Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 176283, 14 pages
doi:10.1155/2012/176283

Research Article

Existence and Iterative Algorithm of Solutions for a System of Generalized Nonlinear Mixed Variational-Like Inequalities

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Received 29 December 2011; Accepted 16 January 2012

Academic Editor: Yonghong Yao

We introduce and study a system of generalized nonlinear mixed variational-like inequality problems (SGNMVLIPs) in Banach spaces. The auxiliary principle technique is applied to study the existence and iterative algorithm of solutions for the SGNMVLIP. First, the existence of solutions of the auxiliary problems for the SGNMVLIP is shown. Second, an iterative algorithm for solving the SGNMVLIP is constructed by using this existence result. Finally, not only the existence of solutions of the SGNMVLIP is shown but also the convergence of iterative sequences generated by the algorithm is also proven. The technique and results presented in this paper generalize and unify the corresponding techniques and results given in the literature.

1. Introduction and Preliminaries

Variational inequality theory, which was introduced by Stampacchia [1] in 1964, is an important part of nonlinear analysis. Various kinds of iterative algorithms to solve the variational inequalities have been developed by many authors, see [2–8] and the references therein. Variational-like inequality introduced by Parida et al. [9] is an important generalization of the variational inequalities and has significant applications in nonconvex optimization. It is worth mentioning that the projection method cannot be extended for constructing iterative algorithms for variational-like inequalities. To overcome this drawback, one uses usually the auxiliary principle technique which deals with finding a suitble auxiliary problems for the original problem. Further, this auxiliary problem is used to construct an algorithm for solving the original problems. Glowinski et al. [10] introduced this technique and used it to study the existence of a solution of variational-like inequality. Later, many authors extended this technique to suggest and analyze a number of algorithms for solving various classes of variational inequalities (see [11–19]).

Recently, the auxiliary principle technique was extended by Ding et al. [15] to study the existence and iterative algorithm of solutions of generalized strongly nonlinear mixed variational-like inequalities in Banach spaces. On the other hand, the auxiliary principle technique was also extended by Kazmi and Khan [16] who studied a system of generalized variational-like inequality problems in Hilbert spaces.

In this paper, we still extend the auxiliary principle technique to study a system of generalized nonlinear mixed variational-like inequality problems (SGNMVLIPs) in Banach spaces. At first, the existence of solutions of the auxiliary problems for the SGNMVLIP is shown. Next, an iterative algorithm for solving SGNMVLIP is constructed by using this existence and uniqueness result. Finally, we prove the existence of solutions of the SGNMVLIP and the convergence of the algorithm. These results improve and generalize many corresponding results given in [12, 13, 16–18].

Throughout the paper unless otherwise stated, let $I = \{1, 2\}$ be an index set. For each $i \in I$, let B_i be a real Banach spaces with norm $\|\cdot\|_i$, let B_i^* be the topological dual space of B_i, and let $\langle \cdot, \cdot \rangle_i$ be the generalized duality pairing between B_i and B_i^*. Let $F_i : B_1 \times B_2 \to B_i^*$, $\eta_i : B_i \times B_i \to B_i$ be nonlinear mappings, then we consider the following system of generalized nonlinear mixed variational-like inequality problems (SGNMVLIPs): for given $\omega_i^* \in B_i^*$, find $(x, y) \in B_1 \times B_2$ such that

$$\left\langle F_1(x,y) - \omega_1^*, \eta_1(v_1, x) \right\rangle_1 + b_1(x, v_1) - b_1(x, x) \geq 0 \quad \forall v_1 \in B_1, \tag{1.1}$$

$$\left\langle F_2(x,y) - \omega_2^*, \eta_2(v_2, x) \right\rangle_2 + b_2(y, v_2) - b_2(y, y) \geq 0 \quad \forall v_2 \in B_2, \tag{1.2}$$

where for each $i \in I$, the bifunction $b_i : B_i \times B_i \to \mathbb{R}$ is a real-valued nondifferential function with the following properties:

(i) b_i is linear in the first argument;

(ii) b_i is convex in the second argument;

(iii) b_i is bounded, that is, there exists a constant $\gamma_i > 0$ such that

$$b_i(u_i, v_i) \leq \gamma_i \|u_i\|_i \|v_i\|_i, \quad \forall u_i, v_i \in B_i; \tag{1.3}$$

(iv) $b_i(u_i, v_i) - b_i(u_i, w_i) \leq b_i(u_i, v_i - w_i)$, for all $u_i, v_i, w_i \in B_i$;

Remark 1.1 (see [15]). (1) It follows from property (i), for any $u_i, v_i \in B_i$, $b_i(-u_i, v_i) = -b_i(u_i, v_i)$. By property (iii), we have $b_i(-u_i, v_i) \leq \gamma_i \|u_i\|_i \|v_i\|_i$, and hence

$$|b_i(u_i, v_i)| \leq \gamma_i \|u_i\|_i \|v_i\|_i, \quad \forall u_i, v_i \in B_i. \tag{1.4}$$

This shows that for any $u_i, v_i \in B_i$, $b_i(u_i, 0) = b_i(0, v_i) = 0$.

(2) It follows from properties (iii) and (iv), for any $u_i, v_i, w_i \in B_i$,

$$\begin{aligned} b_i(u_i, v_i) - b_i(u_i, w_i) &\leq \gamma_i \|u_i\|_i \|v_i - w_i\|_i, \\ b_i(u_i, w_i) - b_i(u_i, v_i) &\leq \gamma_i \|u_i\|_i \|w_i - v_i\|_i. \end{aligned} \tag{1.5}$$

Therefore, we have

$$|b_i(u_i, v_i) - b_i(u_i, w_i)| \leq \gamma_i \|u_i\|_i \|v_i - w_i\|_i. \tag{1.6}$$

This implies that b_i is continuous with respect to the second argument.

Some Special Cases

(1) If for each $i \in I$, B_i is a real Hilbert space and $\omega_i^* = 0$, then the SGNMVLIP (1.1) and (1.2) reduce to the problems: find $(x, y) \in B_1 \times B_2$ such that

$$\begin{aligned} &\langle F_1(x, y), \eta_1(v_1, x)\rangle_1 + b_1(x, v_1) - b_1(x, x) \geq 0 \quad \forall v_1 \in B_1, \\ &\langle F_2(x, y), \eta_2(v_2, x)\rangle_2 + b_2(y, v_2) - b_2(y, y) \geq 0 \quad \forall v_2 \in B_2. \end{aligned} \tag{1.7}$$

The problem (1.7) has been studied by Kazmi and Khan [16].

