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*Research Article*

# A Class of G-Semipreinvex Functions and Optimality

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A class of $G$-semipreinvex functions, which are some generalizations of the semipreinvex functions, and the $G$-convex functions, is introduced. Examples are given to show their relations among $G$-semipreinvex functions, semipreinvex functions and $G$-convex functions. Some characterizations of $G$-semipreinvex functions are also obtained, and some optimality results are given for a class of $G$-semipreinvex functions. Ours results improve and generalize some known results.

## 1. Introduction

Generalized convexity has been playing a central role in mathematical programming and optimization theory. The research on characterizations of generalized convexity is one of most important parts in mathematical programming and optimization theory. Many papers have been published to study the problems of how to weaken the convex condition to guarantee the optimality results. Schaible and Ziemba [1] introduced $G$-convex function which is a generalization of convex function and studied some characterizations of $G$-convex functions. Hanson [2] introduced invexity which is an extension of differentiable convex function. Ben-Israel and Mond [3] considered the functions for which there exists $\eta : R^n \times R^n \to R^n$ such that, for any $x, y \in R^n$, $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda) f(y). \tag{1.1}$$

Weir et al. [4, 5] named such kinds of functions which satisfied the condition (1.1) as preinvex functions with respect to $\eta$. Further study on characterizations and generalizations of convexity and preinvexity, including their applications in mathematical programming, has been done by many authors (see [6–18]). As a generalization of preinvexity, Yang and Chen [15] introduced semipreinvex functions and discussed the applications in prevariational inequality. Yang et al. [16] investigated some properties of semipreinvex functions. As a generalization of $G$-convex functions and preinvex functions, Antczak [17] introduced $G$-preinvex functions and obtained some optimality results for a class of constrained optimization problems. As a generalization of $B$-vexity and semipreinvexity, Long and Peng [18] introduced the concept of semi-$B$-preinvex functions. Zhao et al. [19] introduced $r$-semipreinvex functions and established some optimality results for a class of nonlinear programming problems.

Motivated by the results in [17–19], in this paper, we propose the concept of $G$-semipreinvex functions and obtain some important characterizations of $G$-semipreinvexity. At the same time, we study some optimality results under $G$-semipreinvexity. Our results unify the concepts of $G$-convexity, preinvexity, $G$-preinvexity, semipreinvexity, and $r$-semipreinvexity.

## 2. Preliminaries and Definitions

*Definition 2.1* (see [1]). Let $G$ be a continuous real-valued strictly monotonic function defined on $D \subset R$. A real-valued function $f$ defined on a convex set $X \subset R^n$ is said to be $G$-convex if for any $x, y \in X$, $\lambda \in [0,1]$,

$$f(y + \lambda(x - y)) \le G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))), \tag{2.1}$$

where $G^{-1}$ is the inverse of $G$, $f(X) \subset D$.

*Remark 2.2.* Every convex functions is $G$-convex, but the converse is not necessarily true.

*Example 2.3.* Let $X = [-1,1]$, $f : X \to R$, $I_f(X)$ be the range of real-valued function $f$, and let $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan(|x| + 1), \qquad G(t) = \tan(t). \tag{2.2}$$

Then, we can verify that $f$ is a $G$-convex function. But $f$ is not a convex function because the following inequality

$$f(y + \lambda(x - y)) > \lambda f(x) + (1 - \lambda) f(y) \tag{2.3}$$

holds for $x = 1/4$, $y = 3/4$, and $\lambda = 1/2$.

Weir et al. [4, 5] presented the concepts of invex sets and preinvex functions as follows.

*Definition 2.4* (see [4, 5]). A set $X \subseteq R^n$ is said to be invex if there exists a vector-valued function $\eta : X \times X \to R^n$ such that for any $x, y \in X$, $\lambda \in [0,1]$,

$$y + \lambda\eta(x, y) \in X. \tag{2.4}$$

*Definition 2.5* (see [4, 5]). Let $X \subseteq R^n$ be invex with respect to vector-valued function $\eta : X \times X \to R^n$. Function $f(x)$ is said to be preinvex with respect to $\eta$ if for any $x, y \in X$, $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x,y)) \le \lambda f(x) + (1-\lambda) f(y). \tag{2.5}$$

*Remark 2.6.* Every convex function is a preinvex function with respect to $\eta = x - y$, but the converse is not necessarily true.

