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Research Article

New Approximate Analytical Solutions of the Falkner-Skan Equation

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We propose an iterative method for solving the Falkner-Skan equation. The method provides approximate analytical solutions which consist of coefficients of the previous iterate solution. By some examples, we show that the presented method with a small number of iterations is competitive with the existing method such as Adomian decomposition method. Furthermore, to improve the accuracy of the proposed method, we suggest an efficient correction method. In practice, for some examples one can observe that the correction method results in highly improved approximate solutions.

1. Introduction

We consider the well-known Falkner-Skan equation

$$y'''(x) + y(x)y''(x) + \beta\left\{1 - y'(x)^2\right\} = 0, \quad 0 \le x < \infty, \tag{1.1}$$

subjected to boundary conditions

$$y(0) = y'(0) = 0, \qquad y'(\infty) = 1. \tag{1.2}$$

This boundary value problem arises in the research of viscous flow past a wedge of angle $\beta\pi$. $\beta > 0$ corresponds to flow toward the wedge and $\beta < 0$ does to flow away from the wedge. The special case, $\beta = 0$, is called the Blasius equation where the wedge reduces to a flat plate.

It is well known that a unique smooth solution of the Falkner-Skan equation (1.1) with the condition (1.2) exists for $0 \leq \beta \leq 1$ [1, 2]. For $-0.1988 < \beta < 0$, there exists two solutions, that is, one with $y''(0) > 0$ and the other one with $y''(0) < 0$, as shown in the literature [3]. For $\beta > 1$ the solution is unique under the restriction $0 < y'(x) < 1$ [4, 5].

In order to obtain approximate solutions of the Falkner-Skan equation, analytical methods such as Adomian decomposition methods [6–10], variational iterative methods [11–15], and homotopy analysis methods [16–18] can be referred. The objective of this work is to present a new analytical method which provides a simple form of iterate solutions and can be a match for existing methods in accuracy.

In the next section we develop an iterative method based on a decomposition of the Falkner-Skan equation (1.1) which was recently introduced in [19] for a particular case, $\beta = 0$. In the result we derive an iterative formula producing approximate analytical solutions in the form of polynomial series without requiring any differentiations or integrations of the previous iterate solutions. The degree of the nth iterate solution increases very rapidly, in fact, like $O(2^n)$ as $n \to \infty$. In Section 3, for some values of β we compare the presented solutions with the existing iterate solutions. In Section 4, to improve the accuracy of the presented solutions we suggest a correction method which is composed of the successive differences of the iterations.

2. Iteration for Approximate Analytical Solutions

To develop a new iterative method generating analytical series solutions, we consider the following one-point boundary conditions instead of the boundary conditions in (1.2).

$$y(0) = y'(0) = 0, \qquad y''(0) = \alpha, \tag{2.1}$$

where the curvature α of the solution is assumed to be known. Actually, the value of α can be obtained by numerical evaluation [8, 20, 21]. In this section we derive iterate solutions based on the method which was introduced in [19] for a special case, $\beta = 0$.

First, for $y = y(x)$ the Falkner-Skan equation (1.1) becomes

$$\left(y'' + yy'\right)' = \left(1 + \beta\right)\left(y'\right)^2 - \beta. \tag{2.2}$$

From the boundary conditions in (2.1), it follows that

$$y'' + yy' = \left(1 + \beta\right)\int_0^x y'(t)^2 dt - \beta x + \alpha := A(x), \tag{2.3}$$

and thus

$$y' = \int_0^x A(t)dt - \frac{1}{2}y^2 := B(x). \tag{2.4}$$

In the result we have

$$y(x) = \int_0^x B(t)dt. \tag{2.5}$$

Denoting by $y_n(x)$ the nth iterate solution and substituting it into the right-hand side of (2.5), with a given initial solution $y_0(x)$, we have an iteration formula

$$y_{n+1}(x) = \int_0^x B_n(t)dt, \quad n \geq 0, \tag{2.6}$$

where

$$\begin{aligned} B_n(x) &= \int_0^x A_n(t)dt - \frac{1}{2}y_n(x)^2, \\ A_n(x) &= (1+\beta)\int_0^x y_n'(t)^2 dt - \beta x + \alpha. \end{aligned} \tag{2.7}$$

