Research Article

Homomorphisms of Approximation Spaces

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The notion of the homomorphism of approximation spaces is introduced. Some properties of homomorphism are investigated, and some characterizations of homomorphism are given. Furthermore, the notion of approximation subspace of approximation spaces is introduced. The relations between approximation subspaces and homomorphisms are studied.

1. Introduction

The theory of rough sets [1], proposed by Pawlak, is an effective tool to conceptualize and analyze various types of data. The theory and applications of rough sets have impressively developed. There are many applications of rough set theory especially in data analysis, artificial intelligence, and cognitive sciences [2–4]. Some basic aspects of the research of rough sets and several applications have been presented by Pawlak and Skowron [5–7].

In theory, in recent years scholars have put forward many extended rough set models combining with other soft computing theories or relaxing the relation on the universe or broadening the boundary, such as statistical rough set [8], fuzzy rough set [9–11], probabilistic rough set [12], variable precision rough set [13, 14], Bayesian rough set [15] and grey rough set [16]. It is worth noting that proposed in the early 90s, the decision-theoretic rough set model (DTRS) aims to loosen restrictions of conventional rough approximations [17–20]. It is one of the most important probabilistic rough set models. DTRS has gained research attention in recent years [21–27].

In a word, researchers have proposed many models based on rough set theory, but there are a few researches focusing on comparison between approximation spaces corresponding to rough set models. In fact, through the comparison between approximation spaces, we can find the difference between them and give the classification of approximation

spaces, which can help a decision maker to choose a suitable rough set model for data analysis. The comparison between approximation spaces, in mathematics, can be explained as a mapping between two approximation spaces. A homomorphism may be viewed as a special mapping between approximation spaces, which preserves or transfers properties from one approximation space to another. In this paper, we define the concept of homomorphism between two approximation spaces based on binary relations and study some properties of homomorphism. In addition, the notion of approximation subspace is introduced, and the relationship between homomorphisms and approximation spaces is investigated.

The paper is organized as follows. The next section reviews some basic notions of rough sets based on binary relations and some results to be used in the following sections. In Section 3, the concept of homomorphism of approximation spaces is defined, and its main properties are examined. In Section 4, we define the concept of approximation subspace and investigate union, intersection, and complement of approximation spaces. In addition, Theorem 4.14 shows that we can induce an isomorphism of approximation spaces by means of a homomorphism.

2. Review of Relation Models and Their Properties

Let U be a finite and nonempty universe. A binary relation R over a universe U is a subset of the Cartesian product $U \times U$. For two elements $x, y \in U$, if xRy, we say that y is R related to x, x is a predecessor of y, and y is a successor of x. Given a binary relation, the successor neighborhood of x is defined by $xR = \{y \mid xRy\}$ and the predecessor neighborhood of y by $Ry = \{x \mid xRy\}$.

Definition 2.1 (see [28]). Let U be a finite and nonempty universe and $R \subseteq U \times U$ a binary relation on the universe. The pair (U, R) is called an approximation space. For any subset X of the universe U, a pair of lower and upper approximations, $\underline{R}X$ and $\overline{R}X$, are defined as follows:

$$\underline{R}X = \{x \in U \mid xR \subseteq X\}, \qquad \overline{R}X = \{x \in U \mid xR \cap X \neq \emptyset\}. \tag{2.1}$$

Definition 2.2 (see [29, 30]). Let U be a finite and nonempty universe and R a binary relation on U. The relation R is referred to as serial if, for all $x \in U$, there exists a $y \in U$ such that xRy. R is referred to as inverse serial if, for all $x \in U$, there exists a $y \in U$ such that yRx. R is referred to as reflexive if, for all $x \in U$ the relationship xRx holds. R is referred to as symmetric if, for all $x, y \in U$, xRy implies yRx. R is referred to as transitive if, for all $x, y, z \in U$, xRy and yRz imply xRz. R is referred to as Euclidean if for all $x, y, z \in U$, xRy and xRz imply yRz. Since the approximation operators are defined through the successor neighborhood, it is more convenient to express equivalently the conditions on a binary relation as follows:

Serial: for all $x \in U$, $xR \neq \emptyset$,

Inverse serial: for all $x \in U$, there exists a $y \in U$ such that $x \in yR$,

Reflexive: for all $x \in U$, $x \in xR$,

Symmetric: for all $x, y \in U$, if $x \in yR$, then $y \in xR$,

Transitive: for all $x, y \in U$, if $y \in xR$, then $yR \subseteq xR$,

Euclidean: for all $x, y \in U$, if $y \in xR$, then $xR \subseteq yR$.

Theorem 2.3 (see [28]). *Let U be a finite and nonempty universe and $R \subseteq U \times U$ a binary relation on the universe. The approximation operators satisfy the following properties: for subsets $X, Y \subseteq U$,*

$$
\begin{array}{llll}
(L0) & \underline{R}X = \sim \overline{R} \sim X, & (U0) & \overline{R}X = \sim \underline{R} \sim X, \\
(L1) & \underline{R}U = U, & (U1) & \overline{R}\emptyset = \emptyset, \\
(L2) & \underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y, & (U2) & \overline{R}(X \cup Y) = \overline{R}X \cup \overline{R}Y, \\
(L3) & \underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}Y, & (U3) & \overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y, \\
(L4) & X \subseteq Y \Longrightarrow \underline{R}X \subseteq \underline{R}Y, & (U4) & X \subseteq Y \Longrightarrow \overline{R}X \subseteq \overline{R}Y.
\end{array}
\tag{2.2}
$$

With respect to certain special types, say, serial, inverse serial, reflexive, symmetric, transitive, and Euclidean binary relations on the universe of discourse U, the approximation operators have additional properties [28, 29, 31].

Theorem 2.4. *Let U be a finite and nonempty universe and $R \subseteq U \times U$ a binary relation on the universe, then the following assertions hold.*

(1) *R is serial $\Longleftrightarrow$ (U0) $\overline{R}U = U$.*

(2) *R is inverse serial $\Longleftrightarrow$ $\overline{R}\{x\} \neq \emptyset$, for all $x \in U$.*

(3) *R is reflexive $\Longleftrightarrow$ (U6) $X \subseteq \overline{R}X$, for all $X \in U$.*

(4) *R is symmetric $\Longleftrightarrow$ (U7) $X \subseteq \underline{R}\,\overline{R}X$, for all $X \in U$.*

(5) *R is transitive $\Longleftrightarrow$ (L8) $\underline{R}X \subseteq \underline{R}\,\underline{R}X$, for all $X \in U$.*

(6) *R is Euclidean $\Longleftrightarrow$ (U9) $\overline{R}X \subseteq \underline{R}\,\overline{R}X$, for all $X \in U$.*

3. The Properties of Homomorphisms and Isomorphisms

In this section, we introduce the notion of the homomorphism of approximation spaces and study on the properties of the homomorphisms.

We can study the relations between the approximation operators of one approximation space and the approximation operators of the other approximation space by means of mappings. For this purpose, in the following we present the concept of the homomorphism of approximation spaces.

Definition 3.1. Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping from U_1 to U_2, R_1 a binary relation on U_1, and R_2 a binary relation on U_2. f is called a lower homomorphism from the approximation space (U_1, R_1) to the approximation space (U_2, R_2) if, for all $X \subseteq U_1$, $f(\underline{R_1}X) \subseteq \underline{R_2}f(X)$. f is called an upper homomorphism from (U_1, R_1) to (U_2, R_2) if, for all $X \subseteq U_1$, $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$. f is called a homomorphism from (U_1, R_1) to (U_2, R_2) if, for all $X \subseteq U_1$, $f(\underline{R_1}X) \subseteq \underline{R_2}f(X)$ and $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$. An injective homomorphism from (U_1, R_1) to (U_2, R_2) is called a monomorphism, a surjective homomorphism an epimorphism, and a bijective homomorphism an isomorphism.

Remark 3.2. In the above definition, it is easy to show that f is a homomorphism from (U_1, R_1) to (U_2, R_2) if and only if f is a lower homomorphism from (U_1, R_1) to (U_2, R_2) and f is an upper homomorphism from (U_1, R_1) to (U_2, R_2).

For better understanding of the definition we illustrate it by the following examples.

Example 3.3. Let (U_1, R_1) and (U_2, R_2) be two approximation spaces, where $U_1 = \{x_1, x_2\}$, $U_2 = \{y_1, y_2, y_3, y_4\}$, $R_1 = \{(x_1, x_1), (x_2, x_2), (x_1, x_2), (x_2, x_1)\}$, and $R_2 = \{(y_1, y_1), (y_2, y_3), (y_4, y_3)\}$. f is a mapping from U_1 to U_2 and $f(x_1) = y_1, f(x_2) = y_1$. It is easy to check that for all $X \subseteq U_1$, if $X \neq U_1$, then $\underline{R_1}X = \emptyset$. Hence, for all $X \subset U_1$, $f(\underline{R_1}X) = \emptyset \subseteq \underline{R_2}f(X)$. If $X = U_1$, then $\underline{R_1}X = U_1$. Therefore, $f(\underline{R_1}X) = \{y_1\} \subseteq \{y_1\} = \underline{R_2}f(X)$ when $X = U_1$. We have shown that for all $X \subseteq U_1$, $f(\underline{R_1}X) \subseteq \underline{R_2}f(X)$. It follows from Definition 3.1 that f is a lower homomorphism from (U_1, R_1) to (U_2, R_2).