*Example 2.7.* Let $X = [-1,1]$. $f : X \to R$ be defined by

$$f(x) = \arctan(|x| + 1). \tag{2.6}$$

Then, we can verify that $f$ is a preinvex function with respect to $\eta$, where

$$\eta(x,y) = \begin{cases} -y - x^2 + 2x, & 0 \le x \le 1,\ 0 \le y \le 1, \\ -y - x, & -1 \le x < 0,\ 0 \le y \le 1, \\ -y - x, & 0 \le x \le 1,\ -1 \le y < 0, \\ -y + x, & -1 \le x < 0,\ -1 \le y < 0. \end{cases} \tag{2.7}$$

But $f$ is not convex a function in Example 2.3.

Antczak [17] introduced the concept of $G$-preinvex functions as follows.

*Definition 2.8* (see [17]). Let $X$ be a nonempty invex (with respect to $\eta$) subset of $R^n$. A function $f : X \to R$ is said to be (strictly) $G$-preinvex at $y$ with respect to $\eta$ if there exists a continuous real-valued increasing function $G : I_f(X) \to R$ such that for all $x \in X$ $(x \neq y)$, $\lambda \in [0,1]$,

$$\begin{gathered} f(y + \lambda\eta(x,y)) \le G^{-1}(\lambda(G(f(x))) + (1-\lambda)G(f(y))), \\ \Big(f(y + \lambda\eta(x,y)) < G^{-1}(\lambda(G(f(x))) + (1-\lambda)G(f(y)))\Big). \end{gathered} \tag{2.8}$$

If (2.8) is satisfied for any $y \in X$, then $f$ is said to be (strictly) a $G$-preinvex function on $X$ with respect to $\eta$.

*Remark 2.9.* Every preinvex function with respect to $\eta$ is $G$-preinvex function with respect to the same $\eta$, where $G(x) = x$. Every $G$-convex function is $G$-preinvex function with respect to $\eta(x, y, \lambda) = x - y$. However, the converse is not necessarily true.

*Example 2.10.* Let $X = [-1,1]$. $f : X \to R$, $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan(2 - |x|), \qquad G(t) = \tan t. \tag{2.9}$$

Then, we can verify that $f$ is a $G$-preinvex function with respect to $\eta$, where

$$\eta(x,y) = \begin{cases} -y - x^2 + 2x, & 0 \le x \le 1,\ 0 \le y \le 1, \\ -y - x^2 - 2x, & -1 \le x < 0,\ 0 \le y \le 1, \\ -y - x, & 0 \le x \le 1,\ -1 \le y < 0, \\ -y + x, & -1 \le x < 0,\ -1 \le y < 0. \end{cases} \tag{2.10}$$

But $f$ is not a preinvex function because the following inequality

$$f(y + \lambda\eta(x,y)) > \lambda f(x) + (1-\lambda) f(y) \tag{2.11}$$

holds for $x = 0$, $y = 1$, and $\lambda = 1/2$.

And $f(x)$ is not a $G$-convex function because the following inequality

$$f(y + \lambda(x-y)) > G^{-1}(\lambda G(f(x)) + (1-\lambda) G(f(y))) \tag{2.12}$$

holds for $x = 1$, $y = -1$, and $\lambda = 1/2$.

*Definition 2.11* (see [15]). A set $X \subseteq R^n$ is said to be a semiconnected set if there exists a vector-valued function $\eta : X \times X \times [0,1] \to R^n$ such that for any $x, y \in X$, $\lambda \in [0,1]$,

$$y + \lambda\eta(x,y,\lambda) \in X. \tag{2.13}$$

*Definition 2.12* (see [15]). Let $X \subseteq R^n$ be a semiconnected set with respect to a vector-valued function $\eta : X \times X \times [0,1] \to R^n$. Function $f$ is said to be semipreinvex with respect to $\eta$ if for any $x, y \in X$, $\lambda \in [0,1]$, $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$,

$$f(y + \lambda\eta(x,y,\lambda)) \le \lambda f(x) + (1-\lambda) f(y). \tag{2.14}$$

Next we present the definition of $G$-semipreinvex functions as follows.