Let δ_n denote a degree of the nth iterate solution $y_n(x)$. Then, from the formulas (2.6) and (2.7), it follows that $\delta_{n+1} = 2\delta_n + 1$ for all $n \geq 0$. Solving this recurrence equation, we have

$$\delta_n = \begin{cases} 4 \cdot 2^n - 1, & \text{when } \beta \neq 0, \\ 3 \cdot 2^n - 1, & \text{when } \beta = 0. \end{cases} \tag{2.8}$$

For an initial solution $y_0(x)$ satisfying the conditions in (2.1), we may set the nth iterate solution as

$$y_n(x) = \sum_{k=0}^{\delta_n - 2} a_{n,k} x^{k+2}, \tag{2.9}$$

for any integer $n \geq 0$. Then, by performing integrations in (2.6) and (2.7) directly, we have

$$\begin{aligned} A_n(x) &= (1+\beta)\int_0^x \left(\sum_{k=0}^{\delta_n - 2}(k+2)a_{n,k}t^{k+1}\right)^2 dt - \beta x + \alpha \\ &= \alpha - \beta x + (1+\beta)\sum_{k=0}^{\delta_n-2}\sum_{j=0}^{\delta_n-2}\frac{(k+2)(j+2)a_{n,k}a_{n,j}}{k+j+3}x^{k+j+3}, \\ y_{n+1}'(x) &= B_n(x) \\ &= \alpha x - \frac{\beta}{2}x^2 + \sum_{k=0}^{\delta_n-2}\sum_{j=0}^{\delta_n-2}\left\{\frac{(1+\beta)(k+2)(j+2)}{(k+j+3)(k+j+4)} - \frac{1}{2}\right\}a_{n,k}a_{n,j}x^{k+j+4}. \end{aligned} \tag{2.10}$$

Thus, by integrating $B_n(x)$, we have

$$y_{n+1}(x) = \frac{\alpha x^2}{2} - \frac{\beta x^3}{6} + \sum_{k=0}^{\delta_n-2}\sum_{j=0}^{\delta_n-2}\left\{\frac{(1+\beta)(k+2)(j+2)}{(k+j+3)(k+j+4)} - \frac{1}{2}\right\}\frac{a_{n,k}a_{n,j}}{k+j+5}x^{k+j+5}. \tag{2.11}$$

On the other hand, referring to the boundary conditions in (2.1) and (1.1), we may take an initial solution as

$$y_0(x) = \frac{\alpha}{2}x^2 - \frac{\beta}{6}x^3. \tag{2.12}$$

3. Examples

3.1. A Case of $\beta = 0$ (Blasius Problem)

A case of $\beta = 0$ is called the Blasius problem, and a well-known power series for the solution is

$$S(x) = \sum_{k=0}^{\infty}(-1)^k\frac{c_k\alpha^{k+1}}{(3k+2)!}x^{3k+2}, \quad x \sim 0, \tag{3.1}$$

where the coefficients c_k can be computed from the recurrence [1]:

$$c_k = \begin{cases} 1, & k = 0, 1, \\ \displaystyle\sum_{j=0}^{k-1}\binom{3k-1}{3j}c_j c_{k-j-1}, & k \geq 2. \end{cases} \tag{3.2}$$

In fact, $\alpha \approx 0.46959999$ and the series becomes

$$S(x) = \frac{\alpha}{2}x^2 - \frac{\alpha^2}{120}x^5 + \frac{11\alpha^3}{40320}x^8 - \frac{73\alpha^4}{7983360}x^{11} + \frac{111\alpha^5}{358758400}x^{14} + \cdots. \tag{3.3}$$

The Blasius series, however, converges only for $|x| < \rho \approx 4.02346$ [16, 22, 23]. In this paper we denote by $S_n(x)$ a truncated Blasius series to the nth term of $S(x)$.

For the Blasius problem the first four iterate solutions generated by the formula (2.11), with $y_0(x) = \alpha x^2/2$, are as follows:

