Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 136134, 12 pages
doi:10.1155/2012/136134

Research Article

Strong Convergence Theorems for a Countable Family of Total Quasi-ϕ-Asymptotically Nonexpansive Nonself Mappings

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Received 3 June 2012; Revised 22 July 2012; Accepted 23 July 2012

Academic Editor: Naseer Shahzad

The purpose of this paper is to introduce a class of total quasi-ϕ-asymptotically nonexpansive-nonself mappings and to study the strong convergence under a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper extend and improve the corresponding results announced by some authors recently.

1. Introduction

Throughout this paper, we assume that E is a real Banach space, C is a nonempty closed and convex subset of E, E^* is the dual space of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f^* \in E^*, \langle x, f^* \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad x \in E. \tag{1.1}$$

Recall that a Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex*, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be *smooth*, if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{$**$}$$

exists for all $x, y \in U$. And E is said to be *uniformly smooth*, if the above limit is exists uniformly for $x, y \in U$.

In the sequel, we shall denote the fixed point set of a mapping T by $F(T)$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x.

A mapping $T : C \to C$ is said to be *nonexpansive*, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

Recall that a subset C of E is said to be *retract* of E, if there exists a continuous mapping $P : E \to C$ such that $Px = x$, for all $x \in C$.

It is well known that every nonempty closed and convex subset of a uniformly convex Banach space is a retract of E. A mapping $P : E \to C$ is said to be a *retraction*, if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A mapping $P : E \to C$ is said to be a *nonexpansive retraction*, if it is nonexpansive and it is a retraction from E to C.

In the sequel, we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E. Throughout this paper we assume that $\phi : E \times E \to \mathcal{R}^+$ is the Lyapunov function which is defined by

$$\phi(x, y) = \|x\| - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.4}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E, \tag{1.5}$$

$$\phi\Big(x, J^{-1}(\lambda J y + (1 - \lambda) J z)\Big) \leq \lambda \phi(x, y) + (1 - \lambda)\phi(x, z), \quad \forall x, y \in E. \tag{1.6}$$

Following Alber [1], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg\inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{1.7}$$

Lemma 1.1 (see [1]). *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(1) $\phi(x, \Pi_C y) + \phi(\Pi_c y, y) \leq \phi(x, y)$ *for all* $x \in C$ *and* $y \in E$;

(2) *If* $x \in E$ *and* $z \in C$, *then* $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \geq 0$, *for all* $y \in C$;

(3) *For* $x, y \in E$, $\phi(x, y) = 0$ *if and only if* $x = y$.

Remark 1.2. If E is a real Hilbert space H, then $\phi(x,y) = \|x-y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

A mapping $T : C \to C$ is said to be *closed*, if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Definition 1.3. Let $P : E \to C$ be the nonexpansive retraction.

(1) $T : C \to E$ is said to be *quasi-ϕ-nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and

$$\phi(u,Tx) \leq \phi(u,x), \quad \forall x \in C,\ u \in F(T). \tag{1.8}$$

(2) $T : C \to E$ is said to be *quasi-ϕ-asymptotically nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$\phi\left(u, T(PT)^{n-1}x\right) \leq k_n \phi(u,x), \quad \forall x \in C,\ u \in F(T),\ n \geq 1. \tag{1.9}$$

(3) $T : C \to E$ is said to be *total quasi-ϕ-asymptotically nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and there exists nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ with $\rho(0) = 0$ such that for all $x \in C$, $u \in F(T)$

$$\phi\left(u, T(PT)^{n-1}x\right) \leq \phi(u,x) + \nu_n \rho(\phi(u,x)) + \mu_n, \quad \forall n \geq 1. \tag{1.10}$$