Example 3.4. Let (U_1, R_1) and (U_2, R_2) be two approximation spaces, where $R_2 = U_2 \times U_2$. f is a mapping from U_1 to U_2. It is easy to check that for all $X \subseteq U_1$, if $X \neq \emptyset$ then $\overline{R_2}f(X) = U_2$. Hence for all $X \subseteq U$ and $X \neq \emptyset$, $f(\overline{R_1}X) \subseteq U_2 = \overline{R_2}f(X)$. If $X = \emptyset$, then by the part (U_1) of Theorem 2.3 we have that $\overline{R_1}X = \emptyset$. Therefore, if $X = \emptyset$, then $f(\overline{R_1}X) = \emptyset \subseteq \overline{R_2}f(X)$. We have shown that for all $X \subseteq U_1$, $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$. It follows from Definition 3.1 that f is an upper homomorphism from (U_1, R_1) to (U_2, R_2).

Notation 1. Let U denote a finite and nonempty set and $R \subseteq U \times U$ an equivalence relation on U. Then let U/R denote the quotient set consisting of equivalence classes of R.

Example 3.5. Let (U_1, R_1) and (U_2, R_2) be two approximation spaces, where $U_1 = \{x_1, x_2, x_3\}$, $U_2 = \{y_1, y_2\}$, $U_1/R_1 = \{\{x_1, x_2\}, \{x_3\}\}$, and $U_2/R_2 = \{\{y_1\}, \{y_2\}\}$. f is a mapping from U_1 to U_2 and $f(x_1) = f(x_2) = y_1$, $f(x_3) = y_2$. We first show that f is a lower homomorphism from (U_1, R_1) to (U_2, R_2). By

$$\underline{R_1}\emptyset = \underline{R_1}\{x_1\} = \underline{R_1}\{x_2\} = \emptyset, \tag{3.1}$$

we can get that $f(\underline{R_1}\emptyset) \subseteq \underline{R_2}f(\emptyset)$, $f(\underline{R_1}\{x_1\}) \subseteq \underline{R_2}f(\{x_1\})$, and $f(\underline{R_1}\{x_2\}) \subseteq \underline{R_2}f(\{x_2\})$. By

$$\underline{R_1}\{x_1, x_3\} = \underline{R_1}\{x_2, x_3\} = \underline{R_1}\{x_3\} = \{x_3\}, \qquad \underline{R_2}\{y_1, y_2\} = \{y_1, y_2\}, \qquad \underline{R_2}\{y_2\} = \{y_2\}, \tag{3.2}$$

we can get that

$$f\left(\underline{R_1}\{x_1, x_3\}\right) = \{y_2\} \subseteq \{y_1, y_2\} = \underline{R_2}f(\{x_1, x_3\}),$$
$$f\left(\underline{R_1}\{x_2, x_3\}\right) = \{y_2\} \subseteq \{y_1, y_2\} = \underline{R_2}f(\{x_2, x_3\}), \tag{3.3}$$
$$f\left(\underline{R_1}\{x_3\}\right) = \{y_2\} \subseteq \{y_2\} = \underline{R_2}f(\{x_3\}).$$

In addition, by

$$\underline{R_1}\{x_1, x_2\} = \{x_1, x_2\}, \qquad \underline{R_1}U_1 = U_1, \qquad \underline{R_2}\{y_1\} = \{y_1\}, \qquad \underline{R_2}U_2 = U_2, \tag{3.4}$$

we get that $f(\underline{R_1}\{x_1, x_2\}) = \{y_1\} \subseteq \{y_1\} = \underline{R_2}f(\{x_1, x_2\})$, $f(\underline{R_1}U_1) = U_2 \subseteq U_2 = \underline{R_2}f(U_1)$. We have shown that for all $X \subseteq U_1$, $f(\underline{R_1}X) \subseteq \underline{R_2}f(X)$. It follows from Definition 3.1 that f is a lower homomorphism from (U_1, R_1) to (U_2, R_2).

In the following, we will prove that f is also an upper homomorphism from (U_1, R_1) to (U_2, R_2). By $\overline{R_1}\emptyset = \emptyset$, we get that $f(\overline{R_1}\emptyset) \subseteq \overline{R_2}f(\emptyset)$. By $\overline{R_1}\{x_3\} = \{x_3\}$ and $\overline{R_2}\{y_2\} = \{y_2\}$, we can get that $f(\overline{R_1}\{x_3\}) = \{y_2\} \subseteq \{y_2\} = \overline{R_2}f(\{x_3\})$. By

$$\overline{R_1}\{x_1\} = \overline{R_1}\{x_2\} = \overline{R_1}\{x_1, x_2\} = \{x_1, x_2\}, \qquad \overline{R_2}\{y_1\} = \{y_1\}, \tag{3.5}$$

we can get that

$$f\left(\overline{R_1}\{x_1\}\right) = \{y_1\} \subseteq \{y_1\} = \overline{R_2}f(\{x_1\}),$$
$$f\left(\overline{R_1}\{x_2\}\right) = \{y_1\} \subseteq \{y_1\} = \overline{R_2}f(\{x_2\}), \tag{3.6}$$
$$f\left(\overline{R_1}\{x_1, x_2\}\right) = \{y_1\} \subseteq \{y_1\} = \overline{R_2}f(\{x_1, x_2\}).$$

By

$$\overline{R_1}\{x_1, x_3\} = \overline{R_1}\{x_2, x_3\} = \overline{R_1}U_1 = U_1, \qquad \overline{R_2}U_2 = U_2, \tag{3.7}$$

we can get that

$$f\left(\overline{R_1}\{x_1, x_3\}\right) = U_2 \subseteq U_2 = \overline{R_2}f(\{x_1, x_3\}),$$
$$f\left(\overline{R_1}\{x_2, x_3\}\right) = U_2 \subseteq U_2 = \overline{R_2}f(\{x_2, x_3\}), \tag{3.8}$$
$$f\left(\overline{R_1}U_1\right) = U_2 \subseteq U_2 = \overline{R_2}f(U_1).$$

We have proved that for all $X \subseteq U_1$, $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$. It follows from Definition 3.1 that f is an upper homomorphism from (U_1, R_1) to (U_2, R_2).

In summary, by Remark 3.2, f is a homomorphism from (U_1, R_1) to (U_2, R_2).

Example 3.6. Let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be approximation spaces, where $U_1 = \{x_1, x_2, x_3\}$, $U_2 = \{y_1, y_2, y_3\}$, $U_1/R_1 = \{\{x_1, x_2\}, \{x_3\}\}$, and $U_2/R_2 = \{\{y_1, y_2\}, \{y_3\}\}$. f is a mapping from U_1 to U_2 and $f(x_1) = y_1, f(x_2) = y_2$, and $f(x_3) = y_3$. By Definition 3.1, it is easy to show that f is an isomorphism from (U_1, R_1) to (U_2, R_2).

For better understanding of the upper homomorphism we give the following theorem. In fact, we present a criterion for judging an upper homomorphism.

Theorem 3.7. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping, R_1 a binary relation on U_1 and R_2 a binary relation on U_2, then f is an upper homomorphism from (U_1, R_1) to (U_2, R_2) if and only if, for all $x \in U_1$, $f(\overline{R_1}\{x\}) \subseteq \overline{R_2}f(\{x\})$.*

Proof. The necessity follows directly from Definition 3.1.

Conversely, for all $X \subseteq U$, by the part (U_2) of Theorem 2.3, we have that

$$f\left(\overline{R_1}X\right) = f\left(\bigcup_{x \in X} \overline{R_1}\{x\}\right) = \bigcup_{x \in X} f\left(\overline{R_1}\{x\}\right),$$
$$\overline{R_2}f(X) = \overline{R_2}f\left(\bigcup_{x \in X}\{x\}\right) = \overline{R_2}\left(\bigcup_{x \in X} f(\{x\})\right) = \bigcup_{x \in X} \overline{R_2}f(\{x\}). \tag{3.9}$$

Applying the condition for all $x \in U_1$, $f(\overline{R_1}\{x\}) \subseteq \overline{R_2}f(\{x\})$, we conclude that

$$\bigcup_{x \in X} f\left(\overline{R_1}\{x\}\right) \subseteq \bigcup_{x \in X} \overline{R_2}f(\{x\}). \tag{3.10}$$

Hence,

$$f\left(\overline{R_1}X\right) \subseteq \overline{R_2}f(X). \tag{3.11}$$

We have proved that for all $X \subseteq U$, $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$. It follows from Definition 3.1 that f is an upper homomorphism from (U_1, R_1) to (U_2, R_2). □

In the following, when $f : U_1 \rightarrow U_2$ is a bijective from U_1 to U_2, some characterizations of homomorphisms are given.