$$
\begin{aligned}
y_1(x) &= \frac{\alpha x^2}{2} - \frac{\alpha^2 x^5}{120}, \\
y_2(x) &= \frac{\alpha x^2}{2} - \frac{\alpha^2 x^5}{120} + \frac{11\alpha^3 x^8}{40320} - \frac{\alpha^4 x^{11}}{712800}, \\
y_3(x) &= \frac{\alpha x^2}{2} - \frac{\alpha^2 x^5}{120} + \frac{11\alpha^3 x^8}{40320} - \frac{5\alpha^4 x^{11}}{532224} + \frac{10033\alpha^5 x^{14}}{87178291200} \\
&\quad - \frac{5449\alpha^6 x^{17}}{3908653056000} + \frac{83\alpha^7 x^{20}}{8935557120000} - \frac{\alpha^8 x^{23}}{49080898944000}, \\
y_4(x) &= \frac{\alpha x^2}{2} - \frac{\alpha^2 x^5}{120} + \frac{11\alpha^3 x^8}{40320} - \frac{5\alpha^4 x^{11}}{532224} + \frac{9299\alpha^5 x^{14}}{29059430400} - \frac{2173649\alpha^6 x^{17}}{355687428096000} \\
&\quad + \frac{13722337\alpha^7 x^{20}}{115852476579840000} - \frac{27184438601\alpha^8 x^{23}}{1292600836944248832000} \\
&\quad + \frac{12320831753849\alpha^9 x^{26}}{403291461126605635584000000} + \cdots - \frac{\alpha^{16} x^{47}}{4631724332758783424102400000000}.
\end{aligned}
\tag{3.4}
$$

It should be noted that the degree of the presented solution $y_n(x)$ increases like $O(2^n)$ as shown in (2.8) while that of the truncated Blasius series $S_n(x)$ is $3n - 1$, or it increases like $O(n)$ as $n \to \infty$.

For comparison between $y_n(x)$ and $S_n(x)$, Figure 1 includes graphs of the approximations to the numerical solution $y^*(x)$, which is taken for an exact solution, and their errors over an interval $[0, \rho_0]$. We set $\rho_0 = 4.02346$ which is close to the radius of convergence, ρ of the Blasius series solution mentioned above. Figure 1 shows that the presented solutions $y_n(x)$ approximate the exact solution better than the truncated Blasius series $S_n(x)$.

3.2. A Case of $\beta \neq 0$

We refer to another analytical solution obtained by the Adomian decomposition method as follows:

$$
y_n^A(x) = y_0(x) + \sum_{k=1}^{n} u_k(x), \quad n = 1, 2, \ldots, \tag{3.5}
$$

with

$$
u_{k+1}(x) = -L^{-1}(\mathcal{A}_k(x)), \quad k = 0, 1, 2, \ldots, \tag{3.6}
$$

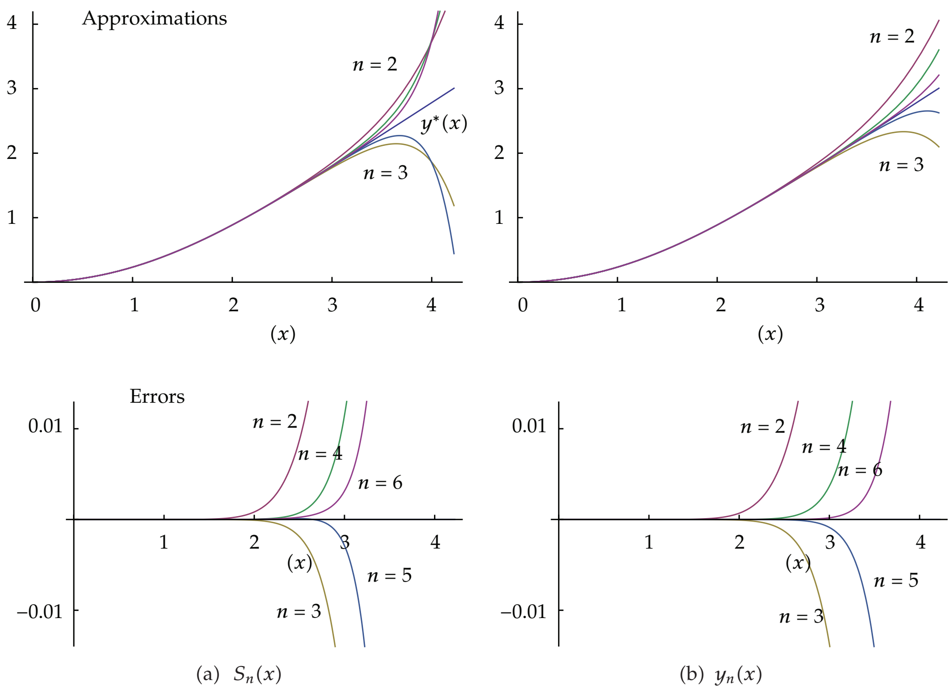


Figure 1: Truncated Blasius series $S_n(x)$ and their errors in (a) and presented solutions $y_n(x)$ and their errors in (b), $n = 2, 3, 4, 5, 6$, for the Blasius equation ($\beta = 0$).