(4) A countable family of nonself mappings $\{T_i\} : C \to E$ is said to be *uniformly total quasi-ϕ-asymptotically nonexpansive,* if $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ with $\rho(0) = 0$ such that for each $i \geq 1$ and all $x \in C$, $u \in \bigcap_{i=1}^{\infty} F(T_i)$

$$\phi\left(u, T_i(PT_i)^{n-1}x\right) \leq \phi(u,x) + \nu_n \rho(\phi(u,x)) + \mu_n, \quad \forall n \geq 1. \tag{1.11}$$

Remark 1.4. From the definitions, it is easy to know that

(1) If T is a quasi-ϕ-nonexpansive nonself mapping, then it must be a quasi-ϕ-asymptotically nonexpansive nonself mapping with $\{k_n = 1\}$.

(2) Taking $\rho(t) = t$, $t > 0$, $\nu_n = (k_n - 1)$ and $\mu_n = 0$, then (1.9) can be rewritten as

$$\phi\left(u, T(PT)^{n-1}x\right) \leq \phi(u,x) + \nu_n \rho(\phi(u,x)) + \mu_n, \quad \forall n \geq 1,\ x \in C,\ u \in F(T). \tag{1.12}$$

This implies that each quasi-ϕ-asymptotically nonexpansive nonself mapping must be a total quasi-ϕ-asymptotically nonexpansive nonself mapping, but the converse is not true.

A nonself mapping $T : C \to E$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le L\|x-y\|, \quad \forall x, y \in C,\ n \ge 1. \tag{1.13}$$

Lemma 1.5 (see [2]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ (as $n \to \infty$) and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ (as $n \to \infty$).*

Lemma 1.6. *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed and convex subset E. Let $T : C \to E$ be a closed and total quasi-ϕ-asymptotically nonexpansive nonself mapping with nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ such that $\nu_n \to 0$, $\mu_n \to 0$ and $\rho(0) = 0$. Then the fixed point set $F(T)$ is a closed and convex subset of C.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to u$ (as $n \to \infty$). Since $Tx_n = x_n \to u$, by the closeness of T, we have $u = Tu$, that is, $u \in F(T)$. This shows that $F(T)$ is a closed set in C.

Next, we prove that $F(T)$ is convex. For any $x, y \in F(T)$, $t \in (0,1)$, putting $q = tx + (1-t)y$, we prove that $q \in F(T)$. Indeed, let $\{u_n\}$ be a sequence generated by

$$\begin{aligned} &u_1 = Tq, \quad u_2 = TPTq = TPu_1, \quad u_3 = T(PT)^2q = TPu_2, \dots, \\ &u_n = T(PT)^{n-1}q = TPu_{n-1}, \dots, \end{aligned} \tag{1.14}$$

we have

$$\begin{aligned} \phi(q, u_n) &= \|q\|^2 - 2\langle q, Ju_n\rangle + \|u_n\|^2 \\ &= \|q\|^2 - 2t\langle x, Ju_n\rangle - 2(1-t)\langle y, Ju_n\rangle + \|u_n\|^2 \\ &= \|q\|^2 + t\phi(x, u_n) + (1-t)\phi(y, u_n) - t\|x\|^2 - (1-t)\|y\|^2. \end{aligned} \tag{1.15}$$

Since

$$\begin{aligned} &t\phi(x, u_n) + (1-t)\phi(y, u_n) \\ &\quad \le t(\phi(x, q) + \nu_n\rho(\phi(x, q)) + \mu_n) + (1-t)(\phi(y, q) + \nu_n\rho(\phi(y, q)) + \mu_n) \\ &\quad = t\left(\|x\|^2 - 2\langle x, Jq\rangle + \|q\|^2 + \nu_n\rho(\phi(x, q)) + \mu_n\right) \\ &\qquad + (1-t)\left(\|y\|^2 - 2\langle y, Jq\rangle + \|q\|^2 + \nu_n\rho(\phi(y, q)) + \mu_n\right) \\ &\quad = t\|x\|^2 + (1-t)\|y\|^2 - \|q\|^2 + t\nu_n\rho(\phi(x, q)) + (1-t)\nu_n\rho(\phi(y, q)) + \mu_n. \end{aligned} \tag{1.16}$$

Substituting (1.16) into (1.15), and simplifying we have

$$\phi(q, u_n) \le t\nu_n\rho(\phi(x, q)) + (1-t)\nu_n\rho(\phi(y, q)) + \mu_n \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{1.17}$$

By Lemma 1.5, we have $u_n \to q$ $(n \to \infty)$. This implies that $u_{n+1} \to q$ $(n \to \infty)$.