*Definition 2.13.* Let $X \subseteq R^n$ be semiconnected set with respect to vector-valued function $\eta : X \times X \times [0,1] \to R^n$. A function $f : X \to R$ is said to be (strictly) $G$-semipreinvex at $y$ with respect to $\eta$ if there exists a continuous real-valued strictly increasing function $G : I_f(X) \to R$ such that for all $x \in X$ $(x \neq y)$, $\lambda \in [0,1]$, $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$,

$$\begin{gathered} f(y + \lambda\eta(x,y,\lambda)) \le G^{-1}(\lambda G(f(x)) + (1-\lambda) G(f(y))), \\ \left(f(y + \lambda\eta(x,y,\lambda)) < G^{-1}(\lambda G(f(x)) + (1-\lambda) G(f(y)))\right). \end{gathered} \tag{2.15}$$

If (2.15) is satisfied for any $y \in X$, then $f$ is said to be (strictly) $G$-semipreinvex on $X$ with respect to $\eta$.

*Remark 2.14.* Every semipreinvex function with respect to $\eta$ is a $G$-semipreinvex function with respect to the same $\eta$, where $G(x) = x$. However, the converse is not true.

*Example 2.15.* Let $X = [-6,6]$. Then $X$ is a semiconnected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$, where

$$\eta(x,y,\lambda) = \begin{cases} \dfrac{x-y}{\sqrt[3]{\lambda}}, & -6 \le x < 0,\ -6 \le y < 0,\ x > y,\ 0 < \lambda \le 1, \\ \lambda^2(x-y), & 0 \le x \le 6,\ 0 \le y \le 6,\ x \ge y, \\ \lambda^2(x-y), & -6 \le x < 0,\ -6 \le y < 0,\ x \le y, \\ x-y, & 0 \le x \le 6,\ 0 \le y \le 6,\ x < y, \\ x-y, & 0 \le x \le 6,\ -6 \le y < 0,\ x < -y, \\ x-y, & -6 \le x < 0,\ 0 \le y \le 6,\ x > -y, \\ 0, & 0 \le x \le 6,\ -6 \le y < 0,\ x \ge -y, \\ 0, & -6 \le x < 0,\ 0 \le y \le 6,\ x \le -y. \end{cases} \tag{2.16}$$

Let $f : X \to R$, $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan\left(x^2 + 2\right), \qquad G(t) = \tan t. \tag{2.17}$$

Then, we can verify that $f$ is a $G$-semipreinvex function with respect to $\eta$. But $f$ is not a semipreinvex function with respect to $\eta$ because the following inequality

$$f(y + \lambda\eta(x,y,\lambda)) > \lambda f(x) + (1-\lambda) f(y) \tag{2.18}$$

holds for $x = 2$, and $y = 4$, $\lambda = 1/2$.

*Example 2.16.* Let $X = [-6,6]$. Then $X$ is a semiconnected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda) = 0$, where

$$\eta(x,y,\lambda) = \begin{cases} \dfrac{x-y}{\varphi(\lambda)}, & -6 \le x < 0,\ -6 \le y < 0,\ x > y,\ 0 < \lambda \le 1, \\ \varphi(\lambda)(x-y), & 0 \le x \le 6,\ 0 \le y \le 6,\ x \ge y, \\ \varphi(\lambda)(x-y), & -6 \le x < 0,\ -6 \le y < 0,\ x \le y, \\ x-y, & 0 \le x \le 6,\ 0 \le y \le 6,\ x < y, \\ x-y, & 0 \le x \le 6,\ -6 \le y < 0,\ x < -y,\ \lambda < \varphi(\lambda) < 1 \\ x-y, & -6 \le x < 0,\ 0 \le y \le 6,\ x > -y, \\ 0, & 0 \le x \le 6,\ -6 \le y < 0,\ x \ge -y, \\ 0, & -6 \le x < 0,\ 0 \le y \le 6,\ x \le -y. \end{cases} \tag{2.19}$$