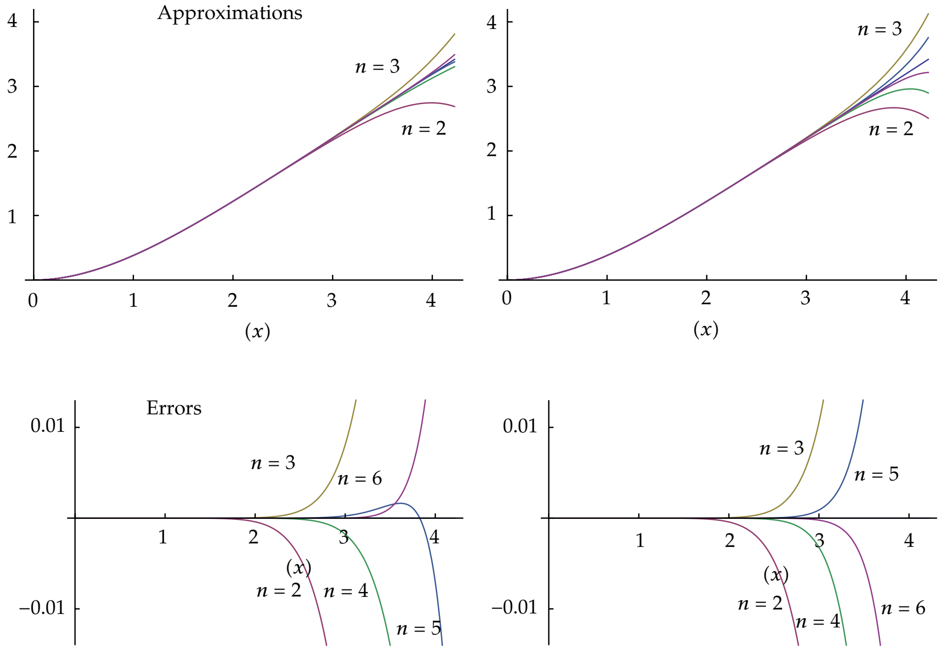
$u_0(x) = y_0(x)$, and Adomian polynomial $\mathcal{A}_k(x)$ generated by the recursive formula [8]:

$$\mathcal{A}_k(x) = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N\left(\sum_{j=0}^{k}\lambda^j u_j(x)\right)\right]_{\lambda=0}, \tag{3.7}$$

where L^{-1} is an inverse operator of $L = d^3/dx^3$ and N is a nonlinear operator defined as $Nz(x) = z(x)z''(x) - \beta z'(x)^2$.

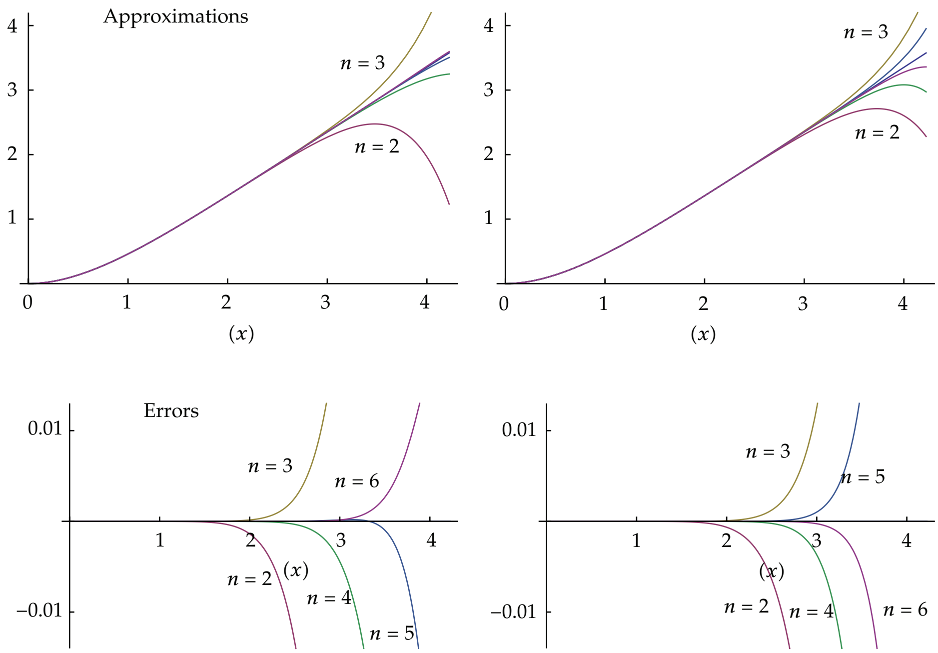
Presented solutions $y_n(x)$, $n = 2, 3, 4, 5, 6$, given in (2.11) and their errors are compared with those of Adomian's solutions $y_n^A(x)$ for the Falkner-Skan equation with $\beta = 0.5$ and $\beta = 1$ in Figures 2 and 3, respectively. We chose $y_0(x)$ in (2.12) as an initial solution and took $\alpha = y''(0) = 0.9277$ for $\beta = 0.5$ and $\alpha = 1.2326$ for $\beta = 1$, as given in [1]. One can see that the presented solutions are competitive with the Adomian solutions. Moreover, it should be noticed that unlike the Adomian solutions, the presented solutions do not require any integration or differentiation.

