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Research Article

New Nonlinear Conditions and Inequalities for the Existence of Coincidence Points and Fixed Points

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Some new fixed point theorems for nonlinear maps are established. By using these results, we can obtain some new coincidence point theorems. Our results are quite different in the literature and references therein.

1. Introduction and Preliminaries

Let us begin with some basic definitions and notation that will be needed in this paper. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively. Let (X, d) be a metric space. For each $x \in X$, and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$. Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, $\mathcal{C}(X)$ the family of all nonempty closed subsets of X, and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : CB(X) \times CB(X) \to [0, \infty)$, defined by

$$\mathcal{H}(A, B) = \max\left\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\right\}, \tag{1.1}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X.

A point v in X is a fixed point of a map T if $v = Tv$ (when $T : X \to X$ is a single-valued map) or $v \in Tv$ (when $T : X \to \mathcal{N}(X)$ is a multivalued map). The set of fixed points of T is denoted by $\mathcal{F}(T)$.

Let $g : X \to X$ be a self-map and $T : X \to \mathcal{N}(X)$ be a multivalued map. A point x in X is said to be a *coincidence point* (see, for instance, [1–4]) of g and T if $gx \in Tx$. The set of coincidence points of g and T is denoted by $\mathcal{COP}(g, T)$.

The celebrated Banach contraction principle (see, e.g., [5]) plays an important role in various fields of applied mathematical analysis. Since then a number of generalizations in various different directions of the Banach contraction principle have been investigated by several authors in the past; see [6–20] and references therein. In 1969, Nadler [6] first proved a famous generalization of the Banach contraction principle for multivalued maps.

Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x \to c} f(x) = \inf_{\varepsilon > 0} \sup_{0<|x-c|<\varepsilon} f(x),$$
$$\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{0<x-c<\varepsilon} f(x). \tag{1.2}$$

Definition 1.1 (see [3, 4, 7–10, 21, 22]). A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-*function* (or $\mathcal{R}$-*function*) if

$$\limsup_{s \to t^+} \varphi(s) < 1 \quad \forall t \in [0,\infty). \tag{1.3}$$

It is obvious that if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. But it is worth to mention that there exist functions which are not $\mathcal{MT}$-functions.

Example A (see [4]). Let $\varphi : [0,\infty) \to [0,1)$ be defined by

$$\varphi(t) := \begin{cases} \dfrac{\sin t}{t}, & \text{if } t \in \left(0, \dfrac{\pi}{2}\right], \\ 0, & \text{otherwise}. \end{cases} \tag{1.4}$$

Since $\limsup_{s \to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

Very recently, Du [4] first proved the following characterizations of $\mathcal{MT}$-functions.

Theorem 1.2 (see [4]). *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*

(b) *For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.*

(c) *For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.*

(d) *For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.*

(e) *For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.*

(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, one has $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

(g) φ *is a* ***function of contractive factor*** *[10]; that is, for any strictly decreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $[0,\infty)$*, one has* $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

In 1989, Mizoguchi and Takahashi [11] proved the following fixed point theorem which is a generalization of Nadler's fixed point theorem and gave a partial answer of Problem 9 in Reich [12]. It is worth to mention that the primitive proof of Mizoguchi-Takahashi's fixed point theorem is quite difficult. Recently, Suzuki [13] gave a very simple proof of Mizoguchi-Takahashi's fixed point theorem.

Theorem MT (Mizoguchi and Takahashi). *Let* (X,d) *be a complete metric space and* $T : X \to \mathcal{CB}(X)$ *be a multivalued map and* $\varphi : [0,\infty) \to [0,1)$ *be a* $\mathcal{MT}$*-function. Assume that*

$$\mathcal{H}(Tx,Ty) \le \varphi(d(x,y))d(x,y) \quad \forall x,\, y \in X, \tag{1.5}$$

then $\mathcal{F}(T) \ne \emptyset$.

In 2007, M. Berinde and V. Berinde [14] proved the following interesting fixed point theorem which generalized Mizoguchi-Takahashi's fixed point theorem.

Theorem BB (M. Berinde and V. Berinde). *Let* (X,d) *be a complete metric space,* $T : X \to \mathcal{CB}(X)$ *be a multivalued map,* $\varphi : [0,\infty) \to [0,1)$ *be a* $\mathcal{MT}$*-function and* $L \ge 0$*. Assume that*

$$\mathcal{H}(Tx,Ty) \le \varphi(d(x,y))d(x,y) + Ld(y,Tx) \quad \forall x,\, y \in X, \tag{1.6}$$

then $\mathcal{F}(T) \ne \emptyset$.

Let (X,d) be a metric space. Recall that a function $p : X \times X \to [0,\infty)$ is called a *w-distance* [3, 5, 7, 15–23], if the following are satisfied:

($w1$) $p(x,z) \le p(x,y) + p(y,z)$ for any $x,y,z \in X$;

($w2$) for any $x \in X$, $p(x,\cdot) : X \to [0,\infty)$ is l.s.c.;

($w3$) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$.

