\le \frac{k^{n-1}}{(1-k)} p(y_1, y_2). \end{aligned} \tag{3.7}$$

By Lemma 2.3, $\{y_n\}$ is a Cauchy sequence. Since X is complete, $\{y_n\}$ converges to $u \in X$. Then, since $p(y_n, \cdot)$ is lower semicontinuous, from (3.7) we have

$$p(y_n, u) \le \liminf_{m \to \infty} p(y_n, y_m) \le \frac{k^{n-1}}{(1-k)} p(y_1, y_2). \tag{3.8}$$

Suppose that $u \notin Su$ or $u \notin Tu$. Then, by (3.3), (3.5), (3.6), and (3.8), we have

$$\begin{aligned} 0 &< \inf \{p(y, u) + p(x, Sx) + p(y, Ty) : x, y \in X\} \\ &\le \inf \{p(y_n, u) + p(x_n, x_{n+1}) + p(y_n, y_{n+1}) : n \in \mathbb{N}\} \\ &\le \inf \left\{ \frac{k^{n-1}}{(1-k)} p(y_1, y_2) + k^{n-1} p(x_1, x_2) + k^{n-1} p(y_1, y_2) : n \in \mathbb{N} \right\} \\ &= \left\{ \frac{2-k}{(1-k)} p(y_1, y_2) + p(x_1, x_2) \right\} \inf \{k^{n-1} : n \in \mathbb{N}\} \\ &= 0. \end{aligned} \tag{3.9}$$

This is a contradiction. Therefore we have $u \in Su$ and $u \in Tu$. □

Theorem 3.2. *Let X be a complete metric space with a metric d and let p be a w-distance on X. Suppose that S and T are two mappings of X into $CB_p(X)$ and $\varphi : X \times X \to [0, \infty)$ is a mapping such that*

$$\max \{p(u_1, u_2), p(v_1, v_2)\} \le q \cdot \varphi(x, y) \tag{3.10}$$

for all $x, y \in X$, $u_1 \in Sx$, $u_2 \in S^2 x$, $v_1 \in Ty$, $v_2 \in T^2 y$, and some $q \in [0,1)$,

$$\sup \left\{ \sup \left(\frac{\varphi(x, y)}{\min [p(x, Sx), p(y, Ty)]} : x \in A,\ y \in B \right) : A, B \subseteq X \right\} < \frac{1}{q}, \tag{3.11}$$

and (3.3) is satisfied. Then S and T have a common fixed point in X.

Proof. By a method similar to that in the proof of Theorem 3.1, the result follows. □

Theorem 3.3. *Let X be a complete metric space with a metric d and let p be a w-distance on X. Suppose that T is a mapping of X into $CB_p(X)$ and $\psi : X \to [0,\infty)$ is a mapping such that*

$$p(u_1,u_2) \leq q \cdot \psi(x) \tag{3.12}$$

for all $x \in X$, $u_1 \in Tx$, $u_2 \in T^2x$ and some $q \in [0,1)$,

$$\sup\left\{\frac{\psi(x)}{p(x,Tx)} : x \in X\right\} < \frac{1}{q}, \qquad \inf\{p(x,u)+p(x,Tx) : x \in X\} > 0, \tag{3.13}$$

for every $u \in X$ with $u \notin Tu$. Then T has a fixed point in X.

Proof. By a method similar to that in the proof of Theorem 3.1, the result follows. □

Theorem 3.4. *Let X be a complete metric space with a metric d and let p be a w-distance on X. Suppose that S and T are self-mapping of X and $\varphi : X \times X \to [0,\infty)$ is a mapping such that*

$$\max\{p(Sx,S^2x),p(Ty,T^2y)\} \leq q \cdot \varphi(x,y) \tag{3.14}$$

for all $x,y \in X$ and some $q \in [0,1)$,

$$\sup\left\{\frac{\varphi(x,y)}{\min[p(x,Sx),p(y,Ty)]} : x,y \in X\right\} < \frac{1}{q}, \qquad \inf\{p(y,u)+p(x,Sx)+p(y,Ty) : x,y \in X\} > 0, \tag{3.15}$$

for every $u \in X$ with $u \neq Su$ or $u \neq Tu$. Then S and T have a common fixed point in X.

Proof. By a method similar to that in the proof of Theorem 3.1, the result follows. □

From Theorem 3.1, we have the following corollary.

Corollary 3.5. *Let X be a complete metric space with a metric d and let p be a w-distance on X. Suppose that S and T are two mappings of X into $CB_p(X)$ and $\varphi : X \times X \to [0,\infty)$ is a mapping such that*

$$\max\Big\{\sup[p(u_1,u_2) : u_1 \in Sx,\ u_2 \in S^2x],\ \sup[p(v_1,v_2) : v_1 \in Tx,\ v_2 \in T^2x]\Big\} \leq q \cdot \varphi(x,y) \tag{3.16}$$

for all $x,y \in X$ and some $q \in [0,1)$, and that (3.3) and (3.11) are satisfied. Then S and T have a common fixed point in X.

