# Research Article

# New Sharp Bounds for the Bernoulli Numbers and Refinement of Becker-Stark Inequalities

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We obtain new sharp bounds for the Bernoulli numbers:  $2(2n)!/(\pi^{2n}(2^{2n}-1)) < |B_{2n}| \le (2(2^{2k}-1)/2^{2k})\zeta(2k)(2n)!/(\pi^{2n}(2^{2n}-1))$ ,  $n=k,k+1,\ldots,k\in N^+$ , and establish sharpening of Papenfuss's inequalities, the refinements of Becker-Stark, and Steckin's inequalities. Finally, we show a new simple proof of Ruehr-Shafer inequality.

#### 1. Introduction

The classical Bernoulli numbers  $B_n$  (n = 1, 2, ...) can be defined by (see [1])

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi.$$
 (1.1)

Reference [2] shows a upper bound for  $|B_{2n}| = (-1)^{n+1}B_{2n}$ 

$$|B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{1-2n}}, \quad n = 1, 2, \dots$$
 (1.2)

On the other hand, [3] presents a lower bound for  $|B_{2n}|$  as follows:

$$|B_{2n}| > \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{-2n}}, \quad n = 1, 2, \dots$$
 (1.3)

On the basis of (1.2) and (1.3), Alzer [4] obtains the further results.

**Theorem A.** For all integers  $n \ge 1$  one has

$$\frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\alpha - 2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\beta - 2n}},\tag{1.4}$$

with the best possible constants  $\alpha = 0$  and  $\beta = 2 + \ln(1 - 6/\pi^2) / \ln 2 \approx 0.6491 \cdots$ 

In this paper, we obtain new bounds for the Bernoulli numbers as follows.

**Theorem 1.1.** *Let*  $k \in \mathbb{N}^+$ , n = k, k + 1, ..., then

$$\frac{2(2n)!}{\pi^{2n}(2^{2n}-1)} < |B_{2n}| \le \frac{2(2^{2k}-1)}{2^{2k}} \zeta(2k) \frac{(2n)!}{\pi^{2n}(2^{2n}-1)}. \tag{1.5}$$

The equality holds in (1.5) if and only if n = k. Furthermore, 2 and  $(2(2^{2k} - 1)/2^{2k})\zeta(2k)$  are the best constants in (1.5).

In the following, we study on some trigonometric inequalities. Mitrinovic [5] gives us a result which belongs to Steckin.

**Theorem B.** *If*  $0 < x < \pi/2$ , then

$$\frac{4}{\pi} \frac{x}{\pi - 2x} < \tan x. \tag{1.6}$$

Now, we show a upper bound for  $\tan x$  and obtain the following sharp Steckin's inequalities.

**Theorem 1.2.** *If*  $0 < x < \pi/2$ , *then* 

$$\frac{4}{\pi} \frac{x}{\pi - 2x} < \tan x < \pi \frac{x}{\pi - 2x} \tag{1.7}$$

or

$$\frac{4}{\pi} \frac{1}{\pi - 2x} < \frac{\tan x}{x} < \pi \frac{1}{\pi - 2x}.\tag{1.8}$$

Furthermore,  $4/\pi$  and  $\pi$  are the best constants in (1.7) and (1.8).

Kuang [6] gives us the further results described as Becker-Stark inequalities

**Theorem C.** *Let* 0 < t < 1, *then* 

$$\frac{4}{\pi} \frac{t}{1 - t^2} < tan \frac{\pi}{2} t < \frac{\pi}{2} \frac{t}{1 - t^2}.$$
 (1.9)

Furthermore,  $4/\pi$  and  $\pi/2$  are the best constants in (1.9).

Let  $x = (\pi/2)t$  in (1.9), then Theorem *C* is equivalent to.

**Theorem D.** *Let*  $0 < x < \pi/2$ , *then* 

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}.\tag{1.10}$$

Furthermore, 8 and  $\pi^2$  are the best constants in (1.10).

Clearly, Becker-Stark inequalities (1.10) are the generalization of the strengthened Steckin's inequalities (1.8).

On the other hand, Papenfuss [7] proposes an open problem described as the following statement.

**Theorem E.** Let  $0 \le x < \pi/2$ , then

$$x\sec^2 x - \tan x \le \frac{8\pi^2 x^3}{(\pi^2 - 4x^2)^2}. (1.11)$$

Bach [8] prove Theorem *E* and obtain a further result.

**Theorem F.** *Let*  $0 \le x < \pi/2$ , *then* 

$$x\sec^2 x - \tan x \le \frac{2\pi^4}{3} \frac{x^3}{(\pi^2 - 4x^2)^2}.$$
 (1.12)

In this section, we first obtain sharp Papenfuss-Bach inequalities described as Theorem 1.3.