A function $p : X \times X \to [0,\infty)$ is said to be a *τ-function* [3, 7, 16, 18–22], introduced and studied by Lin and Du, if the following conditions hold:

($\tau1$) $p(x,z) \le p(x,y) + p(y,z)$ for all $x,y,z \in X$;

($\tau2$) if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x,y_n) \le M$ for some $M = M(x) > 0$, then $p(x,y) \le M$;

($\tau3$) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n,x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n,y_n) = 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$;

($\tau4$) for $x,y,z \in X$, $p(x,y) = 0$ and $p(x,z) = 0$ imply $y = z$.

It is well known that the metric d is a w-distance and any w-distance is a τ-function, but the converse is not true; see [7, 16].

The following results are crucial in this paper.

Lemma 1.3 (see [7, 20]). *Let (X, d) be a metric space and $p : X \times X \to [0, \infty)$ be a function. Assume that p satisfies the condition $(\tau 3)$. If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Let $p : X \times X \to [0, \infty)$ be any function. For each $x \in X$ and $A \subseteq X$, let $p(x, A) = \inf_{y\in A} p(x, y)$.

Lemma 1.4 (see [7, 19, 20]). *Let A be a closed subset of a metric space (X, d) and $p : X \times X \to [0, \infty)$ be any function. Suppose that p satisfies $(\tau 3)$ and there exists $u \in X$ such that $p(u, u) = 0$. Then $p(u, A) = 0$ if and only if $u \in A$.*

Recently, Du [7, 19] first introduced the concepts of τ^0-functions and τ^0-metrics as follows.

Definition 1.5 (see [7, 19]). Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a τ^0-function if it is a τ-function on X with $p(x, x) = 0$ for all $x \in X$.

Remark 1.6. If p is a τ^0-function, then, from $(\tau 4)$, $p(x, y) = 0$ if and only if $x = y$.

Definition 1.7 (see [7, 19]). Let (X, d) be a metric space and p be a τ^0-function (resp., w^0-distance). For any $A, B \in \mathcal{CB}(X)$, define a function $\mathfrak{D}_p : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ by

$$\mathfrak{D}_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}, \tag{1.7}$$

where $\delta_p(A, B) = \sup_{x\in A} p(x, B)$, then $\mathfrak{D}_p$ is said to be the τ^0*-metric* (resp., w^0*-metric*) on $\mathcal{CB}(X)$ induced by p.

Clearly, any Hausdorff metric is a τ^0-metric, but the reverse is not true.

Lemma 1.8 (see [7, 19]). *Let (X, d) be a metric space and $\mathfrak{D}_p$ be a τ^0-metri on $\mathcal{CB}(X)$ induced by a τ^0-function p. Then every τ^0-metric $\mathfrak{D}_p$ is a metric on $\mathcal{CB}(X)$.*

Recently, Du [7] established the following new fixed point theorems for τ^0-metric and $\mathcal{MT}$-functions to extend Berinde-Berinde's fixed point theorem.

Theorem D (Du [7, Theorem 2.1]). *Let (X, d) be a complete metric space, $T : X \to \mathcal{C}(X)$ be a multivalued map and $\varphi : [0, \infty) \to [0, 1)$ a $\mathcal{MT}$-function. Suppose that for each $x \in X$*

$$p(y, Ty) \le \varphi(p(x, y))p(x, y) \quad \forall y \in Tx, \tag{1.8}$$

and T further satisfies one of the following conditions:

(D1) *T is closed;*

(D2) *the map $f : X \to [0, \infty)$ defined by $f(x) = p(x, Tx)$ is l.s.c.;*

(D3) *the map $g : X \to [0, \infty)$ defined by $g(x) = d(x, Tx)$ is l.s.c.;*

(D4) *for each sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = v$, one has $\lim_{n\to\infty} p(x_n, Tv) = 0$;*

(D5) $\inf\{p(x, z) + p(x, Tx) : x \in X\} > 0$ *for every $z \notin \mathcal{F}(T)$.*

Then $\mathcal{F}(T) \neq \emptyset$.

In [7], Du also gave the generalizations of Kannan's fixed point theorem, Chatterjea's fixed point theorem and other new fixed point theorems for nonlinear multivalued contractive maps; see [7] for more detail.

In this paper, we first establish some new types of fixed point theorem. Some applications to the existence for coincidence point and others are also given. Our results are quite different in the literature and references therein.

2. New Inequalities and Nonlinear Conditions for Fixed Point Theorems

In this section, we first establish some new existence theorems for fixed point.

Theorem 2.1. *Let (X, d) be a complete metric space, p be a τ^0-function and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that*

($\mathcal{A}1$) *there exist two functions $\varphi, \tau : [0,\infty) \to [0,1)$ such that τ is an $\mathcal{MT}$-function and for each $x \in X$, it holds*

$$3p(y, Ty) \leq \varphi(p(x,y))p(x,Tx) + \tau(p(x,y))p(x,Ty) \quad \forall y \in Tx. \tag{2.1}$$

($\mathcal{A}2$) *T further satisfies one of the following conditions:*

(H1) *T is closed, that is, $GrT = \{(x,y) \in X \times X : y \in Tx\}$, the graph of T is closed in $X \times X$;*

(H2) *the map $f : X \to [0,\infty)$ defined by $f(x) = p(x,Tx)$ is l.s.c.;*

(H3) *the map $g : X \to [0,\infty)$ defined by $g(x) = d(x,Tx)$ is l.s.c.;*

(H4) *for any sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = v$, one has $\lim_{n\to\infty} p(x_n, Tv) = 0$;*

(H5) $\inf\{p(x,z) + p(x,Tx) : x \in X\} > 0$ *for every $z \notin \mathcal{F}(T)$.*

Then $\mathcal{F}(T) \neq \emptyset$.