Theorem 3.8. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \rightarrow U_2$ a bijective from U_1 to U_2, R_1 a binary relation on U_1, and R_2 a binary relation on U_2, then the following assertions hold:*

(1) *f is an upper homomorphism from (U_1, R_1) to (U_2, R_2) if and only if for all $X \subseteq U_1$, $f(\underline{R_1}X) \supseteq \underline{R_2}f(X)$,*

(2) *f is a lower homomorphism from (U_1, R_1) to (U_2, R_2) if and only if for all $X \subseteq U_1$, $f(\overline{R_1}X) \supseteq \overline{R_2}f(X)$,*

(3) *f is a homomorphism from (U_1, R_1) to (U_2, R_2) if and only if for all $X \subseteq U_1$, $f(\underline{R_1}X) \supseteq \underline{R_2}f(X)$ and $f(\overline{R_1}X) \supseteq \overline{R_2}f(X)$,*

Proof. (1) Since f is a bijective, it follows that for all $X \subseteq U_1$, $f(\sim X) = \sim f(X)$. Hence, f is an upper homomorphism from (U_1, R_1) to (U_2, R_2) $\Leftrightarrow$ for all $X \subseteq U_1$, $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$

$\Leftrightarrow$ for all $X \subseteq U_1$, $f(\overline{R_1}(\sim X)) \subseteq \overline{R_2}f(\sim X)$,

$\Leftrightarrow$ for all $X \subseteq U_1$, $f(\sim \underline{R_1}(X)) \subseteq \overline{R_2}(\sim f(X))$,

$\Leftrightarrow$ for all $X \subseteq U_1$, $\sim f(\underline{R_1}(X)) \subseteq \sim \underline{R_2}(f(X))$,

$\Leftrightarrow$ for all $X \subseteq U_1$, $f(\underline{R_1}(X)) \supseteq \underline{R_2}(f(X))$.

This finishes the proof.

(2) It is similar to the proof of (1).

(3) By (1), (2), and Remark 3.2, (3) holds. □

As natural consequences of the above theorem we can obtain the following conclusions.

Theorem 3.9. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \rightarrow U_2$ a bijective, R_1 a binary relation on U_1, and R_2 a binary relation on U_2, then the following assertions are equivalent:*

(1) *f is an isomorphism from (U_1, R_1) to (U_2, R_2),*

(2) *for all $X \subseteq U_1$, $f(\underline{R_1}X) = \underline{R_2}f(X)$,*

(3) *for all $X \subseteq U_1$, $f(\overline{R_1}X) = \overline{R_2}f(X)$,*

Proof. The proof follows directly from Definition 3.1, Theorems 2.4, and 3.8. □

The following example shows that if the above mapping f is surjective and satisfies the condition for all $X \subseteq U_1$, $f(\underline{R_1}X) = \underline{R_2}f(X)$, then f is not necessarily an isomorphism.

Example 3.10. Let (U_1, R_1) and (U_2, R_2) be two approximation spaces, where $U_1 = \{x_1, x_2\}$, $U_2 = \{y_1\}$, $U_1/R_1 = \{\{x_1\}, \{x_2\}\}$, and $U_2/R_2 = \{\{y_1\}\}$. f is a mapping from U_1 to U_2 and $f(x_1) = f(x_2) = y_1$, then $\underline{R_1}\emptyset = \emptyset$, $\underline{R_1}\{x_1\} = \{x_1\}$, $\underline{R_1}\{x_2\} = \{x_2\}$, $\underline{R_1}U_1 = U_1$ and $\underline{R_2}\emptyset = \emptyset$, $\underline{R_2}U_2 = U_2$. Hence,

$$
\begin{aligned}
f\left(\underline{R_1}\emptyset\right) &= \emptyset = \underline{R_2}f(\emptyset), \\
f\left(\underline{R_1}\{x_1\}\right) &= U_2 = \underline{R_2}f(\{x_1\}), \\
f\left(\underline{R_1}\{x_2\}\right) &= U_2 = \underline{R_2}f(\{x_2\}), \\
f\left(\underline{R_1}U_1\right) &= U_2 = \underline{R_2}f(U_1).
\end{aligned}
\tag{3.12}
$$

We have shown that for all $X \subseteq U_1$, $f(\underline{R_1}X) = \underline{R_2}f(X)$.

In the following, we shall prove that for all $X \subseteq U_1$, $f(\overline{R_1}X) = \overline{R_2}f(X)$. It is easy to check that $\overline{R}\emptyset = \emptyset$, $\overline{R}\{x_1\} = \{x_1\}$, $\overline{R}\{x_2\} = \{x_2\}$, $\overline{R}U_1 = U_1$ and $\overline{R_2}\emptyset = \emptyset$, $\overline{R_2}U_2 = U_2$. Hence,

$$
\begin{aligned}
f\left(\overline{R_1}\emptyset\right) &= \emptyset = \overline{R_2}f(\emptyset), \\
f\left(\overline{R_1}\{x_1\}\right) &= U_2 = \overline{R_2}f(\{x_1\}), \\
f\left(\overline{R_1}\{x_2\}\right) &= U_2 = \overline{R_2}f(\{x_2\}), \\
f\left(\overline{R_1}U_1\right) &= U_2 = \overline{R_2}f(U_1).
\end{aligned}
\tag{3.13}
$$

We have shown that for all $X \subseteq U_1$, $f(\overline{R_1}X) = \overline{R_2}f(X)$.

Thus, by Definition 3.1, f is a homomorphism from (U_1, R_1) to (U_2, R_2), but f is not an isomorphism.

Corollary 3.11. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \rightarrow U_2$ a bijective, R_1 a binary relation on U_1, and R_2 a binary relation on U_2, then f is an isomorphism from (U_1, R_1) to (U_2, R_2) if and only if f^{-1} is an isomorphism from (U_2, R_2) to (U_1, R_1).*

Proof. Since f is a bijective, it follows that f^{-1} is a bijective from U_2 to U_1. For all $Y \subseteq U_2$, by Theorem 3.9, we have that $f(\underline{R_1}f^{-1}(Y)) = \underline{R_2}f(f^{-1}(Y)) = \underline{R_2}Y$. Hence, $\underline{R_1}f^{-1}(Y) = f^{-1}(\underline{R_2}Y)$. We have proved that for all $Y \subseteq U_2$, $f^{-1}(\underline{R_2}Y) = \underline{R_1}f^{-1}(Y)$. It follows from Theorem 3.9 that f^{-1} is an isomorphism from (U_2, R_2) to (U_1, R_1). The proof of the sufficiency is similar to the proof of the necessity. □

Notation 2. The symbolism $(U_1, R_1) \cong (U_2, R_2)$ signifies that there is at least one isomorphism from approximation space (U_1, R_1) to approximation space (U_2, R_2). By Corollary 3.11, we conclude that $(U_1, R_1) \cong (U_2, R_2) \Leftrightarrow (U_2, R_2) \cong (U_1, R_1)$.

In the following, some properties of homomorphisms are given. These properties reveal the difference and relationship between two approximation spaces.

Theorem 3.12. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a surjective from U_1 to U_2, R_1 a binary relation on U_1, and R_2 a binary relation on U_2. If f is an upper homomorphism from (U_1, R_1) to (U_2, R_2), then the following assertions hold:*

(1) *If R_1 is serial, then R_2 is serial,*

(2) *If R_1 is inverse serial, then R_2 is inverse serial,*

(3) *If R_1 is reflexive, then R_2 is reflexive.*

Proof. (1) Since R_1 is serial, it follows from part (1) of Theorem 2.4 that $\overline{R_1}U_1 = U_1$. By the condition, f is an upper homomorphism from (U_1, R_1) to (U_2, R_2), hence we conclude that $f(U_1) = f(\overline{R_1}U_1) \subseteq \overline{R_2}f(U_1)$. In addition, since f is a surjective, it follows that $f(U_1) = U_2$. Thus $U_2 \subseteq \overline{R_2}f(U_1) = \overline{R_2}U_2$. On the other hand, clearly, $U_2 \supseteq \overline{R_2}U_2$. Hence, $\overline{R_2}U_2 = U_2$ and so by part (1) of Theorem 2.4, R_2 is serial.