**Theorem 1.3.** *Let*  $0 < x < \pi/2$ , *then* 

$$\frac{64x^3}{(\pi^2 - 4x^2)^2} < x\sec^2 x - \tan x < \frac{2\pi^4}{3} \frac{x^3}{(\pi^2 - 4x^2)^2}.$$
 (1.13)

Furthermore, 64 and  $2\pi^4/3$  are the best constants in (1.13).

The inequalities (1.13) are equivalent to

$$\frac{64x}{(\pi^2 - 4x^2)^2} < \frac{x\sec^2 x - \tan x}{x^2} < \frac{2\pi^4}{3} \frac{x}{(\pi^2 - 4x^2)^2}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{1.14}$$

That is,

$$\frac{64x}{(\pi^2 - 4x^2)^2} < \left(\frac{\tan x}{x}\right)' < \frac{2\pi^4}{3} \frac{x}{(\pi^2 - 4x^2)^2}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{1.15}$$

Then, integrating the three functions in (1.15) from 0 to x, where  $x \in (0, \pi/2)$ , we obtain the following refinement of Becker-Stark inequalities.

**Theorem 1.4** (Refinement of Becker-Stark Inequalities). *Let*  $0 < x < \pi/2$ , *then* 

$$\frac{8}{\pi^2 - 4x^2} + \left(1 - \frac{8}{\pi^2}\right) < \frac{\tan x}{x} < \frac{\pi^4}{12} \frac{1}{\pi^2 - 4x^2} + \left(1 - \frac{\pi^2}{12}\right). \tag{1.16}$$

An application of Theorem 1.4 leads to Theorem 1.5 ( the refinement of Steckin's inequalities).

**Theorem 1.5** (Refinement of Steckin's Inequalities). *If*  $0 < x < \pi/2$ , *then* 

$$\frac{4}{\pi} \frac{1}{\pi - 2x} + \left(1 - \frac{8}{\pi^2}\right) < \frac{\tan x}{x} < \frac{\pi^3}{12} \frac{1}{\pi - 2x} + \left(1 - \frac{\pi^2}{12}\right). \tag{1.17}$$

Finally, we will show a new proof of Ruehr-Shafer inequality.

**Theorem G** (Ruehr-Shafer Inequality, see [8]). Let  $0 \le x < \pi/2$ , then

$$x\sec^2 x - \tan x \le 2\pi^2 \frac{\tan x - x}{\pi^2 - 4x^2}.$$
 (1.18)

#### 2. Two Lemmas

**Lemma 2.1** (see [9, Lemma 2.1]). The function  $(1-(1/2^n))\zeta(n)$  (n = 1, 2, ...) is decreasing, where  $\zeta(n)$  is the Riemann's zeta function.

**Lemma 2.2** (see [10]). Let  $l_n$  and  $m_n$  (n = 1, 2, ...) be real numbers, and let the power series  $L(x) = \sum_{n=1}^{\infty} l_n x^n$  and  $M(x) = \sum_{n=1}^{\infty} m_n x^n$  be convergent for |x| < R. If  $m_n > 0$  for n = 1, 2, ... and if  $l_n/m_n$  is strictly decreasing for n = 1, 2, ..., then the function L(x)/M(x) is strictly decreasing on (0, R).

#### 3. Proof of Theorem 1.1

Using the representation

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|, \quad n = 1, 2, \dots,$$
(3.1)

(cf. [11, page 266]), we have

$$G(n) = \frac{|B_{2n}|\pi^{2n}(2^{2n}-1)}{(2n)!} = 2\left(1 - \frac{1}{2^{2n}}\right)\zeta(2n), \quad n = k, k+1, \dots; \ k \in \mathbb{N}^+.$$
 (3.2)

From Lemma 2.1, we know that G(n) is decreasing and  $G(k) = (2(2^{2k}-1)/2^{2k})\zeta(2k)$ ,  $G(+\infty) = 2\lim_{n\to\infty} \zeta(2n) = 2$ . Then, the proof of Theorem 1.1 is complete.

#### 4. Proofs of Theorem 1.3 and G

## 4.1. Proof of Theorem 1.3.

The following power series expansion can be found in [12]:

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n} (2^{2n} - 1)}{(2n)!} (-1)^{n-1} B_{2n} x^{2n-1} = \sum_{n=1}^{\infty} \frac{2^{2n} (2^{2n} - 1)}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \frac{\pi}{2}.$$
 (4.1)