For the so-called decelerated flow (or $\beta < 0$), it is well known that a unique solution with $\alpha > 0$ exists when $-0.1988 < \beta < 0$ [4]. So we take an example of $\beta = -0.14$ with $\alpha = 0.2365$ as given in [8]. Comparison between the presented solutions $y_n(x)$ and the Adomian solutions $y_n^A(x)$ for this case is given in Figure 4. Therein we can find superiority of the presented solutions in accuracy.



(a) $y_n^A(x)$ (b) $y_n(x)$

Figure 2: The Adomian solutions $y_n^A(x)$ and their errors in (a) and presented solutions $y_n(x)$ and their errors in (b) for the Falkner-Skan equation with $\beta = 0.5$.



(a) $y_n^A(x)$ (b) $y_n(x)$

Figure 3: The Adomian solutions $y_n^A(x)$ and their errors in (a) and presented solutions $y_n(x)$ and their errors in (b) for the Falkner-Skan equation with $\beta = 1$.

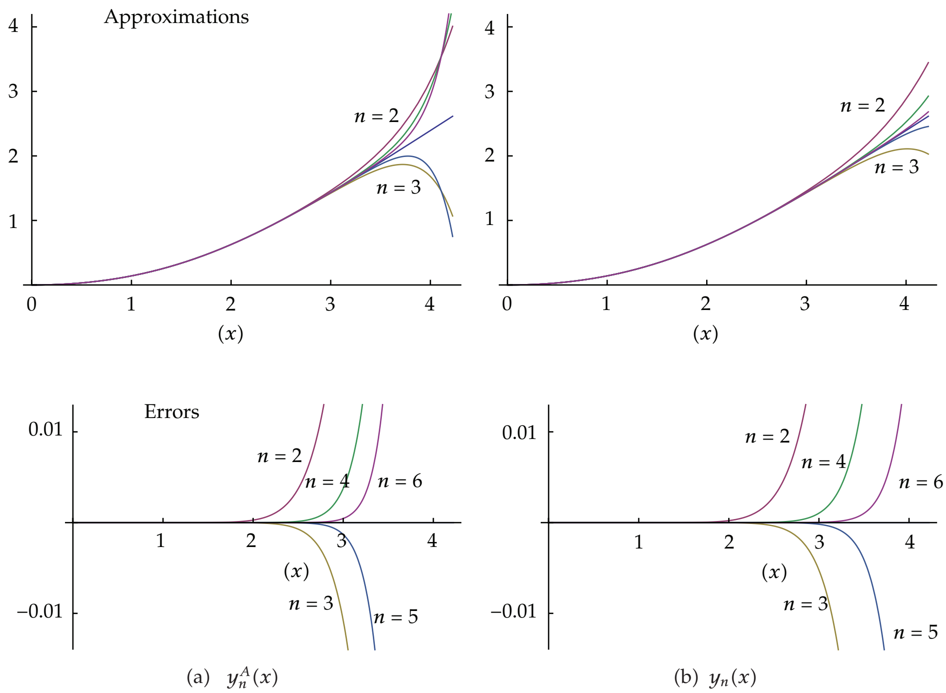


Figure 4: Adomian's solutions $y_n^A(x)$ and their errors in (a) and presented solutions $y_n(x)$ and their errors in (b) for the Falkner-Skan equation with $\beta = -0.14$.