Let $f : X \to R$, $G : I_f(X) \to R$ be defined by

$$f(x) = \arctan\left(x^2 + k\right), \quad G(t) = \tan t, \quad \forall k \in R. \tag{2.20}$$

Then, we can verify that $f(x)$ is a $G$-semipreinvex function with respect to classes of functions $\eta$. But $f(x)$ is not semipreinvex function with respect to $\eta$ because the following inequality

$$f(y+\lambda\eta(x,y,\lambda)) > \lambda f(x) + (1-\lambda)f(y) \tag{2.21}$$

holds for $x=2$, $y=4$, and $\lambda=1/2$.

*Remark 2.17.* Every a $G$-convex function is $G$-semipreinvex function with respect to $\eta(x,y,\lambda)=x-y$. But the converse is not true.

*Example 2.18.* Let $X=(-6,6)$, it is easy to check that $X$ is a semiconnected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda\to 0}\lambda\eta(x,y,\lambda)=0$, where

$$\eta(x,y,\lambda)=\begin{cases}\lambda(x-y), & 0\le x<6,\ 0\le y<6,\ x<y,\\ \lambda(x-y), & -6<x<0,\ -6<y<0,\ x>y,\\ \dfrac{x-y}{\sqrt{\lambda}}, & 0\le x<6,\ 0\le y<6,\ x\ge y,\ 0<\lambda\le 1,\\ \dfrac{x-y}{\sqrt{\lambda}}, & -6<x<0,\ -6<y<0,\ x\le y,\ 0<\lambda\le 1,\\ -x-y, & 0\le x<6,\ -6<y<0,\ x\ge -y,\\ -x-y, & -6<x<0,\ 0\le y<6,\ x\le -y,\\ 0, & 0\le x<6,\ -6<y<0,\ x<-y,\\ 0, & -6<x<0,\ 0\le y<6,\ x>-y.\end{cases} \tag{2.22}$$

Let $f: X\to R$, $G: I_f(X)\to R$ be defined by

$$f(x)=\arctan(6-|x|), \qquad G(t)=\tan t. \tag{2.23}$$

Then, we can verify that $f$ is a $G$-semipreinvex function with respect to $\eta$. But $f$ is not a $G$-convex function, because the following inequality

$$f(y+\lambda(x-y)) > G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) \tag{2.24}$$

holds for $x=1$, $y=-1$, and $\lambda=1/2$.

## 3. Some Properties of $G$-Semipreinvex Functions

In this section, we give some basic characterizations of $G$-semipreinvex functions.

**Theorem 3.1.** *Let $f$ be a $G_1$-semipreinvex function with respect to $\eta$ on a nonempty semiconnected set $X\subset R^n$ with respect to $\eta$, and let $G_2$ be a continuous strictly increasing function on $I_f(X)$. If the function $g(t)=G_2G_1^{-1}(t)$ is convex on the image under $G_1$ of the range of $f$, then $f$ is also $G_2$-semipreinvex function on $X$ with respect to the same function $\eta$.*

*Proof.* Let $X$ be a nonempty semiconnected subset of $R^n$ with respect to $\eta$, and we assume that $f$ is $G_1$-semipreinvex with respect to $\eta$. Then, for any $x, y \in X$, $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x,y,\lambda)) \le G_1^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(y))). \tag{3.1}$$

Let $G_2$ be a continuous strictly increasing function on $I_f(\mathrm{X})$. Then,

$$G_2(f(y + \lambda\eta(x,y,\lambda))) \le G_2G^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(y))). \tag{3.2}$$

By the convexity of $g(t) = G_2G_1^{-1}$, it follows the following inequality

$$\begin{aligned} G_2G^{-1}(\lambda G_1(f(x)) + (1-\lambda)G_1(f(y))) &\le \lambda G_2G_1^{-1}\Big(G_1(f(x)) + (1-\lambda)G_2G_1^{-1}(G_1 f(y))\Big) \\ &= \lambda G_2(f(x)) + (1-\lambda)G_2(f(y)) \end{aligned} \tag{3.3}$$

for all $x, y \in X$, $\lambda \in [0,1]$. Therefore,

$$\begin{aligned} &G_1^{-1}[\lambda(G_1(f(x))) + (1-\lambda)G_1(f(y))] \\ &\quad \le G_2^{-1}[\lambda(G_2(f(x))) + (1-\lambda)G_2(f(y))]. \end{aligned} \tag{3.4}$$