(2) If index set $I = \{1\}$, $B_1 = B_2 = B$ and $F_1(x, x) = F_2(x, x) = Tx - Ax$ for each $x \in B$, where $A, T : B \to B^*$ are two single-valued mappings, then the SGNMVLIP (1.1) and (1.2) reduce to the problem: find $x \in B$ such that

$$\langle Tx - Ax - \omega^*, \eta(v, x)\rangle + b(x, v_1) - b(x, x) \geq 0 \quad \forall v \in B. \tag{1.8}$$

The problem (1.8) with $\omega^* = 0$ was introduced and studied by Ding [17] in reflexive Banach spaces.

In brief, for appropriate and suitable choice of the mappings $F_1, F_2, \eta_1, \eta_2, b_1, b_2$, and the linear continuous functionals ω_1^* and ω_2^*, one can obtain a number of the known classes of variational inequalities as special cases from SGNMVLIP (1.1) and (1.2) (see [6–8]).

We need the following basic concepts, basic assumptions and basic results which will be used in the sequel.

Definition 1.2. Let D be a nonempty subset of a Banach space E with the dual space E^*. Let $g : D \to E^*$ and $\eta : D \times D \to E$ be two mappings, then

(i) g is said to be η-strongly monotone if there exists a constant $\sigma > 0$ such that

$$\langle g(u) - g(v), \eta(u, v)\rangle \geq \sigma \|u - v\|^2, \quad \forall u, v \in D, \tag{1.9}$$

(ii) g is said to be Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\|g(u) - g(v)\| \leq \mu \|u - v\|, \quad \forall u, v \in D, \tag{1.10}$$

(iii) η is said to be Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta \|u - v\|, \quad \forall u, v \in D. \tag{1.11}$$

Definition 1.3. Let D be a nonempty subset of a Banach space E with the dual space E^*. Let $F : D \times D \to E^*$, $\eta : D \times D \to E$ be two mappings, then F is said to be

(i) (λ, ξ)-Lipschitz continuous if there exist constants $\lambda, \xi > 0$ such that

$$\|F(u_1, v_1) - F(u_2, v_2)\| \leq \lambda \|u_1 - u_2\| + \xi \|v_1 - v_2\|, \quad \forall u_1, v_1, u_2, v_2 \in D, \tag{1.12}$$

(ii) η-strongly monotone in the first argument if there exists a constant $\varepsilon_1 > 0$ such that

$$\langle F(u_1, v) - F(u_2, v), \eta(u_1, u_2) \rangle \geq \varepsilon_1 \|u_1 - u_2\|^2, \quad \forall u_1, u_2, v \in D, \tag{1.13}$$

(iii) η-strongly monotone in the second argument if there exists a constant $\varepsilon_2 > 0$ such that

$$\langle F(u, v_1) - F(u, v_2), \eta(v_1, v_2) \rangle \geq \varepsilon_2 \|v_1 - v_2\|^2, \quad \forall v_1, v_2, u \in D. \tag{1.14}$$

Assumption 1.4. For each $i \in I$, the mapping $\eta_i : B_i \times B_i \to B_i$ satisfies the following conditions:

(1) $\eta_i(u, v) = \eta_i(u, z) + \eta_i(z, v)$, for all $u, v, z \in B_i$;

(2) η_i is affine in the second argument;

(3) for each fixed $u \in B_i$, the function $v \mapsto \eta_i(u, v)$ is continuous from the weak topology to the weak topology.

Remark 1.5. It follows from Assumption 1.4 (1) that $\eta_i(u, v) = -\eta_i(v, u)$ and $\eta_i(u, u) = 0$ for any $u, v \in B_i$. Moreover, we can prove that η_i is also affine in the first argument by Assumption 1.4 (1) and (2).

Lemma 1.6 (see [20]). *Let X be a nonempty close convex subset of a Hausdorff linear topological space E, and let $\phi, \psi : X \times X \to \mathbb{R}$ be mappings satisfying the following conditions:*

(i) *for each $(x, y) \in X \times X$, $\psi(x, y) \leq \phi(x, y)$ and $\psi(x, x) \geq 0$ for each $x \in X$;*

(ii) *for each $x \in X$, $\phi(x, y)$ is upper semicontinuous with respect to y;*

(iii) *for each $y \in X$, the set $\{x \in X : \psi(x, y) < 0\}$ is convex;*

(iv) *there exists a nonempty compact set $\Omega \subset X$ and $x_0 \in \Omega$ such that $\psi(x_0, y) < 0$ for any $y \in X \setminus \Omega$.*

Then, there exists an $\overline{y} \in \Omega$ such that $\phi(x, \overline{y}) \geq 0$ for any $x \in X$.

2. Auxiliary Problems and Algorithm

In this section, we introduce the auxiliary problems to study the SGNMVLIP (1.1) and (1.2), and we give an existence theorem for a solution of the auxiliary problems. By using the existence theorem, we construct the iterative algorithm for solving the SGNMVLIP (1.1) and (1.2).

For each $i \in I$, let $g_i : B_i \to B_i^*$ be a single-valued mappings Given $(x_1, x_2) \in B_1 \times B_2$, we consider the following problems $P(x_1, x_2)$: find $(z_1, z_2) \in B_1 \times B_2$ such that

$$\begin{aligned}
&\left\langle g_1(z_1) - g_1(x_1) + \rho\left[F_1(x_1, x_2) - \omega_1^*\right], \eta_1(v_1, z_1)\right\rangle_1 + \rho[b_1(x_1, v_1) - b_1(x_1, z_1)] \geq 0 \quad \forall v_1 \in B_1, \\
&\left\langle g_2(z_2) - g_2(x_2) + \rho\left[F_2(x_1, x_2) - \omega_2^*\right], \eta_2(v_2, z_2)\right\rangle_2 + \rho[b_2(x_2, v_2) - b_2(x_2, z_2)] \geq 0 \quad \forall v_2 \in B_2,
\end{aligned} \tag{2.1}$$

where $\rho > 0$ is a constant. The problems are called the auxiliary problems for SGNMVLIP (1.1) and (1.2).