Since $u_{n+1} = T(PT)^n q = TPT(PT)^{n-1} q = TPu_n$ and T is closed, we have $q = TPq$. Since $q \in C$, $Pq = q$, thus $q = Tq$. this implies that $F(T)$ is a convex set in C. □

Concerning the strong and weak convergence of asymptotically nonexpansive self or nonself mappings, relatively nonexpansive, quasi-ϕ-nonexpansive and quasi-ϕ-asymptotically nonexpansive self or nonself mappings have been considered extensively by several authors in the setting of Hilbert or Banach spaces (see e.g., [2–19]).

The purpose of this paper is to modify the Halpern and Mann-type iteration algorithm for a family of of total quasi-ϕ-asymptotically nonexpansive nonself mappings and to have the strong convergence under removing $F(T)$ is a convex set of condition and a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper extend and improve the corresponding results of Chang et al. [4–7], W. P. Guo and W. Guo [8], Hao et al. [9], Kamimura and Takahashi [10], Kiziltunc and Temir [11], Nilsrakoo and Saejung [2], Pathak et al. [12], Qin et al. [13], Su et al. [14], Thianwan [15], Wang et al. [16], Yıldırım and Özdemir [17], Yang and Xie [18], Zegeye et al. [19], Kanjanasamranwong et al. [20], Saewan and Kumam [21–24] and Wattanawitoon and Kumam [25].

2. Main Results

Theorem 2.1. *Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed convex subset E. Let $T_i : C \to E, i = 1, 2, \ldots$ be a family of closed and uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ such that $\nu_n \to 0$, $\mu_n \to 0$ and $\rho(0) = 0$, and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$;

(b) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$
\begin{gathered}
x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\
y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n)\left(\beta_n J x_n + (1-\beta_n) J T_i (PT_i)^{n-1} x_n\right)\right], \quad i \geq 1, \\
C_{n+1} = \left\{ z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \theta_n \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1.
\end{gathered}
\tag{2.1}
$$

where $\theta_n = \nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n$, for all $n \geq 1$, $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i)$. If $\mathcal{F}$ is a nonempty-bounded subset in C, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. We divide the proof of Theorem 2.1 into five steps.

(I) *$\mathcal{F}$ and C_n, $n \geq 1$ are closed and convex subset in C.*

In fact, it follows from Lemma 1.6 that $F(T_i)$, $i \geq 1$ is closed and convex subset of C. Therefore $\mathcal{F}$ is a closed and convex subset in C.

Again by the assumption that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ we have that

$$\begin{aligned} C_{n+1} &= \left\{ z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \theta_n \right\} \\ &= \bigcap_{i\geq 1} \{ z \in C_n : \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \theta_n \} \cap C_n \\ &= \bigcap_{i\geq 1} \Big\{ z \in C_n : 2\alpha_n \langle z, Jx_1 \rangle + 2(1-\alpha_n)\langle z, Jx_n \rangle - 2\langle z, Jy_{n,i} \rangle \\ &\qquad \leq \alpha_n \|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_{n,i}\|^2 + \theta_n \Big\} \cap C_n. \end{aligned} \tag{2.2}$$

This implies that C_{n+1} is closed and convex. The conclusion is proved.

(II) *Now we prove that* $\mathcal{F} \subset C_n$, $n \geq 1$.