Thus, we have

$$f(y + \lambda\eta(x,y,\lambda)) \le G_2^{-1}(\lambda G_2(f(x)) + (1-\lambda)G_2(f(y))). \tag{3.5}$$

□

**Theorem 3.2.** *Let $f$ be a $G$-semipreinvex function with respect to $\eta$ on a nonempty semiconnected set $X \subset R^n$ with respect to $\eta$. If the function $G$ is concave on $I_f(X)$, then $f$ is semipreinvex function with respect to the same function $\eta$.*

*Proof.* Let $y, z \in I_f(X)$, from the assumption $G$ is concave on $I_f(X)$, we have

$$G(z + \lambda(y - z)) \ge \lambda G(y) + (1-\lambda)G(z), \quad \lambda \in [0,1]. \tag{3.6}$$

Let

$$G(y) = x, \qquad G(z) = u, \qquad y = G^{-1}(x), \qquad z = G^{-1}(u), \tag{3.7}$$

then

$$\begin{aligned} G\Big(G^{-1}(u) + \lambda\Big(G^{-1}(x) - G^{-1}(u)\Big)\Big) &\ge \lambda G\Big(G^{-1}(x)\Big) + (1-\lambda)G\Big(G^{-1}(u)\Big) \\ &= \lambda x + (1-\lambda)u. \end{aligned} \tag{3.8}$$

It follows that

$$G^{-1}G\Big(\lambda G^{-1}(x)+(1-\lambda)G^{-1}(u)\Big)\geq G^{-1}(\lambda x+(1-\lambda)u). \tag{3.9}$$

Then,

$$\lambda G^{-1}(x)+(1-\lambda)G^{-1}(u)\geq G^{-1}(\lambda x+(1-\lambda)u). \tag{3.10}$$

This means that $G^{-1}$ is convex. Let $G_1 = G$, $G_2 = t$, then $g(t) = G_2G_1^{-1}(t)$ is convex. Hence by Theorem 3.1, $f$ is $G_2$-semipreinvex with respect to $\eta$. But $G_2$ is the identity function; hence, $f$ is a semipreinvex function with respect to the same function $\eta$. □

**Theorem 3.3.** *Let $X$ be a nonempty semiconnected set with respect to $\eta$ subset of $R^n$ and let $f_i : X \to R$, $i\in I$, be finite collection of $G$-semipreinvex function with respect to the same $\eta$ and $G$ on $X$. Define $f(x) = \sup(f_i(x) : i \in I)$, for every $x \in X$. Further, assume that for every $x \in X$, there exists $i^* = i(x) \in I$, such that $f(x) = f_{i^*}(x)$. Then $f$ is $G$-semipreinvex function with respect to the same function $\eta$.*

*Proof.* Suppose that the result is not true, that is, $f$ is not $G$-semipreinvex function with respect to $\eta$ on $X$. Then, there exists $x, y \in X$, $\lambda \in [0,1]$ such that

$$f(y+\lambda\eta(x,y,\lambda)) > G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(y))). \tag{3.11}$$

We denote $z = y+\lambda\eta(x,y,\lambda)$ there exist $i(z) := i_z \in I$, $i(x) := i_x \in I$, and $i(y) := i_y \in I$, satisfying

$$f(z)=f_{i_z}(z),\qquad f(x)=f_{i_x}(x),\qquad f(y)=f_{i_y}(y). \tag{3.12}$$

Therefore, by (3.11),

$$f_{i_z}(z) > G^{-1}\Big(\lambda G(f_{i_x}(x))+(1-\lambda)G\big(f_{i_y}(y)\big)\Big). \tag{3.13}$$

By the condition, we obtain

$$f_{i_z}(z) \leq G^{-1}(\lambda G(f_{i_z}(x))+(1-\lambda)G(f_{i_z}(y))). \tag{3.14}$$