Remark 2.1. If for each $i \in I$, B_i is a real Hilbert spaces, $\omega_i^* = 0$ and $g_i = I$ are the identity mappings on B_i, then the auxiliary problems reduce to Kazmi and Khan's auxiliary problems in [16].

Theorem 2.2. *For each $i \in I$, let $\eta_i : B_i \times B_i \to B_i$ be Lipschtiz continuous with constant $\delta_i > 0$, and let $g_i : B_i \to B_i^*$ be η_i-strongly monotone and Lipschtiz continuous with constant $\sigma_i > 0$ and $\mu_i > 0$, respectively. Let $b_i : B_i \times B_i \to \mathbb{R}$ satisfy the properties* (i)–(iv). *If Assumption 1.4 holds, then the auxiliary problems $P(x_1, x_2)$ have a unique solution.*

Proof. For each $i \in I$, define the mappings $\phi_i, \psi_i : B_i \times B_i \to \mathbb{R}$ by

$$\begin{aligned}
\phi_i(v_i, z_i) &= \left\langle g_i(v_i) - g_i(x_i) + \rho\left[F_i(x_1, x_2) - \omega_i^*\right], \eta_i(v_i, z_i)\right\rangle_i + \rho[b_i(x_i, v_i) - b_i(x_i, z_i)], \\
\psi_i(v_i, z_i) &= \left\langle g_i(z_i) - g_i(x_i) + \rho\left[F_i(x_1, x_2) - \omega_i^*\right], \eta_i(v_i, z_i)\right\rangle_i + \rho[b_i(x_i, v_i) - b_i(x_i, z_i)],
\end{aligned} \tag{2.2}$$

respectively.

We claim that the mappings ϕ_i, ψ_i satisfy all conditions of Lemma 1.6 in the weak topology. Indeed, since g_i is η_i-strongly monotone with constant $\sigma_i > 0$ and Remark 1.5, it is clear that ϕ_i and ψ_i satisfy condition (i) of Lemma 1.6. Since the bifunction b_i is convex in the second argument and η_i is affine in the second argument, it follows from Assumption 1.4 (3) and Remark 1.1 (2) that $z_i \mapsto \phi_i(v_i, z_i)$ is weakly upper semicontinuous. By Assumption 1.4 (1) and (2), and the property (ii) of b_i, it is easy to prove that the set $\{v_i \in B_i : \psi_i(v_i, z_i) < 0\}$ is convex, hence the conditions (ii) and (iii) of Lemma 1.6 hold.

Let $w_i = \sigma_i^{-1}[\delta_i\|g_i(x_i) - g_i(0)\|_i + \delta_i\rho\|F_i(x_1, x_2) - \omega_i^*\|_i + \rho\gamma_i\|x_i\|_i]$ and $K_i = \{z_i \in B_i : \|z_i\|_i \leq w_i\}$, then K_i is a weakly compact subset of B_i. For any $z_i \in B_i \setminus K_i$, take $v_{i_0} = 0 \in K_i$. From Assumption 1.4 (1), Remark 1.1 (1), Lipschitz continuity of η_i, and the η_i-strongly monotonicity of g_i, we have

$$\begin{aligned}
\psi_i(v_{i_0}, z_i) &= \psi_i(0, z_i) \\
&= \left\langle g_i(z_i) - g_i(x_i) + \rho\left[F_i(x_1, x_2) - \omega_i^*\right], \eta_i(0, z_i)\right\rangle_i + \rho[b_i(x_i, 0) - b_i(x_i, z_i)] \\
&= -\left\langle g_i(0) - g_i(z_i), \eta_i(0, z_i)\right\rangle_i - \left\langle g_i(x_i) - g_i(0), \eta_i(0, z_i)\right\rangle_i \\
&\quad + \rho\left\langle \left[F_i(x_1, x_2) - \omega_i^*\right], \eta_i(0, z_i)\right\rangle_i + \rho[b_i(x_i, 0) - b_i(x_i, z_i)] \\
&\leq -\sigma_i\|z_i\|_i^2 + \delta_i\left\|g_i(x_i) - g_i(0)\right\|_i\|z_i\|_i + \delta_i\rho\left\|F_i(x_1, x_2) - \omega_i^*\right\|_i\|z_i\|_i + \rho\gamma_i\|x_i\|_i\|z_i\|_i \\
&= -\sigma_i\|z_i\|_i\left\{\|z_i\|_i - \sigma_i^{-1}\left[\delta_i\left\|g_i(x_i) - g_i(0)\right\|_i + \delta_i\rho\left\|F_i(x_1, x_2) - \omega_i^*\right\|_i + \rho\gamma_i\|x_i\|_i\right]\right\} < 0.
\end{aligned} \tag{2.3}$$

Therefore, the condition (iv) of Lemma 1.6 holds. By Lemma 1.6 there exists an $z_i^* \in B_i$ such that $\phi_i(v_i, z_i^*) \geq 0$ for all $v_i \in B_i$, that is

$$\langle g_i(v_i) - g_i(x_i) + \rho[F_i(x_1,x_2) - \omega_i^*], \eta_i(v_i, z_i^*)\rangle_i + \rho[b_i(x_i,v_i) - b_i(x_i,z_i^*)] \geq 0 \quad \forall v_i \in B_i. \tag{2.4}$$

For arbitrary $t \in (0,1]$ and $v_i \in B_i$, let $y_{i,t} = tv_i + (1-t)z_i^*$. Replacing v_i by $y_{i,t}$ in (2.4) and utilizing Assumption 1.4 (1) and (2), Remark 1.5, and the property (ii) of b_i, we obtain

$$\begin{aligned} 0 &\leq \langle g_i(y_{i,t}) - g_i(x_i) + \rho[F_i(x_1,x_2) - \omega_i^*], \eta_i(y_{i,t}, z_i^*)\rangle_i + \rho[b_i(x_i,y_{i,t}) - b_i(x_i,z_i^*)] \\ &= -\langle g_i(y_{i,t}) - g_i(x_i) + \rho[F_i(x_1,x_2) - \omega_i^*], \eta_i(z_i^*, y_{i,t})\rangle_i + \rho[b_i(x_i,y_{i,t}) - b_i(x_i,z_i^*)] \\ &\leq -t\langle g_i(y_{i,t}) - g_i(x_i) + \rho[F_i(x_1,x_2) - \omega_i^*], \eta_i(z_i^*, v_i)\rangle_i + \rho t[b_i(x_i,v_i) - b_i(x_i,z_i^*)]. \end{aligned} \tag{2.5}$$