(2) By part (2) of Theorem 2.4, we need to prove only that for all $y \in U_2$, $\overline{R_2}\{y\} \neq \emptyset$. Let $y \in U_2$. Since f is a surjective, it follows that there exists $x \in U_1$ such that $f(x) = y$. By the condition, R_1 is inverse serial, therefore by part (2) of Theorem 2.4, we have that $\overline{R_1}\{x\} \neq \emptyset$ and so $f(\overline{R_1}\{x\}) \neq \emptyset$. Since f is an upper homomorphism from (U_1, R_1) to (U_2, R_2), it follows from Definition 3.1 that $\emptyset \neq f(\overline{R_1}\{x\}) \subseteq \overline{R_2}f(\{x\}) = \overline{R_2}\{f(x)\} = \overline{R_2}\{y\}$. Thus, $\overline{R_2}\{y\} \neq \emptyset$. This finishes the proof of (2).

(3) By the part (3) of Theorem 2.4, we need to prove only that for all $Y \subseteq U_2$, $Y \subseteq \overline{R_2}Y$. Let $Y \subseteq U_2$. Since f is a surjective, it follows that there exists $X \subseteq U_1$ such that $f(X) = Y$. By the condition, R_1 is reflexive, therefore by the part (3) of Theorem 2.4, we have that $X \subseteq \overline{R_1}X$ and so $Y = f(X) \subseteq f(\overline{R_1}X)$. Since f is an upper homomorphism from (U_1, R_1) to (U_2, R_2), it follows from Definition 3.1 that $Y \subseteq f(\overline{R_1}X) \subseteq \overline{R_2}f(X) = \overline{R_2}Y$. Thus, for all $Y \subseteq U_2$, $Y \subseteq \overline{R_2}Y$. This finishes the proof of (3). □

Theorem 3.13. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping from U_1 to U_2, R_1 a binary relation on U_1 and R_2 a binary relation on U_2. f is an epimorphism from (U_1, R_1) to (U_2, R_2). If R_1 is symmetric, then R_2 is symmetric.*

Proof. By the part (4) of Theorem 2.4, we need to prove only that for all $Y \subseteq U_2$, $Y \subseteq \underline{R_2}\overline{R_2}Y$. Let $Y \subseteq U_2$. Since f is a surjective, it follows that there exists $X \subseteq U_1$ such that $f(X) = Y$. By the condition, R_1 is symmetric, therefore by part (4) of Theorem 2.4, we have that $X \subseteq \underline{R_1}\,\overline{R_1}X$ and so $Y = f(X) \subseteq f(\underline{R_1}\,\overline{R_1}X)$. Since f is a homomorphism from (U_1, R_1) to (U_2, R_2), it follows from Definition 3.1 that $f(\underline{R_1}\overline{R_1}X) \subseteq \underline{R_2}\,f(\overline{R_1}X)$ and $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$. Thus, by the part (4) of Theorem 2.3, we have that $Y \subseteq f(\underline{R_1}\,\overline{R_1}X) \subseteq \underline{R_2}f(\overline{R_1}X) \subseteq \underline{R_2}\,\overline{R_2}f(X) = \underline{R_2}\,\overline{R_2}Y$. Hence for all $Y \subseteq U_2$, $Y \subseteq \underline{R_2}\,\overline{R_2}Y$. This finishes the proof of theorem. □

The following theorem gives main properties of isomorphisms on approximation spaces.

Theorem 3.14. *Let U_1 and U_2 be finite and nonempty universes, R_1 a binary relation on U_1 and R_2 a binary relation on U_2. If $(U_1, R_1) \cong (U_2, R_2)$, then the following assertions hold:*

(1) *R_1 is serial if and only if R_2 is serial,*

(2) *R_1 is inverse serial if and only if R_2 is inverse serial,*

(3) *R_1 is reflexive if and only if R_2 is reflexive,*

(4) R_1 is transitive if and only if R_2 is transitive,

(5) R_1 is symmetric if and only if R_2 is symmetric,

(6) R_1 is Euclidean if and only if R_2 is Euclidean.

Proof. By Theorem 3.12 and Notation 2, (1), (2) and (3) hold.

(4) We first prove the necessity. By part (5) of Theorem 2.4, we need only prove that for all $Y \subseteq U_2$, $\underline{R_2}Y \subseteq \underline{R_2}\ \underline{R_2}Y$. By the condition $(U_1, R_1) \cong (U_2, R_2)$, we may assume that f is an isomorphism from (U_1, R_1) to (U_2, R_2). Let $Y \subseteq U_2$. Since f is a surjective, it follows that there exists $X \subseteq U_1$ such that $f(X) = Y$. By the condition, R_1 is transitive, therefore by part (5) of Theorem 2.4, we have that $\underline{R_1}X \subseteq \underline{R_1}\ \underline{R_1}X$ and so $f(\underline{R_1}X) \subseteq f(\underline{R_1}\underline{R_1}X)$. Since f an isomorphism from (U_1, R_1) to (U_2, R_2), it follows from Theorem 3.9 that $f(\underline{R_1}X) = \underline{R_2}f(X)$ and $f(\underline{R_1}\ \underline{R_1}X) = \underline{R_2}f(\underline{R_1}X)$. Thus, $\underline{R_2}Y = \underline{R_2}f(X) = f(\underline{R_1}X) \subseteq f(\underline{R_1}\ \underline{R_1}X) = \underline{R_2}f(\underline{R_1}X) = \underline{R_2}\ \underline{R_2}f(X) = \underline{R_2}\ \underline{R_2}Y$. Hence for all $Y \subseteq U_2$, $\underline{R_2}Y \subseteq \underline{R_2}\underline{R_2}Y$. So R_2 is transitive. By Notation 2, the proof of the sufficiency is similar to the proof of the necessity.

(5) By theorem 3.13 and Notation 2, (5) holds.

(6) We first prove the necessity. By part (6) of Theorem 2.4, we need only prove that for all $Y \subseteq U_2$, $\overline{R_2}Y \subseteq \underline{R_2}\ \overline{R_2}Y$. By the condition $(U_1, R_1) \cong (U_2, R_2)$, we may assume that f is an isomorphism from (U_1, R_1) to (U_2, R_2). Let $Y \subseteq U_2$. Since f is a surjective, it follows that there exists $X \subseteq U_1$ such that $f(X) = Y$. By the condition, R_1 is Euclidean, therefore by the part (6) of Theorem 2.4, we have that $\overline{R_1}X \subseteq \underline{R_1}\overline{R_1}X$ and so $f(\overline{R_1}X) \subseteq f(\underline{R_1}\ \overline{R_1}X)$. Since f an isomorphism from (U_1, R_1) to (U_2, R_2), it follows from Theorem 3.9 that $f(\overline{R_1}X) = \overline{R_2}f(X)$ and $f(\underline{R_1}\ \overline{R_1}X) = \underline{R_2}f(\overline{R_1}X)$. Thus, $\overline{R_2}Y = \overline{R_2}f(X) = f(\overline{R_1}X) \subseteq f(\underline{R_1}\ \overline{R_1}X) = \underline{R_2}f(\overline{R_1}X) = \underline{R_2}\ \overline{R_2}f(X) = \underline{R_2}\ \overline{R_2}Y$. Hence, for all $Y \subseteq U_2$, $\overline{R_2}Y \subseteq \underline{R_2}\overline{R_2}Y$. So R_2 is Euclidean. By Notation 2, the proof of the sufficiency is similar to the proof of the necessity. $\square$

4. Approximation Subspaces and Homomorphisms

In mathematics, subspaces are similar to original space and independent of the original space, such as, linear subspaces of linear space subspaces of topological space and. According to ideas, the notion of approximation subspace of approximation space is introduced in this section. For this purpose, we first introduce the following notation.

Notation 3. Let U be a finite and nonempty universe, $R \subseteq U \times U$ a binary relation on U, and $S \subseteq U$. Then let $R|_S$ denote the set $R \cap (S \times S)$, that is, $R|_S = R \cap (S \times S)$. Clearly, $R|_S$ is a binary relation on S.

Definition 4.1. Let U be a finite and nonempty universe, R a binary relation on U and $S \subseteq U$. Then the pair $(S, R|_S)$ is called an approximation subspace of the approximation space (U, R) if for all $X \subseteq S$, $\underline{R|_S}X = \underline{R}X$ and $\overline{R|_S}X = \overline{R}X$.

Remark 4.2. Let U be a finite and nonempty universe, $R \subseteq U \times U$ a binary relation on U, and $S \subseteq U$. i is a mapping from S to U and for all $x \in S$, $i(x) = x$. If $(S, R|_S)$ is an approximation subspace of (U, R), then by Definitions 3.1 and 4.1, it is easy to check that i is a homomorphism from $(S, R|_S)$ to (U, R).