4. Improvement of the Approximate Solution

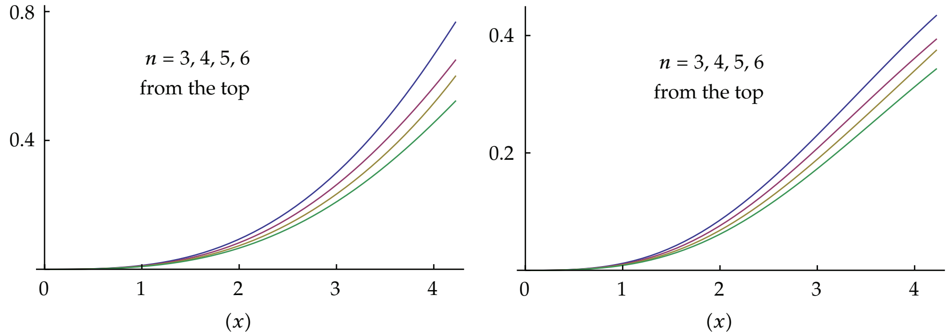
In Figure 1, for the case of $\beta = 0$, one can see that both the proposed solutions $y_n(x)$ and the truncated Blasius series $S_n(x)$ overshoot and undershoot, alternately, as n increases. This tendency, in the proposed solutions, is continued to the cases of other values of β as can be observed in Figures 2–4. Considering this feature, we propose a correction method as follows.

Let $y_j = y_j(x)$, $j = n-1, n, n+1$, be three successive iterate solutions obtained by the formula (2.11). To improve the accuracy of the iterate solutions, we suggest two corrections of the form

$$\hat{y}_{n+1} = \theta_n y_n + (1-\theta_n) y_{n+1},$$
$$\hat{y}_n = \theta_n y_{n-1} + (1-\theta_n) y_n \tag{4.1}$$

for a weight function $\theta_n = \theta_n(x)$ such that $0 \le \theta_n(x) < 1$. Setting a constraint $\hat{y}_{n+1} = \hat{y}_n$, we can determine θ_n as

$$\theta_n = \frac{y_n - y_{n+1}}{2y_n - y_{n-1} - y_{n+1}} = 1 - \frac{1}{1+r_n}, \tag{4.2}$$



(a) $r_n(x)$ (b) $\theta_n(x)$

Figure 5: Graphs of the ratios $r_n(x)$ and the weight functions $\theta_n(x)$, $n = 3, 4, 5, 6$ ($\beta = 0$).

Table 1: Comparison of the L_2-norm errors of the truncated Blasius series $S_{n+1}(x)$, the presented solution $y_{n+1}(x)$, and its correction $\widehat{y}_n(x)$ for the Blasius problem ($\beta = 0$).

n	$E(S_{n+1})$	$E(y_{n+1})$	$E(\widehat{y}_n)$
2	3.9×10^{-1}	2.0×10^{-1}	2.4×10^{-3}
3	3.8×10^{-1}	1.1×10^{-1}	9.5×10^{-3}
4	3.4×10^{-1}	5.6×10^{-2}	1.9×10^{-3}
5	3.3×10^{-1}	2.5×10^{-2}	1.7×10^{-3}
6	3.0×10^{-1}	1.0×10^{-2}	4.6×10^{-4}