Hence, we derive

$$\langle g_i(y_{i,t}) - g_i(x_i) + \rho[F_i(x_1,x_2) - \omega_i^*], \eta_i(v_i, z_i^*)\rangle_i + \rho[b_i(x_i,v_i) - b_i(x_i,z_i^*)] \geq 0. \tag{2.6}$$

Let $t \to 0^+$, by the Lipschitz continuity of g_i, we have

$$\langle g_i(z_i^*) - g_i(x_i) + \rho[F_i(x_1,x_2) - \omega_i^*], \eta_i(v_i, z_i^*)\rangle_i + \rho[b_i(x_i,v_i) - b_i(x_i,z_i^*)] \geq 0. \tag{2.7}$$

Therefore, (z_1^*, z_2^*) is a solution of the auxiliary problems $P(x_1,x_2)$.

Now, let (z_1', z_2') be another solution of the auxiliary problems $P(x_1,x_2)$ which is different from (z_1^*, z_2^*), then we have

$$\langle g_i(z_i') - g_i(x_i) + \rho[F_i(x_1,x_2) - \omega_i^*], \eta_i(v_i, z_i')\rangle_i + \rho[b_i(x_i,v_i) - b_i(x_i,z_i')] \geq 0. \tag{2.8}$$

Taking $v_i = z_i'$ in (2.7) and $v_i = z_i^*$ in (2.8) and adding these two inequalities, we obtain

$$\langle g_i(z_i') - g_i(z_i^*), \eta_i(z_i^*, z_i')\rangle_i \geq 0. \tag{2.9}$$

Since g_i is η_i-strongly monotone, we obtain

$$\sigma_i \|z_i^* - z_i'\|_i^2 \leq \langle g_i(z_i^*) - g_i(z_i'), \eta_i(z_i^*, z_i')\rangle_i \leq 0, \tag{2.10}$$

and so $(z_1', z_2') = (z_1^*, z_2^*)$. This completes the proof. □

By virtue of Theorem 2.2, we now construct an iterative algorithm for solving the SGNMVLIP (1.1) and (1.2).

For given $(x_0, y_0) \in B_1 \times B_2$, from Theorem 2.2, we know that the auxiliary problems $P(x_0, y_0)$ have a solution $(x_1, y_1) \in B_1 \times B_2$, that is,

$$\begin{aligned} &\langle g_1(x_1) - g_1(x_0) + \rho[F_1(x_0,y_0) - \omega_1^*], \eta_1(v_1, x_1)\rangle_1 + \rho[b_1(x_0,v_1) - b_1(x_0,x_1)] \geq 0 \quad \forall v_1 \in B_1, \\ &\langle g_2(y_1) - g_2(y_0) + \rho[F_2(x_0,y_0) - \omega_2^*], \eta_2(v_2, y_1)\rangle_2 + \rho[b_2(y_0,v_2) - b_2(y_0,y_1)] \geq 0 \quad \forall v_2 \in B_2. \end{aligned} \tag{2.11}$$

Again by Theorem 2.2, the auxiliary problems $P(x_1, y_1)$ have a solution $(x_2, y_2) \in B_1 \times B_2$, that is,

$$\begin{aligned}
&\left\langle g_1(x_2) - g_1(x_1) + \rho\left[F_1(x_1, y_1) - \omega_1^*\right], \eta_1(v_1, x_2)\right\rangle_1 + \rho[b_1(x_1, v_1) - b_1(x_1, x_2)] \geq 0 \quad \forall v_1 \in B_1, \\
&\left\langle g_2(y_2) - g_2(y_1) + \rho\left[F_2(x_1, y_1) - \omega_2^*\right], \eta_2(v_2, y_2)\right\rangle_2 + \rho\left[b_2(y_1, v_2) - b_2(y_1, y_2)\right] \geq 0 \quad \forall v_2 \in B_2.
\end{aligned} \tag{2.12}$$

By induction, we can get the iterative algorithm for solving the SGNMVLIP (1.1) and (1.2) as follows.

Algorithm 2.3. For given $(x_0, y_0) \in B_1 \times B_2$, there exists a sequence $\{(x_n, y_n)\}$ such that

$$\begin{aligned}
&\left\langle g_1(x_{n+1}) - g_1(x_n) + \rho\left[F_1(x_n, y_n) - \omega_1^*\right], \eta_1(v_1, x_{n+1})\right\rangle_1 + \rho[b_1(x_n, v_1) - b_1(x_n, x_{n+1})] \geq 0 \\
&\qquad \forall v_1 \in B_1;
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&\left\langle g_2(y_{n+1}) - g_2(y_n) + \rho[F_2(x_n, y_n) - \omega_2^*], \eta_2(v_2, y_{n+1})\right\rangle_2 + \rho\left[b_2(y_n, v_2) - b_2(y_n, y_{n+1})\right] \geq 0 \\
&\qquad \forall v_2 \in B_2, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{2.14}$$

where $\rho > 0$ is a constant.

3. Existence of Convergence Theorem

In this section, we will prove not only that the sequence $\{(x_n, y_n)\}$ generated by Algorithm 2.3 converges strongly to (x^*, y^*), and also that (x^*, y^*) is a solution of SGNMVLIP (1.1) and (1.2).