From Theorem 3.3, we have the following corollaries.

Corollary 3.6. *Let X be a complete metric space with a metric d and let p be a w-distance on X. Suppose that T is a mapping of X into $CB_p(X)$ and $\psi : X \to [0,\infty)$ is a mapping such that*

$$\sup\left[p(u_1,u_2) : u_1 \in Tx,\ u_2 \in T^2x\right] \le q \cdot \psi(x) \tag{3.17}$$

for all $x \in X$ and some $q \in [0,1)$, and that (3.13) is satisfied. Then T has a fixed point in X.

Corollary 3.7. *Let X be a complete metric space with a metric d and let p be a w-distance on X. Suppose that T is a self-mapping of X and $\psi : X \to [0,\infty)$ is a mapping such that*

$$p(Tx,T^2x) \le q \cdot \psi(x) \tag{3.18}$$

for all $x \in X$ and some $q \in [0,1)$,

$$\begin{gathered} \sup\left\{\frac{\psi(x)}{p(x,Tx)} : x \in X\right\} < \frac{1}{q}, \\ \inf\{p(x,u) + p(x,Tx) : x \in X\} > 0, \end{gathered} \tag{3.19}$$

for every $u \in X$ with $u \neq Tu$. Then T has a fixed point in X.

From Corollary 3.7, we have the following corollaries.

Corollary 3.8 (see [3]). *Let X be a complete metric space with a metric d and let p be a w-distance on X. Suppose that T is a self-mapping of X such that*

$$p(Tx,Ty) \le q \cdot \max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\} \tag{3.20}$$

for all $x, y \in X$ and some $q \in [0,1)$, and that

$$\inf\{p(x,u) + p(x,Tx) : x \in X\} > 0 \tag{3.21}$$

for every $u \in X$ with $u \neq Tu$. Then T has a unique fixed point in X.

Proof. By (3.20) and Lemma 2.4(3), we have

$$\sup\{p(T^ix,T^jx) \mid i,\ j \in \mathbb{N} \cup \{0\}\} < \infty \tag{3.22}$$

for every $x \in X$. Thus we may define a function $r : X \times X \to [0,\infty)$ by

$$r(x,y) = \max\{\sup[p(T^ix,T^jx) \mid i,\ j \in \mathbb{N} \cup \{0\}], p(x,y)\} \tag{3.23}$$

for every $x, y \in X$. Clearly, r is a w-distance on X. Let x be a given element of X, then, by using Lemma 2.4(1), (3.20), and (3.23), we have

$$\begin{aligned} r(Tx,T^2x) &= \sup\{p(T^ix,T^jx) \mid i,\ j \in \mathbb{N}\} \\ &\le q \cdot \sup\{p(T^ix,T^jx) \mid i,\ j \in \mathbb{N} \cup \{0\}\} \\ &= q \cdot r(x,Tx). \end{aligned} \tag{3.24}$$

By (3.21) and (3.23), we obtain

$$\inf\{r(x,u)+r(x,Tx) : x \in X\} > 0 \tag{3.25}$$

for every $u \in X$ with $u \neq Tu$. From (3.24), (3.25), and Corollary 3.7, T has a fixed point in X. By (3.20) and Lemma 2.4, it is clear that the fixed point of T is unique. □

Corollary 3.9 (see [2]). *Let X be a complete metric space, let p be a w-distance on X, and let T be a mapping from X into itself. Suppose that there exists $q \in [0,1)$ such that*

$$p(Tx,T^2x) \leq q \cdot p(x,Tx) \tag{3.26}$$

for every $x \in X$ and that

$$\inf\{p(x,y)+p(x,Tx) : x \in X\} > 0 \tag{3.27}$$

for every $y \in X$ with $y \neq Ty$. Then T has a fixed point in X.

Proof. Define $\psi : X \to [0,\infty)$ by

$$\psi(x) = p(x,Tx) \tag{3.28}$$

for all $x \in X$. Thus the conditions of Corollary 3.7 are satisfied. Hence T has a fixed point in X. □

From Corollary 3.8, we have the following corollary.

Corollary 3.10 (see [1]). *Let X be a complete metric space with a metric d and let T be a mapping from X into itself. Suppose that T is a quasicontraction, that is, there exists $q \in [0,1)$ such that*

$$d(Tx,Ty) \leq q \cdot \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\} \tag{3.29}$$

for every $x,y \in X$. Then T has a unique fixed point in X.

Proof. It is clear that the metric d is a w-distance and

$$\inf\{d(x,y)+d(x,Tx) : x \in X\} > 0 \tag{3.30}$$

for every $y \in X$ with $y \neq Ty$. Thus, by Corollary 3.8 or 3.9, T has a unique fixed point in X. □

Acknowledgment. This work was supported by grant No. 2001-1-10100-005-2 from the Basic Research Program of the Korea Science & Engineering Foundation.

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