In fact, it is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 2$. Letting

$$w_{n,i} = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T_i (PT_i)^{n-1} x_n \right), \tag{2.3}$$

it follows from (1.6) that for any $u \in \mathcal{F} \subset C_n$ we have

$$\begin{aligned} \phi(u, y_{n,i}) &= \phi\left(u, J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J w_{n,i})\right) \\ &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, w_{n,i}), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \phi(u, w_{n,i}) &= \phi\left(u, J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T_i (PT_i)^{n-1} x_n\right)\right) \\ &\leq \beta_n \phi(u, x_n) + (1-\beta_n)\phi\left(u, T_i (PT_i)^{n-1} x_n\right) \\ &\leq \beta_n \phi(u, x_n) + (1-\beta_n)\{\phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n\} \\ &= \phi(u, x_n) + (1-\beta_n)(\nu_n \rho(\phi(u, x_n)) + \mu_n) \\ &\leq \phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n, \end{aligned} \tag{2.5}$$

therefore we have

$$\begin{aligned} \sup_{i\geq 1} \phi(u, y_{n,i}) &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\{\phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n\} \\ &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, x_n) + \nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n \\ &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, x_n) + \theta_n, \end{aligned} \tag{2.6}$$

where $\theta_n = \nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n$. This shows that $u \in C_{n+1}$, and so $\mathcal{F} \subset C_{n+1}$. The conclusion is proved.

(III) *Next we prove that* $\{x_n\}$ *is a Cauchy sequence in* C.

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 1.1(2) we have

$$\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0, \quad \forall y \in C_n. \tag{2.7}$$

Again since $\mathcal{F} \subset C_n$, for all $n \geq 1$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in \mathcal{F}. \tag{2.8}$$

It follows from Lemma 1.1(1) that for each $u \in \mathcal{F}$ and for each $n \geq 1$

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \tag{2.9}$$

Therefore $\{\phi(x_n, x_1)\}$ is bounded. By virtue of (1.5), $\{x_n\}$ is also bounded.

Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$, for all $n \geq 1$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence the limit $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. By the construction of C_n, for any positive integer $m \geq n$, we have $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_1 \in C_n$. This shows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \tag{2.10}$$

It follows from Lemma 1.5 that $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a nonempty closed subset of Banach space E, it is complete, without loss of generality, we can assume that $x_n \to x^*$ $(n \to \infty)$.

By the assumption, it is easy to see that

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \left(\nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n \right) = 0. \tag{2.11}$$

(IV) *Now we prove that* $x^* \in \mathcal{F}$.

In fact, since $x_{n+1} \in C_{n+1}$ and $\alpha_n \to 0$, it follows from (2.1) and (2.11) that

$$\sup_{i\geq 1} \phi(x_{n+1}, y_{n,i}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \theta_n \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{2.12}$$

Since $x_n \to x^*$, by virtue of Lemma 1.5 for each $i \geq 1$,we have

$$\lim_{n\to\infty} y_{n,i} = x^*. \tag{2.13}$$

Since $\{x_n\}$ is bounded, $\{T_i\}_{i=1}^{\infty}$ is uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ such that $\nu_n \to 0$, $\mu_n \to 0$, and $\rho(0) = 0$, for any given $u \in \mathcal{F}$, we have

$$\phi\left(u, T_i(PT_i)^{n-1}x_n\right) \le \phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n. \tag{2.14}$$

This implies that $\{T_i(PT_i)^{n-1}x_n\}$ is uniformly bounded. Since

$$\begin{aligned} \|w_{n,i}\| &= \left\|J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T_i(PT_i)^{n-1}x_n\right)\right\| \\ &\le \beta_n \|x_n\| + (1-\beta_n)\left\|T_i\|PT_i\|^{n-1}x_n\right\| \\ &\le \|x_n\| + \left\|T_i(PT_i)^{n-1}x_n\right\|. \end{aligned} \tag{2.15}$$

This implies that $\{w_{n,i}\}$ is also uniformly bounded.