Theorem 3.1. *For each $i \in I$, assume that the following conditions are satisfied:*

(1) $g_i : B_i \to B_i^*$ *is η_i-strongly monotone and Lipschtiz continuous with constant $\sigma_i > 0$ and $\mu_i > 0$, respectively;*

(2) $\eta_i : B_i \times B_i \to B_i$ *is Lipschtiz continuous with constant $\delta_i > 0$;*

(3) $F_i : B_1 \times B_2 \to B_i^*$ *is (λ_i, ξ_i)-Lipschtiz continuous;*

(4) $F_i : B_1 \times B_2 \to B_i^*$ *is η_i-strongly monotone in the ith argument with constant $\varepsilon_i > 0$;*

(5) $b_i : B_i \times B_i \to \mathbb{R}$ *satisfies the properties* (i)–(iv).

If Assumption 1.4 holds and there exists a constant $\rho > 0$ such that

$$\begin{aligned}
P_1 &= 2\sigma_1 - \frac{1}{4\rho\varepsilon_1}\left[\delta_1^2\mu_1^2 + \rho\delta_1^2\mu_1\lambda_1 + 2\rho^2\varepsilon_1(\gamma_1 + \delta_1\xi_1 + 2\delta_1\lambda_1)\right] > 0, \\
P_2 &= 2\sigma_2 - \frac{1}{4\rho\varepsilon_2}\left[\delta_2^2\mu_2^2 + \rho\delta_2^2\mu_2\xi_2 + 2\rho^2\varepsilon_2(\gamma_2 + \delta_2\lambda_2 + 2\delta_2\xi_2)\right] > 0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\frac{1}{4\varepsilon_1}\left[\delta_1^2\mu_1\lambda_1+\rho\left(\delta_1^2\lambda_1^2+2\varepsilon_1\gamma_1\right)\right]+\frac{1}{2}\rho\lambda_2\delta_2< \min\{P_1,P_2\},\\
&\frac{1}{4\varepsilon_2}\left[\delta_2^2\mu_2\xi_2+\rho\left(\delta_2^2\xi_2^2+2\varepsilon_2\gamma_2\right)\right]+\frac{1}{2}\rho\xi_1\delta_1< \min\{P_1,P_2\},
\end{aligned}\tag{3.2}$$

then the sequence $\{(x_n,y_n)\}$ *generated by Algorithm 2.3 converges strongly to* (x^*,y^*), *and* (x^*,y^*) *is a solution of SGNMVLIP* (1.1) *and* (1.2).

Proof. For any $(v_1,v_2)\in B_1\times B_2$, it follows from Algorithm 2.3 that

$$\left\langle g_1(x_n)-g_1(x_{n-1})+\rho\left[F_1(x_{n-1},y_{n-1})-\omega_1^*\right],\eta_1(v_1,x_n)\right\rangle_1+\rho[b_1(x_{n-1},v_1)-b_1(x_{n-1},x_n)]\geq 0,\tag{3.3}$$

$$\left\langle g_2(y_n)-g_2(y_{n-1})+\rho\left[F_2(x_{n-1},y_{n-1})-\omega_2^*\right],\eta_2(v_2,y_n)\right\rangle_2+\rho\left[b_2(y_{n-1},v_2)-b_2(y_{n-1},y_n)\right]\geq 0,\tag{3.4}$$

$$\left\langle g_1(x_{n+1})-g_1(x_n)+\rho\left[F_1(x_n,y_n)-\omega_1^*\right],\eta_1(v_1,x_{n+1})\right\rangle_1+\rho[b_1(x_n,v_1)-b_1(x_n,x_{n+1})]\geq 0,\tag{3.5}$$

$$\left\langle g_2(y_{n+1})-g_2(y_n)+\rho\left[F_2(x_n,y_n)-\omega_2^*\right],\eta_2(v_2,y_{n+1})\right\rangle_2+\rho\left[b_2(y_n,v_2)-b_2(y_n,y_{n+1})\right]\geq 0.\tag{3.6}$$

Taking $v_1=x_{n+1}$ in (3.3) and $v_1=x_n$ in (3.5), respectively, we get

$$\left\langle g_1(x_n)-g_1(x_{n-1})+\rho\left[F_1(x_{n-1},y_{n-1})-\omega_1^*\right],\eta_1(x_{n+1},x_n)\right\rangle_1+\rho[b_1(x_{n-1},x_{n+1})-b_1(x_{n-1},x_n)]\geq 0,\tag{3.7}$$

$$\left\langle g_1(x_{n+1})-g_1(x_n)+\rho\left[F_1(x_n,y_n)-\omega_1^*\right],\eta_1(x_n,x_{n+1})\right\rangle_1+\rho[b_1(x_n,x_n)-b_1(x_n,x_{n+1})]\geq 0.\tag{3.8}$$

Adding (3.7) and (3.8), we obtain

$$\begin{aligned}
&\left\langle g_1(x_n)-g_1(x_{n+1}),\eta_1(x_n,x_{n+1})\right\rangle_1\\
&\quad\leq\left\langle g_1(x_{n-1})-g_1(x_n)-\rho\left[F_1(x_{n-1},y_{n-1})-F_1(x_n,y_n)\right],\eta_1(x_n,x_{n+1})\right\rangle_1\\
&\qquad+\rho[b_1(x_n-x_{n-1},x_n)-b_1(x_n-x_{n-1},x_{n+1})].
\end{aligned}\tag{3.9}$$

From conditions (1) and (2), we have

$$\begin{aligned}
&\left\langle g_1(x_n)-g_1(x_{n+1}),\eta_1(x_n,x_{n+1})\right\rangle_1\geq\sigma_1\|x_n-x_{n+1}\|_1^2,\\
&\left\langle g_1(x_{n-1})-g_1(x_n),\eta_1(x_n,x_{n+1})\right\rangle_1=\left\langle g_1(x_{n-1})-g_1(x_{n+1}),\eta_1(x_n,x_{n+1})\right\rangle_1\\
&\qquad\qquad+\left\langle g_1(x_{n+1})-g_1(x_n),\eta_1(x_n,x_{n+1})\right\rangle_1\\
&\qquad\qquad\leq\delta_1\mu_1\|x_{n+1}-x_{n-1}\|_1\|x_{n+1}-x_n\|_1-\sigma_1\|x_{n+1}-x_n\|_1^2.
\end{aligned}\tag{3.10}$$