From the definition of $G$-semipreinvexity, $G$ is an increasing function on its domain. Then, $G^{-1}$ is increasing. Since $f_{i_z}(x)\leq f_{i_x}(x)$, $f_{i_z}(y)\leq f_{i_y}(y)$, then (3.14) gives

$$f_{i_z}(z) \leq G^{-1}\Big(\lambda G(f_{i_x}(x))+(1-\lambda)G\big(f_{i_y}(y)\big)\Big). \tag{3.15}$$

The inequality (3.15) above contradicts (3.13). □

**Theorem 3.4.** *Let $f$ be a $G$-semipreinvex function with respect to $\eta$ on a nonempty semiconnected set $X \subset R^n$ with respect to $\eta$. Then, the level set $S_\alpha = \{x \in X : f(x) \leq \alpha\}$ is a semiconnected set with respect to $\eta$, for each $\alpha \in R$.*

*Proof.* Let $x, y \in S_\alpha$, for any arbitrary real number $\alpha$. Then, $f(x) \leq \alpha,\ f(y) \leq \alpha$. Hence, it follows that

$$f(y + \lambda\eta(x, y, \lambda)) \leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))) \leq G^{-1}(G(\alpha)) = \alpha. \tag{3.16}$$

Then, by the definition of level set we conclude that $y + \lambda\eta(x, y, \lambda) \in S_\alpha$, for any $\lambda \in [0, 1]$, we conclude that $S_\alpha$ is a semiconnected set with respect to $\eta$.

Let $f$ is a $G$-semipreinvex function with respect to $\eta$, its epigraph $E_f = \{(x, \alpha) : x \in X, \alpha \in R, f(x) \leq \alpha\}$ is said to be $G$-semiconnected set with respect to $\eta$ if for any $(x, \alpha) \in E_f,\ (y, \beta) \in E_f$, $\lambda \in [0, 1]$,

$$\left(y + \lambda\eta(x, y, \lambda), G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta))\right) \in E_f. \tag{3.17}$$

□

**Theorem 3.5.** *Let $X \subset R^n$ with respect to $\eta$ be a nonempty semiconnected set, and let $f$ be a real-valued function defined on $X$. Then, $f$ is a $G$-semipreinvex function with respect to $\eta$ if and only if its epigraph $E_f = \{(x, \alpha) : x \in X, \alpha \in R, f(x) \leq \alpha\}$ is a $G$-semiconnected set with respect to $\eta$.*

*Proof.* Let $(x, \alpha) \in E_f,\ (y, \beta) \in E_f$, then $f(x) \leq \alpha, f(y) \leq \beta$. Thus, for any $\lambda \in [0, 1]$,

$$\begin{aligned} f(y + \lambda\eta(x, y, \lambda)) &\leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))) \\ &\leq G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta)). \end{aligned} \tag{3.18}$$

By the definition of an epigraph of $f$, this means that

$$\left(y + \lambda\eta(x, y, \lambda), G^{-1}(\lambda G(\alpha) + (1 - \lambda)G(\beta))\right) \in E_f. \tag{3.19}$$

Thus, we conclude that $E_f$ is a $G$ semiconnected set with respect to $\eta$.

Conversely, let $E_f$ be a $G$ semiconnected set. Then, for any $x, y \in X$, we have $(x, f(x)) \in E_f$, $(y, f(y)) \in E_f$. By the definition of an epigraph of $f$, the following inequality

$$f(y + \lambda\eta(x, y, \lambda)) \leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))) \tag{3.20}$$

holds for any $\lambda \in [0, 1]$. This implies that $f$ is a $G$-semipreinvex function on $X$ with respect to $\eta$. □

The following results characterize the class of $G$-semipreinvex functions.