Since $\alpha_n \to 0$, from (2.1), for each $i \ge 1$ we have

$$\lim_{n\to\infty} \|Jy_{n,i} - Jw_{n,i}\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - Jw_{n,i}\| = 0. \tag{2.16}$$

Since J^{-1} is uniformly continuous on each bounded subset of E^*, it follows from (2.13) and (2.16) that

$$\lim_{n\to\infty} w_{n,i} = x^* \quad \text{for each } i \ge 1. \tag{2.17}$$

Since J is uniformly continuous on each bounded subset of E, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|Jw_{n,i} - Jx^*\| \\ &= \lim_{n\to\infty} \left\|\beta_n J x_n + (1-\beta_n) J T_i(PT_i)^{n-1}x_n - Jx^*\right\| \\ &= \lim_{n\to\infty} \left\|\beta_n (Jx_n - Jx^*) + (1-\beta_n)\left(JT_i(PT_i)^{n-1}x_n - Jx^*\right)\right\| \\ &= \lim_{n\to\infty} (1-\beta_n)\left\|JT_i(PT_i)^{n-1}x_n - Jx^*\right\|. \end{aligned} \tag{2.18}$$

By condition (b), we have that

$$\lim_{n\to\infty} \left\|JT_i(PT_i)^{n-1}x_n - Jx^*\right\| = 0. \tag{2.19}$$

Since J is uniformly continuous, this shows that $\lim_{n\to\infty} T_i(PT_i)^{n-1}x_n = x^*$ uniformly in $i \ge 1$.

Again by the assumptions that for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous, thus we have

$$\begin{aligned}
&\left\| T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n \right\| \\
&\quad \leq \left\| T_i(PT_i)^n x_n - T_i(PT_i)^n x_{n+1} \right\| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| \\
&\qquad + \|x_{n+1} - x_n\| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\| \\
&\quad \leq (L_i + 1)\|x_n - x_{n+1}\| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\|.
\end{aligned} \tag{2.20}$$

Since $\lim_{n\to\infty} T_i(PT_i)^{n-1} x_n = x^*$ and $x_n \to x^*$, these together with (2.20) imply that $\lim_{n\to\infty} \|T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n\| = 0$ and $\lim_{n\to\infty} T_i(PT_i)^n x_n = x^*$, that is,

$$\lim_{n\to\infty} T_i P(PT_i)^{n-1} x_n = x^*. \tag{2.21}$$

In view continuity of T_iP, it yields that $T_iPx^* = x^*$. Since $x^* \in C$, $Px^* = x^*$. This shows that $Tx^* = x^*$. By the arbitrariness of $i \geq 1$, we have $x^* \in \mathcal{F}$.

(V) *Finally we prove that* $x_n \to x^* = \Pi_{\mathcal{F}} x_1$.

Let $w = \Pi_{\mathcal{F}} x_1$. Since $w \in \mathcal{F} \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$, for all $n \geq 1$. This implies that

$$\phi(x^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(w, x_1). \tag{2.22}$$

In view of the definition of $\Pi_{\mathcal{F}} x_1$, from (2.22) we have $x^* = w$. Therefore $x_n \to x^* = \Pi_{\mathcal{F}} x_1$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E, C, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 2.1. Let $T_i : C \to E$, $i = 1, 2, \ldots$ be a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$, and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered}
x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\
y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1 - \alpha_n)\left(\beta_n J x_n + (1 - \beta_n) J T_i(PT_i)^{n-1} x_n\right)\right], \quad i \geq 1, \\
C_{n+1} = \left\{ z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \theta_n \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{gathered} \tag{2.23}$$

where $\theta_n = (k_n - 1)\sup_{u\in\mathcal{F}} \phi(u, x_n)$, $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i)$. If $\mathcal{F}$ is a nonempty bounded subset in C, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. By Remark 1.4 $T_i : C \to E$, $i = 1, 2, \ldots$ be a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings that it is a family of closed and uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with taking $\rho(t) = t,\ t > 0$, $\nu_n = (k_n - 1)$ and $\mu_n = 0$. Therefore all conditions in Theorem 2.1 are satisfied. By the similar methods as given in the proof of Theorem 2.1, we can prove that the sequence $\{x_n\}$ defined by (2.23) converges strongly to $\Pi_{\mathcal{F}} x_1$. □