From conditions (3) and (4), we obtain

$$\begin{aligned}
&\left\langle F_1(x_n, y_n) - F_1(x_{n-1}, y_{n-1}), \eta_1(x_n, x_{n+1})\right\rangle_1 \\
&\quad = \left\langle F_1(x_n, y_n) - F_1(x_{n+1}, y_{n-1}), \eta_1(x_n, x_{n+1})\right\rangle_1 \\
&\qquad + \left\langle F_1(x_{n+1}, y_{n-1}) - F_1(x_{n-1}, y_{n-1}), \eta_1(x_n, x_{n-1})\right\rangle_1 \\
&\qquad + \left\langle F_1(x_{n+1}, y_{n-1}) - F_1(x_{n-1}, y_{n-1}), \eta_1(x_{n-1}, x_{n+1})\right\rangle_1 \\
&\quad \le \delta_1\lambda_1\|x_{n+1} - x_n\|_1^2 + \delta_1\xi_1\|x_{n+1} - x_n\|_1\left\|y_n - y_{n-1}\right\|_2 \\
&\qquad + \delta_1\lambda_1\|x_{n+1} - x_{n-1}\|_1\|x_n - x_{n-1}\|_1 - \varepsilon_1\|x_{n+1} - x_{n-1}\|_1^2.
\end{aligned} \tag{3.11}$$

From condition (5) and Remark 1.1 (2), we have

$$b_1(x_n - x_{n-1}, x_n) - b_1(x_n - x_{n-1}, x_{n+1}) \le \gamma_1\|x_{n+1} - x_n\|_1\|x_n - x_{n-1}\|_1. \tag{3.12}$$

Therefore, from (3.9)–(3.12), we derive

$$\begin{aligned}
&(2\sigma_1 - \rho\delta_1\lambda_1)\|x_n - x_{n+1}\|_1^2 \\
&\quad \le -\rho\varepsilon_1\|x_{n+1} - x_{n-1}\|_1^2 + \left[\delta_1\mu_1\|x_{n+1} - x_n\|_1 + \rho\delta_1\lambda_1\|x_n - x_{n-1}\|_1\right]\|x_{n+1} - x_{n-1}\|_1 \\
&\qquad + \rho\delta_1\xi_1\|x_{n+1} - x_n\|_1\left\|y_n - y_{n-1}\right\|_2 + \rho\gamma_1\|x_{n+1} - x_n\|_1\|x_n - x_{n-1}\|_1 \\
&\quad \le \frac{1}{4\rho\varepsilon_1}\left[\delta_1\mu_1\|x_{n+1} - x_n\|_1 + \rho\delta_1\lambda_1\|x_n - x_{n-1}\|_1\right]^2 \\
&\qquad + \rho\delta_1\xi_1\|x_{n+1} - x_n\|_1\left\|y_n - y_{n-1}\right\|_2 + \rho\gamma_1\|x_{n+1} - x_n\|_1\|x_n - x_{n-1}\|_1 \\
&\quad = \frac{\delta_1^2\mu_1^2}{4\rho\varepsilon_1}\|x_{n+1} - x_n\|_1^2 + \frac{\rho\delta_1^2\lambda_1^2}{4\varepsilon_1}\|x_n - x_{n-1}\|_1^2 \\
&\qquad + \left[\frac{\delta_1^2\mu_1\lambda_1}{2\varepsilon_1} + \rho\gamma_1\right]\|x_{n+1} - x_n\|_1\|x_n - x_{n-1}\|_1 + \rho\delta_1\xi_1\|x_{n+1} - x_n\|_1\left\|y_n - y_{n-1}\right\|_2 \\
&\quad \le \frac{1}{4\rho\varepsilon_1}\left[\delta_1^2\mu_1^2 + \rho\delta_1^2\mu_1\lambda_1 + 2\rho^2\varepsilon_1(\gamma_1 + \delta_1\xi_1)\right]\|x_{n+1} - x_n\|_1^2 \\
&\qquad + \frac{1}{4\varepsilon_1}\left[\delta_1^2\mu_1\lambda_1 + \rho\left(\delta_1^2\lambda_1^2 + 2\varepsilon_1\gamma_1\right)\right]\|x_n - x_{n-1}\|_1^2 + \frac{1}{2}\rho\delta_1\xi_1\left\|y_n - y_{n-1}\right\|_2^2,
\end{aligned} \tag{3.13}$$

which implies

$$P_1\|x_n - x_{n+1}\|_1^2 \le Q_1\|x_n - x_{n-1}\|_1^2 + R_1\left\|y_n - y_{n-1}\right\|_2^2, \tag{3.14}$$

where $Q_1 = (1/4\varepsilon_1)[\delta_1^2\mu_1\lambda_1 + \rho(\delta_1^2\lambda_1^2 + 2\varepsilon_1\gamma_1)]$, $R_1 = (1/2)\rho\delta_1\xi_1$.

Taking $v_2 = y_{n+1}$ in (3.4) and $v_2 = y_n$ in (3.6), respectively, we get

$$\begin{gathered}\left\langle g_2(y_n)-g_2(y_{n-1})+\rho\left[F_2(x_{n-1},y_{n-1})-\omega_2^*\right],\eta_2(y_{n+1},y_n)\right\rangle_2+\rho\left[b_2(y_{n-1},y_{n+1})-b_2(y_{n-1},y_n)\right]\geq 0,\\ \left\langle g_2(y_{n+1})-g_2(y_n)+\rho\left[F_2(x_n,y_n)-\omega_2^*\right],\eta_2(y_n,y_{n+1})\right\rangle_2+\rho\left[b_2(y_n,y_n)-b_2(y_n,y_{n+1})\right]\geq 0.\end{gathered} \tag{3.15}$$

Adding (3.15), we obtain

$$\begin{aligned}&\left\langle g_2(y_n)-g_2(y_{n+1}),\eta_2(y_n,y_{n+1})\right\rangle_2\\ &\quad\leq\left\langle g_2(y_{n-1})-g_2(y_n)-\rho\left[F_2(x_{n-1},y_{n-1})-F_2(x_n,y_n)\right],\eta_2(y_n,y_{n+1})\right\rangle_2\\ &\qquad+\rho\left[b_2(y_n-y_{n-1},y_n)-b_2(y_n-y_{n-1},y_{n+1})\right].\end{aligned} \tag{3.16}$$