**Theorem 3.6.** *Let $X \subseteq R^n$ be a semiconnected set with respect to $\eta : X \times X \times [0,1] \to R^n$; $f : X \to R$ is a $G$-semipreinvex function with respect to the same $\eta$ if and only if for all $x, y \in X$, $\lambda \in [0,1]$, and $u, v \in R$,*

$$f(x) \leq u, \qquad f(y) \leq v \Longrightarrow f(y + \lambda\eta(x,y,\lambda)) \leq G^{-1}(\lambda G(u) + (1-\lambda)G(v)). \tag{3.21}$$

*Proof.* Let $f$ be $G$-semipreinvex functions with respect to $\eta$, and let $f(x) \leq u$, $f(y) \leq v$, $0 < \lambda < 1$. From the definition of $G$-semipreinvexity, we have

$$\begin{aligned} f(y + \lambda\eta(x,y,\lambda)) &\leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) \\ &\leq G^{-1}(\lambda G(u) + (1-\lambda)G(v)). \end{aligned} \tag{3.22}$$

Conversely, let $x, y \in X$, $\lambda \in [0,1]$. For any $\delta > 0$,

$$\begin{aligned} f(x) &< f(x) + \delta, \\ f(y) &< f(y) + \delta. \end{aligned} \tag{3.23}$$

By the assumption of theorem, we can get that for $0 < \lambda < 1$,

$$\begin{aligned} f(y + \lambda\eta(x,y,\lambda)) &\leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) \\ &\leq G^{-1}(\lambda G(f(x) + \delta) + (1-\lambda)G(f(y) + \delta)). \end{aligned} \tag{3.24}$$

Since $G$ is a continuous real-valued increasing function, and $\delta > 0$ can be arbitrarily small, let $\delta \to 0$, it follows that

$$f(y + \lambda\eta(x,y,\lambda)) \leq G^{-1}(\lambda G(u) + (1-\lambda)G(v)). \tag{3.25}$$

□

## 4. $G$-Semipreinvexity and Optimality

In this section, we will give some optimality results for a class of $G$-semipreinvex functions.

**Theorem 4.1.** *Let $f : X \to R$ be a $G$-semipreinvex function with respect to $\eta$, and we assume that $\eta$ satisfies the following condition: $\eta(x,y,\lambda) \neq 0$, when $x \neq y$. Then, each local minimum point of the function $f$ is its point of global minimum.*

*Proof.* Assume that $\overline{y} \in X$ is a local minimum point of $f$ which is not a global minimum point. Hence, there exists a point $\overline{x} \in X$ such that $f(\overline{x}) < f(\overline{y})$. By the $G$-semipreinvexity of $f$ with respect to $\eta$, we have

$$f(\overline{y} + \lambda\eta(\overline{x},\overline{y},\lambda)) \leq G^{-1}(\lambda G(f(\overline{x})) + (1-\lambda)G(f(\overline{y}))), \quad \lambda \in [0,1]. \tag{4.1}$$

Then, for $\lambda \in [0,1]$,

$$\begin{aligned} f(\overline{y} + \lambda\eta(\overline{x},\overline{y},\lambda)) &< G^{-1}(\lambda G(f(\overline{y})) + (1-\lambda)G(f(\overline{y}))) \\ &= G^{-1}(G(f(\overline{y}))) \\ &= f(\overline{y}). \end{aligned} \tag{4.2}$$

Thus, we have

$$f(\overline{y} + \lambda\eta(\overline{x},\overline{y},\lambda)) < f(\overline{y}). \tag{4.3}$$

This is a contradiction with the assumption. □

**Theorem 4.2.** *The set of points which are global minimum of $f$ is a semiconnected set with respect to $\eta$.*

*Proof.* Denote by $A$ the set of points of global minimum of $f$, and let $x, y \in A$. Since $f$ is $G$-semipreinvex with respect to $\eta$, then

$$f(y + \lambda\eta(x,y,\lambda)) \le G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))), \quad \lambda \in [0,1] \tag{4.4}$$

is satisfied. Since $f(x) = f(y)$, we have

$$f(y + \lambda\eta(x,y,\lambda)) \le G^{-1}(\lambda G(f(y)) + (1-\lambda)G(f(y))). \tag{4.5}$$