Lemma 4.3. *Let U be a finite and nonempty universe and $R \subseteq U \times U$ a binary relation on U. If R is an equivalence relation on U, then $R|_S$ is an equivalence relation on S.*

Proof. (1) Since R is an equivalence relation on U, it follows that for all $x \in S$, $(x, x) \in R$. Hence, $(x, x) \in R \cap (S \times S) = R|_S$ and so $R|_S$ is reflexive.

(2) Let $(x, y) \in R|_S$, then $(x, y) \in S \times S$ and $(x, y) \in R$. Clearly, $(y, x) \in S \times S$. Since R is an equivalence relation on U, it follows that $(y, x) \in R$. Hence $(y, x) \in R \cap (S \times S) = R|_S$. Therefore $R|_S$ is symmetric.

(3) Let $(x, y), (y, z) \in R|_S$, then $(x, y), (y, z) \in S \times S$ and $(x, y), (y, z) \in R$. Clearly $(x, z) \in S \times S$. Since R is an equivalence relation on U, it follows that $(x, z) \in R$. Hence, $(x, z) \in R \cap (S \times S) = R|_S$. Therefore, $R|_S$ is transitive.

By (1), (2), and (3), we conclude that $R|_S$ is an equivalence relation on S.   □

Note 1. Let $R \subseteq U \times U$ be an equivalence relation on U, that is, R is reflexive, symmetric, and transitive. The pair (U, R) is called an approximation space. The equivalence relation R partitions the universe U into disjoint subsets called equivalence classes. Elements in the same equivalence class are said to be indistinguishable. Equivalent classes of R are called elementary sets. A union of elementary sets is called an R-definable (composed) set [32]. The empty set is considered to be a R-definable set [33]. Let $X \subseteq U$, then it is easy to check that X is an R-definable (composed) set if and only if $\underline{R}X = X = \overline{R}X$. Clearly, if $X, Y \subseteq U$ are R-definable sets, then $X \cup Y$, $X \cap Y$ and$\sim X$ are R-definable sets.

Lemma 4.4. *Let U be a finite and nonempty universe, $R \subseteq U \times U$ an equivalence relation on U and $S \subseteq U$, then S is an R-definable set if and only if, for all $x \in S$, $[x]_{R|_S} = [x]_R$.*

Proof. We first prove the necessity. Let $x \in S$. Since S is an R-definable set, it follows that $\underline{R}S = S = \overline{R}S$ and so $[x]_R \subseteq S$. Hence, for all $y \in [x]_R$, we get that $(x, y) \in S \times S$ and $(x, y) \in R$, which implies $(x, y) \in R \cap (S \times S) = R|_S$ and so $y \in [x]_{R|_S}$. Thus $[x]_R \subseteq [x]|_{R|_S}$. On the other hand, by $R|_S = R \cap (S \times S)$, it is clear that $[x]_R \supseteq [x]_{R|_S}$. So $[x]_{R|_S} = [x]_R$.

Conversely, suppose that S is not an R-definable set, then $\underline{R}S \neq \overline{R}S$. Hence, there exists $y \in U$ such that $[y]_R \nsubseteq S$ and $[y]_R \cap S \neq \emptyset$. Choosing $x \in [y]_R \cap S$, namely, $x \in [y]_R$ and $x \in S$. Since R is an equivalence relation on U, it follows that $[x]_R = [y]_R$. Hence, $[x]_R \nsubseteq S$ and $x \in S$. By Lemma 4.3, we have that $R|_S$ is an equivalence relation on S. Thus, $[x]_{R|_S} \subseteq S$ and so $[x]_{R|_S} \neq [x]_R$. We have proved that supposing that S is not crisp (exact with respect to R), then there exists $x \in S$ such that $[x]_{R|_S} \neq [x]_R$. This is a contradiction with the condition for all $x \in S$, $[x]_{R|_S} = [x]_R$. It follows that S is an R-definable set.   □

When R is an equivalence relation in the approximation space (U, R), some characterizations of approximation subspace are given in the following theorem.

Theorem 4.5. *Let U be a finite and nonempty universe, $R \subseteq U \times U$ an equivalence relation on U and $S \subseteq U$, then the following assertions are equivalent:*

(1) *S is a R-definable set,*

(2) *for all $X \subseteq S$, $\overline{R|_S}X = \overline{R}X$,*

(3) *$(S, R|_S)$ is an approximation subspace of the approximation space (U, R),*

(4) *for all $X \subseteq S$, $\underline{R|_S}X = \underline{R}X$.*

Proof. (1) $\Rightarrow$ (2) Let $X \subseteq S$, then by part (U_4) of Theorem 2.3, we have that $\overline{R}X \subseteq \overline{R}S$. Since R is an equivalence relation on U and S is an R–definable set, it follows that $\overline{R}S = S$. Hence, $\overline{R}X \subseteq \overline{R}S = S$ and so for all $x \in \overline{R}X$, we have that $x \in S$ and $[x]_R \cap X \neq \emptyset$. In addition,

by Lemma 4.4, we have that $[x]_{R|_S} = [x]_R$. Hence, $[x]_{R|_S} \cap X \neq \emptyset$ and so $x \in \overline{R|_S}X$. Thus $\overline{R|_S}X \supseteq \overline{R}X$. On the other hand, for all $x \in \overline{R|_S}X$, clearly $x \in S$ and $[x]_{R|_S} \cap X \neq \emptyset$. By Lemma 4.4, we have that $[x]_{R|_S} = [x]_R$. Thus, $[x]_R \cap X \neq \emptyset$, which implies $x \in \overline{R}X$. Hence $\overline{R|_S}X \subseteq \overline{R}X$. Consequently, $\overline{R|_S}X = \overline{R}X$.

(2) $\Rightarrow$ (1) Suppose that S is not an R-definable set, then by Lemma 4.4, there exists $x \in S$, such that $[x]_{R|_S} \neq [x]_R$. By Lemma 4.3, $R|_S$ is an equivalence relation on S, thus $\overline{R|_S}\{x\} = [x]_{R|_S}$. On the other hand, $\overline{R}\{x\} = [x]_R$. Hence, $\overline{R|_S}\{x\} \neq \overline{R}\{x\}$. We have proved that supposing that S is not a R-definable set, then there exists $X = \{x\} \subseteq S$ such that $\overline{R|_S}X \neq \overline{R}X$. This is a contradiction with the condition for all $X \subseteq S$, $\overline{R|_S}X = \overline{R}X$. It follows that S is an R-definable set.

(2) $\Rightarrow$ (3) By Definition 4.1, we need to prove only that for all $X \subseteq S$, $\underline{R|_S}X = \underline{R}X$. Since (1) $\Leftrightarrow$ (2), it follows that S is an R-definable set. By Lemma 4.4, we can get that for all $x \in S$, $[x]_{R|_S} = [x]_R$. For all $x \in \underline{R|_S}X$, clearly $x \in S$. Hence, $x \in \underline{R|_S}X \Rightarrow [x]_{R|_S} \subseteq X \Rightarrow [x]_R \subseteq X \Rightarrow x \in \underline{R}X$. Thus, $\underline{R|_S}X \subseteq \underline{R}X$. On the other hand, since R is an equivalence relation on U, it follows that $\underline{R}X \subseteq X \subseteq S$. Hence, $x \in \underline{R}X$ implies $x \in S$. Thus, $x \in \underline{R}X \Rightarrow [x]_R \subseteq X \Rightarrow [x]_{R|_S} \subseteq X \Rightarrow x \in \underline{R|_S}X$. Therefore $\underline{R|_S}X \supseteq \underline{R}X$. It follows that $\underline{R|_S}X = \underline{R}X$. This completes the proof.

(3) $\Rightarrow$ (4) By Definition 4.1, the proof is obvious.

(4) $\Rightarrow$ (1) Suppose that S is not an R-definable set, then by Lemma 4.4, there exists $x \in S$ such that $[x]_{R|_S} \neq [x]_R$. By $R|_S = R \cap (S \times S)$, it is easy to check that $[x]_{R|_S} \subseteq [x]_R$. Hence, $[x]_{R|_S} \subset [x]_R$ and so $\underline{R}[x]_{R|_S} = \emptyset$. On the other hand, by Lemma 4.3, we have that $R|_S$ is an equivalent relation on S. Thus, $\underline{R|_S}[x]_{R|_S} = [x]_{R|_S}$. Hence, $\underline{R|_S}[x]_{R|_S} \neq \underline{R}[x]_{R|_S}$. We have proved that supposing that S is not an R-definable set, then there exists $X = [x]_{R|_S} \subseteq S$ such that $\underline{R|_S}X \neq \underline{R}X$. This is a contradiction with the condition for all $X \subseteq S$, $\underline{R|_S}X = \underline{R}X$. This completes the proof. $\qquad\square$

Notation 4. Let (U, R) be an approximation space. Let $\mathcal{SP}(A)$ denote the set of all approximation subspaces of (U, R).

Now we consider the union, intersection, and complement of approximation subspaces.