From conditions (1) and (2), we have

$$\begin{aligned}\left\langle g_2(y_n)-g_2(y_{n+1}),\eta_2(y_n,y_{n+1})\right\rangle_2&\geq\sigma_2\left\|y_n-y_{n+1}\right\|_2^2,\\ \left\langle g_2(y_{n-1})-g_2(y_n),\eta_2(y_n,y_{n+1})\right\rangle_2&=\left\langle g_2(y_{n-1})-g_2(y_{n+1}),\eta_2(y_n,y_{n+1})\right\rangle_2\\ &\quad+\left\langle g_2(y_{n+1})-g_2(y_n),\eta_2(y_n,y_{n+1})\right\rangle_2\\ &\leq\delta_2\mu_2\left\|y_{n+1}-y_{n-1}\right\|_2\left\|y_{n+1}-y_n\right\|_2-\sigma_2\left\|y_{n+1}-y_n\right\|_2^2.\end{aligned} \tag{3.17}$$

From conditions (3) and (4), we obtain

$$\begin{aligned}&\left\langle F_2(x_n,y_n)-F_2(x_{n-1},y_{n-1}),\eta_2(y_n,y_{n+1})\right\rangle_2\\ &\quad=\left\langle F_2(x_n,y_n)-F_2(x_{n-1},y_{n+1}),\eta_2(y_n,y_{n+1})\right\rangle_2\\ &\qquad+\left\langle F_2(x_{n-1},y_{n+1})-F_2(x_{n-1},y_{n-1}),\eta_2(y_n,y_{n-1})\right\rangle_2\\ &\qquad+\left\langle F_2(x_{n-1},y_{n+1})-F_2(x_{n-1},y_{n-1}),\eta_2(y_{n-1},y_{n+1})\right\rangle_2\\ &\quad\leq\delta_2\xi_2\left\|y_{n+1}-y_n\right\|_2^2+\delta_2\lambda_2\left\|y_{n+1}-y_n\right\|_2\left\|x_n-x_{n-1}\right\|_1\\ &\qquad+\delta_2\xi_2\left\|y_{n+1}-y_{n-1}\right\|_2\left\|y_n-y_{n-1}\right\|_2-\varepsilon_2\left\|y_{n+1}-y_{n-1}\right\|_2^2.\end{aligned} \tag{3.18}$$

From condition (5) and Remark 1.1 (2), we have

$$b_2(y_n-y_{n-1},y_n)-b_2(y_n-y_{n-1},y_{n+1})\leq\gamma_2\left\|y_{n+1}-y_n\right\|_2\left\|y_n-y_{n-1}\right\|_2. \tag{3.19}$$

Therefore, from (3.16)–(3.19), we derive

$$\begin{aligned}
&(2\sigma_2-\rho\delta_2\xi_2)\|y_n-y_{n+1}\|_2^2\\
&\quad\le -\rho\varepsilon_2\|y_{n+1}-y_{n-1}\|_2^2+\left[\delta_2\mu_2\|y_{n+1}-y_n\|_2+\rho\delta_2\xi_2\|y_n-y_{n-1}\|_2\right]\|y_{n+1}-y_{n-1}\|_2\\
&\qquad+\rho\delta_2\lambda_2\|y_{n+1}-y_n\|_2\|x_n-x_{n-1}\|_1+\rho\gamma_2\|y_{n+1}-y_n\|_2\|y_n-y_{n-1}\|_2\\
&\quad\le \frac{1}{4\rho\varepsilon_2}\left[\delta_2\mu_2\|y_{n+1}-y_n\|_2+\rho\delta_2\xi_2\|y_n-y_{n-1}\|_2\right]^2\\
&\qquad+\rho\delta_2\lambda_2\|y_{n+1}-y_n\|_2\|x_n-x_{n-1}\|_1+\rho\gamma_2\|y_{n+1}-y_n\|_2\|y_n-y_{n-1}\|_2\\
&\quad=\frac{\delta_2^2\mu_2^2}{4\rho\varepsilon_2}\|y_{n+1}-y_n\|_2^2+\frac{\rho\delta_2^2\xi_2^2}{4\varepsilon_2}\|y_n-y_{n-1}\|_2^2\\
&\qquad+\left[\frac{\delta_2^2\mu_2\xi_2}{2\varepsilon_2}+\rho\gamma_2\right]\|y_{n+1}-y_n\|_2\|y_n-y_{n-1}\|_2+\rho\delta_2\lambda_2\|y_{n+1}-y_n\|_2\|x_n-x_{n-1}\|_1\\
&\quad\le\frac{1}{4\rho\varepsilon_2}\left[\delta_2^2\mu_2^2+\rho\delta_2^2\mu_2\xi_2+2\rho^2\varepsilon_2(\gamma_2+\delta_2\lambda_2)\right]\|y_{n+1}-y_n\|_2^2\\
&\qquad+\frac{1}{4\varepsilon_2}\left[\delta_2^2\mu_2\xi_2+\rho\left(\delta_2^2\xi_2^2+2\varepsilon_2\gamma_2\right)\right]\|y_n-y_{n-1}\|_2^2+\frac{1}{2}\rho\delta_2\lambda_2\|x_n-x_{n-1}\|_1^2,
\end{aligned}\tag{3.20}$$

which implies

$$P_2\|y_n-y_{n+1}\|_2^2\le Q_2\|y_n-y_{n-1}\|_2^2+R_2\|x_n-x_{n-1}\|_1^2,\tag{3.21}$$

where $Q_2=(1/4\varepsilon_2)[\delta_2^2\mu_2\xi_2+\rho(\delta_2^2\xi_2^2+2\varepsilon_2\gamma_2)]$, $R_2=(1/2)\rho\delta_2\lambda_2$.

Adding (3.14) and (3.21), we have

$$P_1\|x_n-x_{n+1}\|_1^2+P_2\|y_n-y_{n+1}\|_2^2\le(Q_1+R_2)\|x_n-x_{n-1}\|_1^2+(Q_2+R_1)\|y_n-y_{n-1}\|_2^2.\tag{3.22}$$

Define the norm $\|\cdot\|_*$ on $B_1\times B_2$ by

$$\|(u,v)\|_*=\sqrt{\|u\|_1^2+\|v\|_2^2}\quad\forall(u,v)\in B_1\times B_2,\tag{3.23}$$

it is easy to prove that $(B_1\times B_2,\|\cdot\|_*)$ is a Banach space.