So, for any $\lambda \in [0,1]$,

$$f(y + \lambda\eta(x,y,\lambda)) \le G^{-1}(G(f(y))) = f(y) = f(x). \tag{4.6}$$

Since $x, y \in A$ are points of a global minimum of $f$, it follows that, for any $\lambda \in [0,1]$, the following relation:

$$y + \lambda\eta(x,y,\lambda) \in A \tag{4.7}$$

is satisfied. Then, $A$ is a semiconnected set with respect to $\eta$. □

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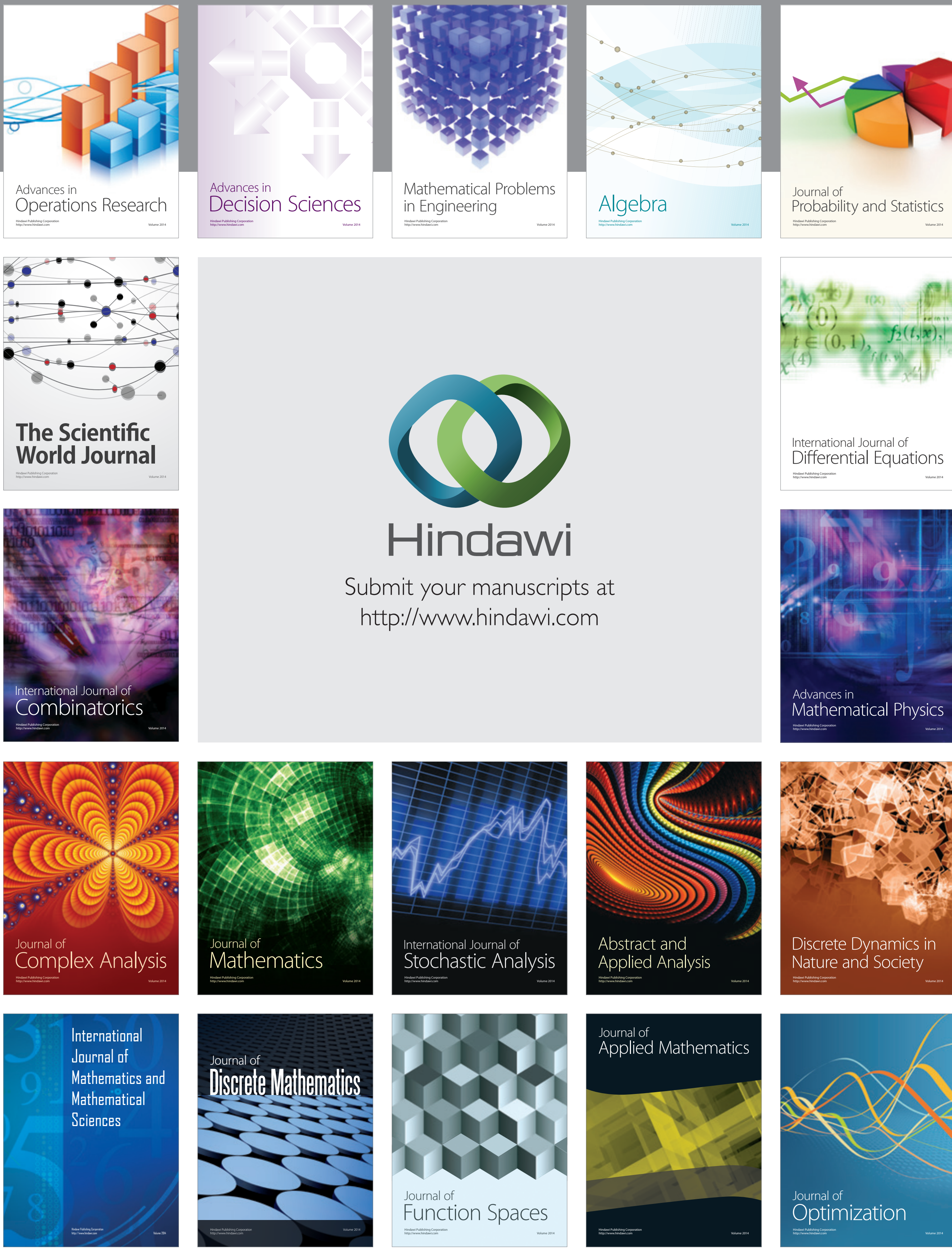
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