Then

$$\sec^{2}x = (\tan x)' = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} (2n - 1)|B_{2n}|x^{2n-2}, \quad |x| < \frac{\pi}{2},$$

$$x\sec^{2}x - \tan x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} (2n - 2)|B_{2n}|x^{2n-1}, \quad |x| < \frac{\pi}{2}.$$

$$(4.2)$$

Let

$$A(x) = \frac{x\sec^2 x - \tan x}{x^3/(\pi^2 - 4x^2)^2} = \frac{L(x)}{M(x)},$$
(4.3)

where

$$L(x) = x\sec^{2}x - \tan x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} (2n - 2)|B_{2n}|x^{2n-1}, \quad |x| < \frac{\pi}{2},$$

$$M(x) = \frac{x^{3}}{(\pi^{2} - 4x^{2})^{2}} = \sum_{n=2}^{\infty} \frac{2n - 2}{32} \left(\frac{2}{\pi}\right)^{2n} x^{2n-1}, \quad |x| < \frac{\pi}{2}.$$

$$(4.4)$$

Then,  $l_n=((2^{2n}(2^{2n}-1)/(2n)!)(2n-2)|B_{2n}|,\ m_n=((2n-2)/32)(2/\pi)^{2n}>0\ (n\ge 2)$  and  $l_n/m_n=64(1-1/2^{2n})\zeta(2n)$ . So,  $l_n/m_n$  is decreasing by Lemma 2.1. Therefore, A(x)=L(x)/M(x) is decreasing on  $(0,\pi/2)$  by Lemma 2.2. At the same time,  $\lim_{x\to 0^+}A(x)=2\pi^4/3$  and  $\lim_{x\to (\pi/2)^-}A(x)=64$ , so 64 and  $2\pi^4/3$  are the best constants in (1.13).

## 4.2. Proof of Theorem G.

By (4.1) and (4.2), we have

$$(x\sec^{2}x - \tan x)(\pi^{2} - 4x^{2}) = 2\pi^{2} \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} (n - 1)|B_{2n}|x^{2n-1}$$

$$-4 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} (2n - 2)|B_{2n}|x^{2n+1},$$

$$2\pi^{2}(\tan x - x) = 2\pi^{2} \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} |B_{2n}|x^{2n-1},$$

$$(4.5)$$

so, (1.18) is equivalent to

$$2\pi^{2} \sum_{n=2}^{\infty} \frac{2^{2n} (2^{2n} - 1)}{(2n)!} (n - 2) |B_{2n}| x^{2n-1} \le 4 \sum_{n=2}^{\infty} \frac{2^{2n} (2^{2n} - 1)}{(2n)!} (2n - 2) |B_{2n}| x^{2n+1}, \tag{4.6}$$

that is,

$$2\pi^{2} \sum_{n=3}^{\infty} \frac{2^{2n} (2^{2n} - 1)}{(2n)!} (n - 2) |B_{2n}| x^{2n-1} \le 4 \sum_{n=2}^{\infty} \frac{2^{2n} (2^{2n} - 1)}{(2n)!} (2n - 2) |B_{2n}| x^{2n+1}, \tag{4.7}$$

or

$$\pi^{2} \sum_{n=2}^{\infty} \frac{2^{2n+2} (2^{2n+2} - 1)}{(2n+2)!} (n-1) |B_{2n+2}| x^{2n+1} \le 4 \sum_{n=2}^{\infty} \frac{2^{2n} (2^{2n} - 1)}{(2n)!} (n-1) |B_{2n}| x^{2n+1}. \tag{4.8}$$

From Lemma 2.1, we have that  $(1 - 1/2^{2n})\zeta(2n)$  is decreasing or

$$\frac{2^{2n+2}-1}{4}\zeta(2n+2) < \left(2^{2n}-1\right)\zeta(2n) \tag{4.9}$$

holds. By (3.1), we get

$$\frac{\pi^2(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}| < \frac{(2^{2n}-1)}{(2n)!}|B_{2n}|,\tag{4.10}$$

so, (4.8) holds.

## 5. Remark

In 2010, Zhu and Hua [9] proved for  $x \in (0, \pi/2)$  that

$$\frac{\pi^2 + ((4(8-\pi^2))/\pi^2)x^2}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + (\pi^2/3 - 4)x^2}{\pi^2 - 4x^2}.$$
 (5.1)

Now, we can compare the results of (5.1) with (1.16). In fact, we can easy check that

$$\frac{8}{\pi^2 - 4x^2} + \left(1 - \frac{8}{\pi^2}\right) = \frac{\pi^2 + \left(\left(4(8 - \pi^2)\right)/\pi^2\right)x^2}{\pi^2 - 4x^2},$$

$$\frac{\pi^4}{12} \frac{1}{\pi^2 - 4x^2} + \left(1 - \frac{\pi^2}{12}\right) = \frac{\pi^2 + \left(\pi^2/3 - 4\right)x^2}{\pi^2 - 4x^2}.$$
(5.2)

So, (5.1) is equivalent to (1.16).

#### References

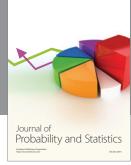
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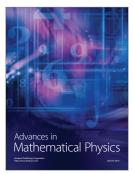




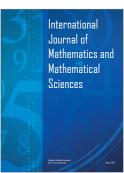


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