where $r_n = r_n(x)$ is a ratio of the successive differences defined as

$$r_n = -\frac{y_{n+1} - y_n}{y_n - y_{n-1}}. \tag{4.3}$$

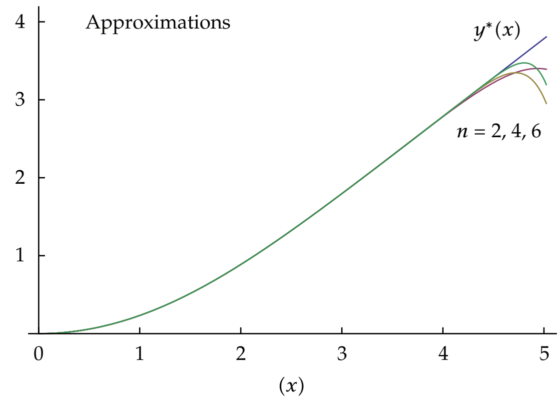
We may surmise that $r_n(x) \geq 0$ because, as mentioned above, the proposed solutions $y_n(x)$ alternately overshoot and undershoot as n increases. In practice, for the Blasius problem ($\beta = 0$), Figure 5(a) depicts the graphs of $r_n(x)$ on the interval $[0, \rho_0]$ for $n = 3, 4, 5, 6$, which shows $r_n(x) \geq 0$ for all $x \geq 0$. Thus from (4.2) we have $0 \leq \theta_n(x) < 1$ so that the correction $\widehat{y}_n(x)$ or $\widehat{y}_{n+1}(x)$ in (4.1) with the weight $\theta_n(x)$ in (4.2) is reasonable. The graph of $\theta_n(x)$ is given in Figure 5(b).

For $\beta = 0$, for example, Figure 6 shows the approximations of $\widehat{y}_n(x)$, $n = 2, 4, 6$, and their errors on an extended interval $[0, \rho_0 + 1]$. Furthermore, Table 1 includes L_2-norm errors of $S_{n+1}(x)$, $y_{n+1}(x)$, and $\widehat{y}_n(x)$ with respect to the numerical solution $y^*(x)$ on the interval $[0, \rho_0]$. The L_2-norm error $E(y)$ is defined as

$$E(y) = \left(\int_0^{\rho_0} \left(y(x) - y^*(x)\right)^2 dx\right)^{1/2}. \tag{4.4}$$

From Figure 6 and Table 1, compared with the results of $y_{n+1}(x)$ in Figure 1, we can see that the correction $\widehat{y}_n(x)$ highly improves the accuracy of the original iterate solution $y_{n+1}(x)$.

In addition, absolute errors of $y_{n+1}(x)$ and $\widehat{y}_n(x)$, $n = 2, 6$, for each $\beta = 0.5, 1, -0.14$ are depicted in Figure 7. Similar to Table 1, Table 2 includes L_2-norm errors of $y^A_{n+1}(x)$, $y_{n+1}(x)$, and $\widehat{y}_n(x)$ on the interval $[0, \rho_0]$. We can also observe that the correction $\widehat{y}_n(x)$ highly improves the accuracy of the original iterate solution $y_{n+1}(x)$.



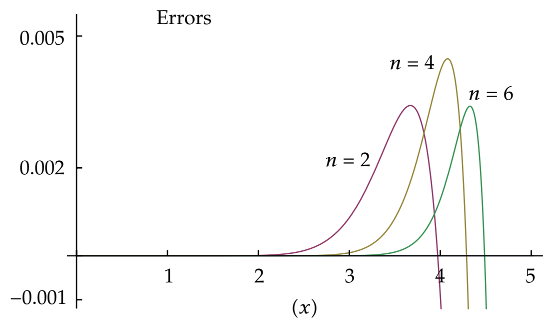


Figure 6: Corrected solutions $\hat{y}_n(x)$ of the presented solutions $y_n(x)$ and their errors, $n = 2, 4, 6$ $(\beta = 0)$.

Table 2: Comparison of the L_2-norm errors of the Adomian solution $y^A_{n+1}(x)$, the presented solution $y_{n+1}(x)$, and its correction $\hat{y}_n(x)$ for various values of β.

β	n	$E(y^A_{n+1})$	$E(y_{n+1})$	$E(\hat{y})$
	2	9.9×10^{-2}	1.6×10^{-1}	5.7×10^{-3}
0.5	4	2.1×10^{-3}	5.1×10^{-2}	2.4×10^{-3}
	6	5.4×10^{-3}	1.2×10^{-2}	5.1×10^{-4}
	2	3.1×10^{-1}	2.0×10^{-1}	4.6×10^{-3}
1.0	4	9.3×10^{-3}	5.5×10^{-2}	1.5×10^{-3}
	6	5.1×10^{-3}	1.2×10^{-2}	4.1×10^{-4}
	2	2.8×10^{-1}	1.2×10^{-1}	2.0×10^{-3}
−0.14	4	2.0×10^{-1}	2.0×10^{-2}	9.7×10^{-4}
	6	1.5×10^{-1}	2.3×10^{-3}	1.3×10^{-4}

5. Conclusion

We have developed a new iterative method generating analytical solutions to the Falkner-Skan equation (1.1) with the boundary conditions (2.1). In practice, for several cases of β, it was shown that the presented method with a small number of iterations is available and efficient, compared with a well-known existing method such as the Adomian decomposition method. Moreover, we have proposed a simple correction method which improves the accuracy and the rate of convergence of the presented method.