From (3.22), by conditions (3.1), we have

$$\|(x_{n+1},y_{n+1})-(x_n,y_n)\|_*^2\le\max\{\theta_1,\theta_2\}\|(x_n,y_n)-(x_{n-1},y_{n-1})\|_*^2,\tag{3.24}$$

where $\theta_1=(Q_1+R_2)/\min\{P_1,P_2\}$, $\theta_2=(Q_2+R_1)/\min\{P_1,P_2\}$. From condition (3.2), which implies $\theta_1,\theta_2\in(0,1)$, hence $\{(x_n,y_n)\}$ is a cauchy sequence, let $(x_n,y_n)\to(x^*,y^*)$ $(n\to\infty)$.

By the Lipschitz continuities of η_1 and g, and $x_n \to x^*$ $(n \to \infty)$, we have

$$\left|\langle g_1(x_{n+1}) - g_1(x_n), \eta_1(v_1, x_{n+1})\rangle_1\right| \leq \delta_1\mu_1\|x_{n+1} - x_n\|_1\|v_1 - x_{n+1}\|_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

Since F_1 is (λ_1, ξ_1)-Lipschitz, $x_n \to x^*$ $(n \to \infty)$ and $y_n \to y^*$ $(n \to \infty)$, then we obtain

$$\begin{aligned}
&\left|\langle F_1(x_n, y_n) - \omega_1^*, \eta_1(v_1, x_{n+1})\rangle_1 - \langle F_1(x^*, y^*) - \omega_1^*, \eta_1(v_1, x^*)\rangle_1\right| \\
&\quad \leq \left|\langle F_1(x_n, y_n) - F_1(x^*, y^*), \eta_1(v_1, x_{n+1})\rangle_1\right| + \left|\langle F_1(x^*, y^*) - \omega_1^*, \eta_1(v_1, x_{n+1}) - \eta_1(v_1, x^*)\rangle_1\right| \\
&\quad \leq \delta_1(\lambda_1\|x_n - x^*\|_1 + \xi_1\|y_n - y^*\|_2)\|v_1 - x_{n+1}\|_1 \\
&\qquad + \delta_1\|F_1(x^*, y^*) - \omega_1^*\|_1\|x_{n+1} - x^*\|_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.26}$$

From condition (5) and Remark 1.1 (2), we have

$$\begin{aligned}
|b_1(x_n, x_{n+1}) - b_1(x^*, x^*)| &\leq |b_1(x_n, x_{n+1}) - b_1(x_n, x^*)| + |b_1(x_n, x^*) - b_1(x^*, x^*)| \\
&\leq \gamma_1\|x_n\|_1\|x_{n+1} - x^*\|_1 + |b_1(x_n - x^*, x^*)| \\
&\leq \gamma_1\|x_n\|_1\|x_{n+1} - x^*\|_1 + \gamma_1\|x_n - x^*\|_1\|x^*\|_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.27}$$

Hence, as $n \to \infty$ in (2.13), we obtain

$$\langle F_1(x^*, y^*) - \omega_1^*, \eta_1(v_1, x^*)\rangle_1 + b_1(x^*, v_1) - b_1(x^*, x^*) \geq 0 \quad \forall v_1 \in B_1. \tag{3.28}$$

It is similar as above, we can obtain

$$\langle F_2(x^*, y^*) - \omega_1^*, \eta_2(v_2, y^*)\rangle_2 + b_2(y^*, v_2) - b_2(y^*, y^*) \geq 0 \quad \forall v_2 \in B_2. \tag{3.29}$$

Therefore, (x^*, y^*) is a solution of SGNMLVIP (1.1) and (1.2). □

Example 3.2. Let $B_1 = B_2 = L^2[0,1] = \{x : [0,1] \to \mathbb{R} \mid x$ be Lebesgue measurable and $\int_0^1 x^2(t)dt < +\infty\}$, then the dual space $B_1^* = B_2^* = L^2[0,1]$. For each $x \in B_i$, $(x, y) \in B_1 \times B_2$, let the norm $\|x\| = (\int_0^1 x^2(t)dt)^{1/2}$, and the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$.

For each $i \in I$, let the mappings $F_i : B_1 \times B_2 \to B_i^*$, $\eta_i : B_i \times B_i \to B_i$, $b_i : B_i \times B_i \to \mathbb{R}$, $g_i : B_i \to B_i^*$, be defined as for any $(x, y) \in B_1 \times B_2$, $(x', y') \in B_i \times B_i$, $z \in B_i$,

$$\begin{gathered}
F_1(x, y) = 19x + \arctan x - y, \qquad F_2(x, y) = 19y + \arctan y - x; \\
\eta_i(x', y') = \frac{9}{10}(x' - y'); \\
b_i(x', y') = \frac{1}{5}\langle x', y'\rangle, \qquad g_i(z) = \frac{1}{20}(19z + \arctan z),
\end{gathered} \tag{3.30}$$

respectively, then we have

(1) F_1 is Lipschitz continuous with constant $(\lambda_1, \xi_1) = (20, 1)$ and η_1-strongly monotone in the first argument with constant $\varepsilon_1 = 18$;

(2) F_2 is Lipschitz continuous with constant $(\lambda_2, \xi_2) = (1, 20)$ and η_2-strongly monotone in the second argument with constant $\varepsilon_2 = 18$;

(3) for each $i \in I$, η_i is Lipschitz continuous with constant $\delta_i = 9/10$;

(4) for each $i \in I$, b_i satisfy properties (i)–(iv) with constant $\gamma_i = 1/5$;

(5) for each $i \in I$, g_i is Lipschitz continuous with constant $\mu_i = 1$ and η_i-strongly monotone with constant $\sigma_i = 9/10$.

After simple calculations, conditions (3.1) and (3.2) imply that $\rho \in (0.01, 0.047)$.

Remark 3.3. Example 3.2 shows that the constant ρ which satisfies the conditions (3.1) and (3.2) can be obtained.

Acknowledgment

This work is partially supported by the National Natural Science Foundation of China Grant (10771050).

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