Definition 4.6. Let (U, R) be an approximation space, where U is a finite and nonempty universe, R is an equivalence relation on U, and $S, H \subseteq U$. If $(S, R|_S), (H, R|_H) \in \mathcal{SP}(A)$, then the union, intersection and complement of approximation subspaces are correspondingly defined as follows:

$$
\begin{aligned}
(S, R|_S) \cup (H, R|_H) &= (S \cup H, R|_S \cup R|_H), \\
(S, R|_S) \cap (H, R|_H) &= (S \cap H, R|_S \cap R|_H), \\
\sim (S, R|_S) &= (\sim S, R|_{\sim S}).
\end{aligned}
\tag{4.1}
$$

Lemma 4.7. *Let (U, R) be an approximation space, where U is a finite and nonempty universe and R is an equivalence relation on U, and let $(S, R|_S), (H, R|_H) \in \mathcal{SP}(A)$, then the following assertions hold:*

(1) $R|_S \cup R|_H = R|_{S \cup H}$,

(2) $R|_S \cap R|_H = R|_{S \cap H}$.

Proof. (1) For all $(x, y) \in R|_{S \cup H}$, clearly, $y \in [x]_{R|_{S \cup H}}$ and $(x, y) \in (S \cup H) \times (S \cup H)$. In addition, by Theorem 4.5, we have that S and H are crisp (exact with respect to R), hence $S \cup H$ is crisp (exact with respect to R), and so by Lemma 4.4, we have that $[x]_{R|_{S \cup H}} = [x]_R$. $(x, y) \in (S \cup H) \times (S \cup H)$ implies $x \in S \cup H$, namely, $x \in S$ or $x \in H$. We may assume that $x \in S$, then by Lemma 4.4, we have $[x]_{R|_S} = [x]_R$. Hence, $[x]_{R|_{S \cup H}} = [x]_{R|_S}$ and so $y \in [x]_{R|_{S \cup H}}$ implies $y \in [x]_{R|_S}$. Thus, $(x, y) \in R|_S$. Hence, $R|_S \cup R|_H \supseteq R|_{S \cup H}$. On the other hand, clearly $R|_S = R \cap (S \times S) \subseteq R \cap ((S \cup H) \times (S \cup H)) = R|_{S \cup H}$ and $R|_H = R \cap (H \times H) \subseteq R \cap ((S \cup H) \times (S \cup H)) = R|_{S \cup H}$. Hence, $R|_S \cup R|_H \subseteq R|_{S \cup H}$. It follows that $R|_S \cup R|_H = R|_{S \cup H}$.

(2) For all $(x, y) \in R|_S \cap R|_H$, clearly, $(x, y) \in R|_S = R \cap (S \times S)$ and $(x, y) \in R|_H = R \cap (H \times H)$. Hence, $(x, y) \in R$, $x, y \in S$ and $x, y \in H$, which implies $(x, y) \in R$ and $x, y \in S \cap H$. Therefore $(x, y) \in R \cap ((S \cap H) \times (S \cap H)) = R|_{S \cap H}$. Thus $R|_S \cap R|_H \subseteq R|_{S \cap H}$. On the other hand, clearly, $R|_S = R \cap (S \times S) \supseteq R \cap ((S \cap H) \times (S \cap H)) = R|_{S \cap H}$ and $R|_H = R \cap (H \times H) \supseteq R \cap ((S \cap H) \times (S \cap H)) = R|_{S \cap H}$. Hence, $R|_S \cap R|_H \supseteq R|_{S \cap H}$. It follows that $R|_S \cap R|_H = R|_{S \cap H}$. $\square$

It is meaningful to notice that the union, intersection and complement of approximation subspaces still are approximation subspaces.

Theorem 4.8. *Let (U, R) be an approximation space, where U is a finite and nonempty universe and R an equivalence relation on U, and $S, H \subseteq U$. If $(S, R|_S), (H, R|_H) \in \mathcal{SP}(A)$, then $(S, R|_S) \cup (H, R|_H) \in \mathcal{SP}(A)$, $(S, R|_S) \cap (H, R|_H) \in \mathcal{SP}(A)$, and $\sim (S, R|_S) \in \mathcal{SP}(A)$.*

Proof. By Lemma 4.7 and Definition 4.6, we have that $(S, R_S) \cup (H, R_H) = (S \cup H, R_S \cup R_H) = (S \cup H, R_{S \cup H})$, $(S, R_S) \cap (H, R_H) = (S \cap H, R_S \cap R_H) = (S \cap H, R_{S \cap H})$. It follows from Note 1 and Theorem 4.5 that $(S, R_S) \cup (H, R_H) \in \mathcal{SP}(A)$, $(S, R_S) \cap (H, R_H) \in \mathcal{SP}(A)$. In addition, by Note 1 and Theorem 4.5, we can get that $\sim (S, R_S) \in \mathcal{SP}(A)$. $\square$

When the binary relations are equivalence relations in approximation spaces, the important properties of homomorphism are given in the following theorem.

Theorem 4.9. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping from U_1 to U_2, R_1 an equivalence relation on U_1, and R_2 an equivalence relation on U_2. f is a homomorphism from (U_1, R_1) to (U_2, R_2) and $X \subseteq U_1$. If X is an R_1-definable set, then $f(X)$ is a R_2-definable set.*

Proof. Since X is an R_1-definable set, it follows that $\overline{R_1}X = \underline{R_1}X$ and so $f(\overline{R_1}X) = f(\underline{R_1}X)$. In addition, since f is a homomorphism from (U_1, R_1) to (U_2, R_2), it follows from Definition 3.1 and Theorem 3.9 that $f(\underline{R_1}X) = \underline{R_2}f(X)$ and $f(\overline{R_1}X) = \overline{R_2}f(X)$. Thus, $\overline{R_2}f(X) = \underline{R_2}f(X)$. Hence, $f(X)$ is an R_2-definable set. $\square$

As natural consequence of the above theorem, we can obtain the following conclusion.

Corollary 4.10. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping from U_1 to U_2, R_1 an equivalence relation on U_1, and R_2 an equivalence relation on U_2. f is a homomorphism from (U_1, R_1) to (U_2, R_2) and $X \subseteq U_1$. If $(X, R|_X)$ is an approximation subspace of (U_1, R_1), then $(f(X), R|_{f(X)})$ is an approximation subspace of (U_2, R_2).*

Proof. The proof follows directly from Theorems 4.5 and 4.9. $\square$

Notation 5. Let U_1 and U_2 be finite and nonempty universes and $f : U_1 \to U_2$ a mapping from U_1 to U_2. Then let K_f denote the binary relation on U_1 and $x, y \in U_1$, $xK_f y \Leftrightarrow f(x) = f(y)$. Clearly, K_f is an equivalence relation on U_1. U_1/K_f denotes the set of all equivalence classes of K_f, that is, $U_1/K_f = \{[x]_{K_f} \mid x \in U_1\}$.

Lemma 4.11. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping, R_1 an equivalence relation on U_1, and R_2 an equivalence relation on U_2. One writes*

$$R_1/K_f = \left\{ \left([x]_{K_f}, [y]_{K_f} \right) \mid [x]_{K_f}, [y]_{K_f} \in U_1/K_f, \exists u \in [x]_{K_f}, \exists v \in [y]_{K_f}, s.t.\ uR_1v \right\}. \quad (4.2)$$

Then R_1/K_f is a reflexive and symmetric relation on U_1/K_f.

Proof. By the definition of R_1/K_f, it is easy to show that if $([x_1]_{K_f}, [y_1]_{K_f}) = ([x]_{K_f}, [y]_{K_f})$, then $([x]_{K_f}, [y]_{K_f}) \in R_1/K_f \Leftrightarrow ([x_1]_{K_f}, [y_1]_{K_f}) \in R_1/K_f$. Hence, R_1/K_f is a binary relation on U_1/K_f. In the following, we shall prove that R_1/K_f is reflexive and symmetric.

(i) Let $[x]_{K_f} \in U_1/K_f$, clearly, $x \in U_1$. Since R_1 is an equivalence relation on U_1, it follows that xR_1x. Hence, by the definition of R_1/K_f, we have that $([x]_{K_f}, [x]_{K_f}) \in R_1/K_f$. Thus, R_1/K_f is reflexive.