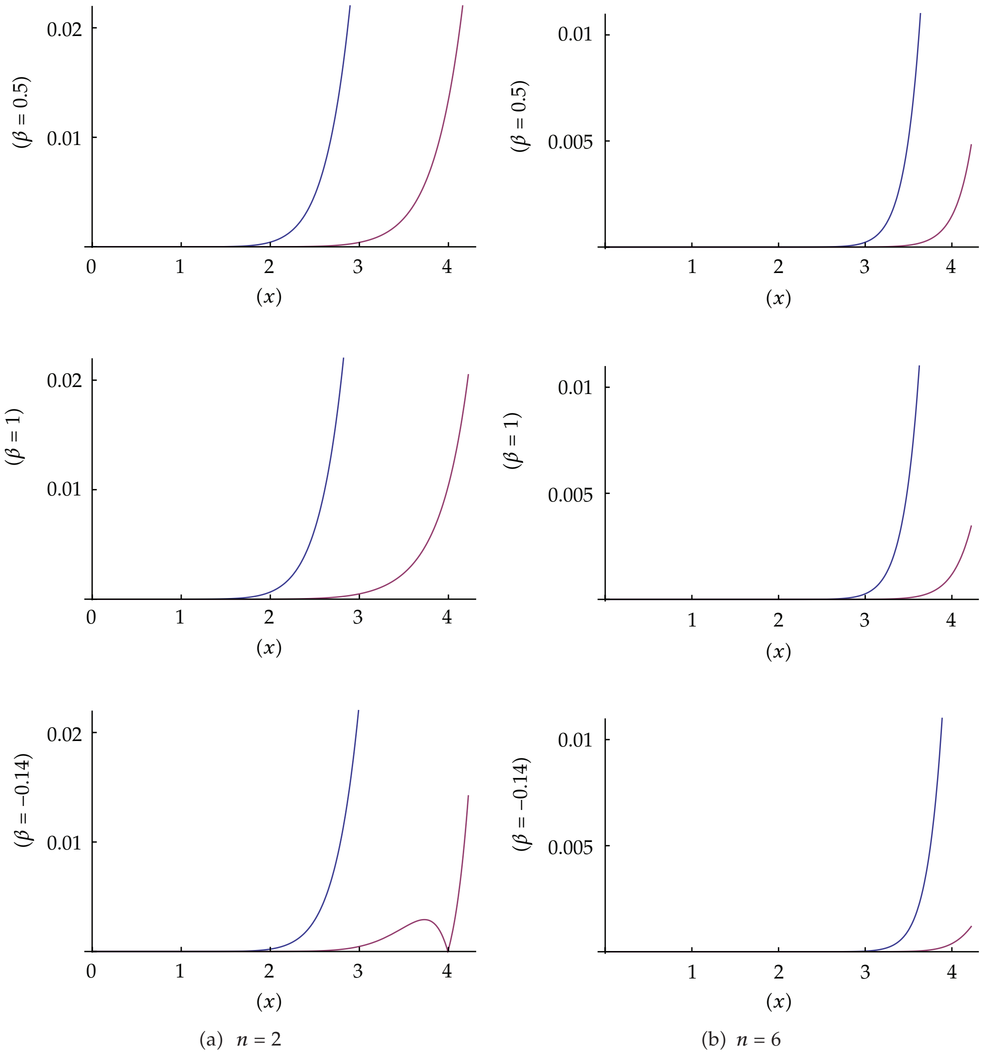


Figure 7: Absolute errors of the presented solutions $y_{n+1}(x)$ (thin lines) and those of the corrected solutions $\hat{y}_n(x)$ (thick lines) for $\beta = 0.5, 1, -0.14$.

Although in this paper we considered $-0.1988 < \beta \leq 1$ with the boundary conditions $y(0) = y'(0) = 0$, the proposed method may be extended to the general case of β with boundary conditions, $y(0) = \gamma$ and $y'(0) = \lambda$, as long as the existence of the solution is assured under some additional restrictions. In addition, we leave theoretical analysis of the error and the radius of convergence for a future work.

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