Theorem 2.3. *Let E, C, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 2.2. Let $T_i : C \to E,\ i = 1, 2, \ldots$ be a family of quasi-ϕ-nonexpansive nonself mappings such that $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{gathered}
x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\
y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n)\left(\beta_n J x_n + (1-\beta_n) J T_i x_n\right)\right], \quad i \geq 1, \\
C_{n+1} = \left\{ z \in C_n : \sup_{i\geq 1} \phi\left(z, y_{n,i}\right) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1.
\end{gathered}
\tag{2.24}
$$

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. By Remark 1.4 $T_i : C \to E,\ i = 1, 2, \ldots$ be a family of quasi-ϕ-nonexpansive nonself mappings that it is a family of uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings with sequence $\{k_n\} = \{1\}$. Hence $\theta_n = (k_n - 1)\sup_{u\in\mathcal{F}} \phi(u, x_n) = 0$ Therefore all conditions in Theorem 2.2 are satisfied. By the similar methods, we can prove that the sequence $\{x_n\}$ defined by (2.24) converges strongly to $\Pi_{\mathcal{F}} x_1$. □

3. Application and Example

In this section we utilize the results presented in Section 2 to prove a strong convergence theorem concerning maximal monotone operators in Hilbert spaces.

Let E be a real Hilbert space and let A be a maximal monotone operator from E to E. For each $r > 0$, we can define a single valued mapping $J_r^A : E \to E$ by $J_r^A = (I + rA)^{-1}$ and such a mapping J_r^A is called the *resolvent of* A. It is easy to prove that J_r^A is a nonexpansive mapping and $A^{-1}(0) = F(J_r^A)$ for all $r > 0$. Therefore it is a uniformly 1-Lipschitz continuous and quasi-ϕ-nonexpansive mapping. Hence for each $p \in F(J_r^A)$ and $w \in E$, we have

$$
\phi\left(p, J_r^A w\right) \leq \phi(p, w), \tag{3.1}
$$

and $F(J_r^A) = A^{-1}(0)$. These show that all conditions in Theorem 2.3 are satisfied. Hence from Theorem 2.3 we have the following.

Theorem 3.1. *Let E be a real Hilbert space. Let A_1, A_2 be two maximal monotone operators from E to E such that $\mathcal{F} = A_1^{-1}(0) \cap A_2^{-1}(0) \neq \emptyset$. Let $J_r^{A_1}$ and $J_r^{A_2}$ be the resolvent of A_1 and A_2, respectively, where $r > 0$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 2.3 and $\{x_n\}$ be the sequence defined by*

$$\begin{aligned}
&x_1 \in E \text{ chosen arbitrarily}; \; C_1 = E, \\
&y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n)\left(\beta_n J x_n + (1-\beta_n) J J_r^{A_i} x_n\right)\right], \quad i = 1,2, \\
&C_{n+1} = \left\{ z \in C_n : \max_{i=1,2} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) \right\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{aligned} \tag{3.2}$$

where P_C is the metric projection from H onto the subset $C \subset H$. Then the sequence $\{x_n\}$ defined by (3.2) *converges strongly to $P_{\mathcal{F}} x_1$.*

Acknowledgments

The authors would like to express their thanks to the referees and the Editor for their helpful comments and suggestions. This Project was supported by the Scientific Research Fund of SiChuan Provincial Education Department (no. 11ZA222).

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