(ii) Let $([x]_{K_f}, [y]_{K_f}) \in R_1/K_f$, then by the definition of R_1/K_f, there exist $u \in [x]_{K_f}$ and $v \in [y]_{K_f}$ such that uR_1v. Since R_1 is an equivalence relation on U_1, it follows that vR_1u. Hence, there exist $v \in [y]_{K_f}$ and $u \in [x]_{K_f}$ such that vR_1u and so by the definition of R_1/K_f, $([y]_{K_f}, [x]_{K_f}) \in R_1/K_f$. Thus, R_1/K_f is symmetric. $\square$

Lemma 4.12. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping, R_1 an equivalence relation on U_1, and R_2 an equivalence relation on U_2. f is a homomorphism from (U_1, R_1) to (U_2, R_2). For all $[x]_{K_f} \in U_1/K_f$, one defines $h([x]_{K_f}) = f(x)$, then h is a mapping from U_1/K_f to U_2 and for all $Y \subseteq U_1/K_f$, $h(Y) = f(\bigcup_{[y]_{K_f} \in Y} [y]_{K_f})$.*

Proof. Let $[x]_{K_f}, [y]_{K_f} \in U_1/K_f$. If $[x]_{K_f} = [y]_{K_f}$, then by Notation 5, we have that $f(x) = f(y)$. Hence, by the definition of h, we can get that $h([x]_{K_f}) = f(x) = f(y) = h([y]_{K_f})$. We have proved that for all $[x]_{K_f}, [y]_{K_f} \in U_1/K_f$, if $[x]_{K_f} = [y]_{K_f}$, then $h([x]_{K_f}) = h([y]_{K_f})$. Therefore, h is a mapping from U_1/K_f to U_2.

We shall prove that for all $Y \subseteq U_1/K_f$, $h(Y) = f(\bigcup_{[y]_{K_f} \in Y} [y]_{K_f})$. By Notation 5, it is easy to verify that for all $[x]_{K_f} \in U_1/K_f$, $f([x]_{K_f}) = \{f(x)\}$. Hence, $h(Y) = h(\bigcup_{[y]_{K_f} \in Y} \{[y]_{K_f}\}) = \bigcup_{[y]_{K_f} \in Y} \{h([y]_{K_f})\} = \bigcup_{[y]_{K_f} \in Y} \{f(y)\} = \bigcup_{[y]_{K_f} \in Y} f([y]_{K_f}) = f(\bigcup_{[y]_{K_f} \in Y} [y]_{K_f})$. It follows that for all $Y \subseteq U_1/K_f$, $h(Y) = f(\bigcup_{[y]_{K_f} \in Y} [y]_{K_f})$. $\square$

Lemma 4.13. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping, R_1 an equivalence relation on U_1, and R_2 an equivalence relation on U_2, then*

$$\forall Y \subseteq U_1/K_f, \quad \overline{R_1/K_f}Y = \bigcup_{[y]_{K_f} \in Y} \left[[y]_{K_f} \right]_{R_1/K_f}. \quad (4.3)$$

Proof. Let $[y]_{K_f} \in Y$. We first show that for all $[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}$, $[u]_{K_f} \in \overline{R_1/K_f}Y$. By Lemma 4.11, R_1/K_f is symmetric, hence $[y]_{K_f} \in [[u]_{K_f}]_{R_1/K_f}$. Thus, $[y]_{K_f} \in [[u]_{K_f}]_{R_1/K_f} \cap Y \neq \emptyset$. This implies $[u]_{K_f} \in \overline{R_1/K_f}Y$. Therefore, $[[y]_{K_f}]_{R_1/K_f} \subseteq \overline{R_1/K_f}Y$. It follows that $\bigcup_{[y]_{K_f} \in Y} [[y]_{K_f}]_{R_1/K_f} \subseteq \overline{R_1/K_f}Y$.

On the other hand, for all $[u]_{K_f} \in \overline{R_1/K_f}Y$, then $[[u]_{K_f}]_{R_1/K_f} \cap Y \neq \emptyset$. Hence, there exists $[y]_{K_f} \in [[u]_{K_f}]_{R_1/K_f} \cap Y \neq \emptyset$ such that $[y]_{K_f} \in [[u]_{K_f}]_{R_1/K_f}$ and $[y]_{K_f} \in Y$. By Lemma 4.11, R_1/K_f is symmetric, hence $[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}$. Therefore, $[u]_{K_f} \in \bigcup_{[y]_{K_f} \in Y}[[y]_{K_f}]_{R_1/K_f}$. It follows that $\overline{R_1/K_f}Y \subseteq \bigcup_{[y]_{K_f} \in Y}[[y]_{K_f}]_{R_1/K_f}$.

Thus, $\overline{R_1/K_f}Y = \bigcup_{[y]_{K_f} \in Y}[[y]_{K_f}]_{R_1/K_f}$. □

An interesting problem is to discuss the isomorphisms of approximation spaces induced by the homomorphisms of approximation spaces. The next theorem shows the very intimate relation between homomorphisms and isomorphisms. The theorem is similar to first isomorphism theorem in group theory.

Theorem 4.14. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \to U_2$ a mapping from U_1 to U_2, R_1 an equivalence relation on U_1 and R_2 an equivalence relation on U_2. If f is an epimorphism from (U_1, R_1) to (U_2, R_2), then $(U_1/K_f, R_1/K_f) \cong (U_2, R_2)$.*

Proof. For all $[x]_{K_f} \in U_1/K_f$, we define $h([x]_{K_f}) = f(x)$, then by Lemma 4.12, we have that h is a mapping from U_1/K_f to U_2. Let $[x]_{K_f}, [y]_{K_f} \in U_1/K_f$. If $[x]_{K_f} \neq [y]_{K_f}$, then by Notation 5, we get that $f(x) \neq f(y)$. Hence, $h([x]_{K_f}) = f(x) \neq f(y) = h([y]_{K_f})$. We have proved that h is injective. In addition, for all $z \in U_2$, since f is surjective, it follows that there exists $x \in U_1$ such that $f(x) = z$. Thus for all $z \in U_2$, there exists $[x]_{K_f} \in U_1/K_f$ such that $h([x]_{K_f}) = f(x) = z$. Therefore, h is surjective. It follows that h is a bijective mapping from U_1/K_f to U_2. By Lemma 4.11, we conclude that R_1/K_f is a binary relation on U_1/K_f, hence $(U_1/K_f, R_1/K_f)$ is an approximation space. Since h is a bijective mapping from U_1/K_f to U_2, it follows from Definition 3.1 and Notation 2 that in order to have $(U_1/K_f, R_1/K_f) \cong (U_2, R_2)$, we need to prove only that h is a homomorphism from $(U_1/K_f, R_1/K_f)$ to (U_2, R_2).

Let $Y \subseteq U_1/K_f$. We shall show that $h(\underline{R_1/K_f}Y) \subseteq \underline{R_2}h(Y)$ and $h(\overline{R_1/K_f}Y) \subseteq \overline{R_2}h(Y)$. For simplicity, we write

$$\bigcup_{[y]_{K_f} \in Y}[y]_{K_f} = X, \tag{4.4}$$

clearly, $X \subseteq U_1$, and by Lemma 4.12, we have that $h(Y) = f(X)$.

(i) We shall prove that for all $[z]_{K_f} \in \underline{R_1/K_f}Y$, $[z]_{K_f} \subseteq \underline{R_1}X$. We need to prove only that for all $u \in [z]_{K_f}$, then $u \in \underline{R_1}X$. By Notation 5, $u \in [z]_{K_f}$ implies $[u]_{K_f} = [z]_{K_f}$. Let $t \in [u]_{R_1}$, then by the definition of $\overline{R_1/K_f}$, we have that $([u]_{K_f}, [t]_{K_f}) \in R_1/K_f$. Hence, $([z]_{K_f}, [t]_{K_f}) = ([u]_{K_f}, [t]_{K_f}) \in R_1/K_f$, that is, $([z]_{K_f}, [t]_{K_f}) \in R_1/K_f$ and so $[t]_{K_f} \in [[z]_{K_f}]_{R_1/K_f}$. We have proved that for all $t \in [u]_{R_1}$, $[t]_{K_f} \in [[z]_{K_f}]_{R_1/K_f}$. Thus, $\bigcup_{t \in [u]_{R_1}}\{[t]_{K_f}\} \subseteq [[z]_{K_f}]_{R_1/K_f}$. In addition, since $[z]_{K_f} \in \underline{R_1/K_f}Y$, it follows that $[[z]_{K_f}]_{R_1/K_f} \subseteq Y$. Hence, $\bigcup_{t \in [u]_{R_1}}\{[t]_{K_f}\} \subseteq Y$ and so $\bigcup_{t \in [u]_{R_1}}[t]_{K_f} \subseteq \bigcup_{[y]_{K_f} \in Y}[y]_{K_f} = X$. This implies $[u]_{R_1} \subseteq \bigcup_{t \in [u]_{R_1}}[t]_{K_f} \subseteq X$, that is, $[u]_{R_1} \subseteq X$. Thus $u \in \underline{R_1}X$. It follows that $[z]_{K_f} \subseteq \underline{R_1}X$.

(ii) We shall prove that $h(\underline{R_1/K_f}Y) \subseteq \underline{R_2}h(Y)$. By (i), we have that for all $[z]_{K_f} \in \underline{R_1/K_f}Y$, $[z]_{K_f} \subseteq \underline{R_1}X$ and so for all $[z]_{K_f} \in \underline{R_1/K_f}Y$, $f([z]_{K_f}) \subseteq f(\underline{R_1}X)$. Therefore, by Lemma 4.12, we have that

$$h\left(\underline{R_1/K_f}Y\right) = f\left(\bigcup_{[z]_{K_f} \in \underline{R_1/K_f}Y}[z]_{K_f}\right) = \bigcup_{[z]_{K_f} \in \underline{R_1/K_f}Y}f\left([z]_{K_f}\right) \subseteq \bigcup_{[z]_{K_f} \in \underline{R_1/K_f}Y}f\left(\underline{R_1}X\right) = f\left(\underline{R_1}X\right),$$

$$\tag{4.5}$$

that is, $h(R_1/K_fY) \subseteq f(\underline{R_1}X)$. In addition, since f is a homomorphism from (U_1, R_1) to (U_2, R_2), it follows that $f(\underline{R_1}X) \subseteq \underline{R_2}f(X)$. Thus,

$$h\left(\underline{R_1/K_f}Y\right) \subseteq f\left(\underline{R_1}X\right) \subseteq \underline{R_2}f(X) = \underline{R_2}h(Y). \tag{4.6}$$

We have proved that $h(\underline{R_1/K_f}Y) \subseteq \underline{R_2}h(Y)$.

(iii) Let $[y]_{K_f} \in Y$. We shall prove that for all $[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}$, $f([u]_{K_f}) \subseteq f(\overline{R_1}X)$. By the definition of R_1/K_f, there exist $s \in [y]_{K_f}$ and $t \in [u]_{K_f}$ such that sR_1t. Hence, $s \in [t]_{R_1} \cap [y]_{K_f} \neq \emptyset$. Since $[y]_{K_f} \in Y$, it follows that $[t]_{R_1} \cap (\bigcup_{[y]_{K_f} \in Y}[y]_{K_f}) \neq \emptyset$, namely, $[t]_{R_1} \cap X \neq \emptyset$. This implies $t \in \overline{R_1}X$ and so $f(t) \in f(\overline{R_1}X)$. In addition, since $t \in [u]_{K_f}$, it follows from Notation 5 that $f([u]_{K_f}) = \{f(t)\}$. Thus, $f([u]_{K_f}) \subseteq f(\overline{R_1}X)$.

(iv) We shall prove that $h(\overline{R_1/K_f}Y) \subseteq \overline{R_2}h(Y)$. By Lemma 4.13, we get that

$$h\left(\overline{R_1/K_f}Y\right) = h\left(\bigcup_{[y]_{K_f} \in Y}\left[[y]_{K_f}\right]_{R_1/K_f}\right) = \bigcup_{[y]_{K_f} \in Y}h\left(\left[[y]_{K_f}\right]_{R_1/K_f}\right). \tag{4.7}$$

By Lemma 4.12, we have that

$$h\left(\left[[y]_{K_f}\right]_{R_1/K_f}\right) = f\left(\bigcup_{[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}}[u]_{K_f}\right) = \bigcup_{[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}}f\left([u]_{K_f}\right). \tag{4.8}$$

Thus, $h(\overline{R_1/K_f}Y) = \bigcup_{[y]_{K_f} \in Y}h([[y]_{K_f}]_{R_1/K_f}) = \bigcup_{[y]_{K_f} \in Y}\bigcup_{[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}}f([u]_{K_f})$, that is, $h(\overline{R_1/K_f}Y) = \bigcup_{[y]_{K_f} \in Y}\bigcup_{[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}}f([u]_{K_f})$. By (iii), we have that for all $[y]_{K_f} \in Y$ and all $[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}$, $f([u]_{K_f}) \subseteq f(\overline{R_1}X)$. Hence,

$$h\left(\overline{R_1/K_f}Y\right) = \bigcup_{[y]_{K_f} \in Y}\bigcup_{[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}}f\left([u]_{K_f}\right) \subseteq \bigcup_{[y]_{K_f} \in Y}\bigcup_{[u]_{K_f} \in [[y]_{K_f}]_{R_1/K_f}}f\left(\overline{R_1}X\right) = f\left(\overline{R_1}X\right), \tag{4.9}$$

that is, $h(\overline{R_1/K_f}Y) \subseteq f(\overline{R_1}X)$. In addition, since f is a homomorphism from (U_1, R_1) to (U_2, R_2), it follows from Definition 3.1 that $f(\overline{R_1}X) \subseteq \overline{R_2}f(X) = \overline{R_2}h(Y)$. Therefore $h(\overline{R_1/K_f}Y) \subseteq f(\overline{R_1}X) \subseteq \overline{R_2}h(Y)$, that is, $h(\overline{R_1/K_f}Y) \subseteq \overline{R_2}h(Y)$.

(v) By (ii), (iv), and Definition 3.1, we conclude that h is a homomorphism $(U_1/K_f, R_1/K_f)$ to (U_2, R_2). Since h is bijective, it follows from Definition 3.1 that h is an isomorphism from $(U_1/K_f, R_1/K_f)$ to (U_2, R_2). Thus, by Notation 2, we conclude that $(U_1/K_f, R_1/K_f) \cong (U_2, R_2)$. This completes the proof. □

In the following, we extend Theorem 4.14 to more general cases.

Corollary 4.15. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \rightarrow U_2$ a mapping, R_1 an equivalence relation on U_1, and R_2 an equivalence relation on U_2. If f is a homomorphism from (U_1, R_1) to (U_2, R_2), then $(U_1/K_f, R_1/K_f) \cong (f(U_1), R_2|_{f(U_1)})$.*

Proof. For all $X \subseteq U_1$, since f is a homomorphism from (U_1, R_1) to (U_2, R_2), it follows from Definition 3.1 that $f(\underline{R_1}X) \subseteq \underline{R_2}f(X)$ and $f(\overline{R_1}X) \subseteq \overline{R_2}f(X)$. In addition, By Corollary 4.10, we have that $(f(U_1), \overline{R_2|_{f(U_1)}})$ is an approximation subspace of (U_2, R_2). Since $f(X) \subseteq f(U_1)$, it follows from Definition 4.1 that $\underline{R_2}f(X) = \underline{R_2|_{f(U_1)}}f(X)$ and $\overline{R_2}f(X) = \overline{R_2|_{f(U_1)}}f(X)$. Thus $f(\underline{R_1}X) \subseteq \underline{R_2|_{f(U_1)}}f(X)$ and $f(\overline{R_1}X) \subseteq \overline{R_2|_{f(U_1)}}f(X)$. We have proved that for all $X \subseteq U_1$, $f(\underline{R_1}X) \subseteq \underline{R_2|_{f(U_1)}}f(X)$ and $f(\overline{R_1}X) \subseteq \overline{R_2|_{f(U_1)}}f(X)$. Hence, by Definition 3.1, we have that f is a homomorphism from (U_1, R_1) to $(f(U_1), R_2|_{f(U_1)})$. Clearly, f is a surjective mapping from U_1 to $f(U_1)$; therefore, f is an epimorphism from (U_1, R_1) to $(f(U_1), R_2|_{f(U_1)})$. It follows from Theorem 4.14 that $(U_1/K_f, R_1/K_f) \cong (f(U_1), R_2|_{f(U_1)})$.   □

Corollary 4.16. *Let U_1 and U_2 be finite and nonempty universes, $f : U_1 \rightarrow U_2$ a mapping, R_1 an equivalence relation on U_1, and R_2 an equivalence relation on U_2. If f is a homomorphism from (U_1, R_1) to (U_2, R_2), then R_1/K_f is an equivalence relation on U_1/K_f.*

Proof. By Corollary 4.15, we have that $(U_1/K_f, R_1/K_f) \cong (f(U_1), R_2|_{f(U_1)})$. by Lemma 4.3, we have that $R_2|_{f(U_1)}$ is an equivalence relation on $f(U_1)$. It follows from Theorem 3.14 that R_1/K_f is an equivalence relation on U_1/K_f.   □

In fact, R_1/K_f may not be an equivalence relation on U_1/K_f when f is not a homomorphism from (U_1, R_1) to (U_2, R_2). The above results show that R_1/K_f is an equivalence relation on U_1/K_f when f is a homomorphism from (U_1, R_1) to (U_2, R_2).

5. Conclusions

In this paper, we present the notion of the homomorphism of approximation spaces. A homomorphism may be viewed as a special mapping between two approximation spaces. By means of this concept, we establish the relationships between two universes. In this way, one can make inference in one universe, based on information about another universe. In addition, we give the notion of approximation subspaces of approximation spaces, and investigate the properties of approximation subspaces by means of homomorphism. In the future, we will introduce a similar notion of homomorphism into covering-based rough set model in order to deepen understanding of this model.

Acknowledgments

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