Proof. Let $u_1 \in X$. If $u_1 \in Tu_1$, then we are done. If $u_1 \notin Tu_1$, then $p(u_1, Tu_1) > 0$ by Lemma 1.4. Choose $u_2 \in Tu_1$. If $u_2 \in Tu_2$, then u_2 is a fixed point of T. Otherwise, if $u_2 \notin Tu_2$, then, by ($\mathcal{A}1$), we have

$$\begin{aligned} 3p(u_2, Tu_2) &\leq \varphi(p(u_1,u_2))p(u_1,Tu_1) + \tau(p(u_1,u_2))p(u_1,Tu_2) \\ &\leq \varphi(p(u_1,u_2))p(u_1,u_2) + \tau(p(u_1,u_2))p(u_1,Tu_2). \end{aligned} \tag{2.2}$$

Since $\varphi(p(u_1,u_2)) + \tau(p(u_1,u_2)) < 2$, there exists $u_3 \in Tu_2$ such that

$$\begin{aligned} 3p(u_2,u_3) &< (2 - \tau(p(u_1,u_2)))p(u_1,u_2) + \tau(p(u_1,u_2))p(u_1,u_3) \\ &\leq (2 - \tau(p(u_1,u_2)))p(u_1,u_2) + \tau(p(u_1,u_2))\left[p(u_1,u_2) + p(u_2,u_3)\right], \end{aligned} \tag{2.3}$$

which implies

$$p(u_2, u_3) < \frac{2}{3 - \tau(p(u_1, u_2))} p(u_1, u_2). \tag{2.4}$$

Since $u_3 \neq u_2$ and p is a τ^0-function, $p(u_2, u_3) > 0$. Since $0 \leq \tau(p(u_1, u_2)) < 1$, we have $2/(3 - \tau(p(u_1, u_2))) \in [2/3, 1)$ and hence

$$0 < p(u_2, u_3) < p(u_1, u_2). \tag{2.5}$$

Continuing in this way, we can construct inductively a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X satisfying $u_{n+1} \in Tu_n$, $\{p(u_n, u_{n+1})\}$ is strictly decreasing in $(0, \infty)$ and

$$p(u_{n+1}, u_{n+2}) < \frac{2}{3 - \tau(p(u_n, u_{n+1}))} p(u_n, u_{n+1}) \tag{2.6}$$

for each $n \in \mathbb{N}$. Since τ is an $\mathcal{MT}$-function, applying Theorem 1.2, we get

$$0 \leq \sup_{n\in\mathbb{N}} \tau(p(u_n, u_{n+1})) < 1. \tag{2.7}$$

Put $\lambda := \sup_{n\in\mathbb{N}} \tau(p(u_n, u_{n+1}))$ and $\xi := 2/(3 - \lambda)$. Then $\lambda \in [0, 1)$, $\xi \in [2/3, 1)$ and

$$\frac{2}{3 - \tau(p(u_n, u_{n+1}))} \leq \xi \quad \forall n \in \mathbb{N}. \tag{2.8}$$

By (2.6) and (2.8), we have

$$p(u_{n+1}, u_{n+2}) < \xi p(u_n, u_{n+1}) \ < \cdots < \xi^n p(u_1, u_2) \quad \text{for each } n \in \mathbb{N}. \tag{2.9}$$

We claim that $\lim_{n\to\infty} \sup\{p(u_n, u_m) : m > n\} = 0$. Let $\alpha_n = (\xi^{n-1}/(1 - \xi))p(u_1, u_2)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$p(u_n, u_m) \leq \sum_{j=n}^{m-1} p(u_j, u_{j+1}) < \alpha_n. \tag{2.10}$$

Since $\xi \in [2/3, 1)$, $\lim_{n\to\infty}\alpha_n = 0$ and, from (2.10), we get

$$\lim_{n\to\infty} \sup\{p(u_n, u_m) : m > n\} = 0. \tag{2.11}$$

Applying Lemma 1.3, $\{u_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $u_n \to v$ as $n \to \infty$. From (τ2) and (2.10), we have

$$p(u_n, v) \leq \alpha_n \quad \forall n \in \mathbb{N}. \tag{2.12}$$

Now, we verify that $v \in \mathcal{F}(T)$. If (H1) holds, since T is closed, $u_n \in Tu_{n-1}$ and $u_n \to v$ as $n \to \infty$, we have $v \in Tv$.

If (H2) holds, by the lower semicontinuity of f, $u_n \to v$ as $n \to \infty$ and (2.11), we obtain

$$\begin{aligned} p(v, Tv) &= f(v) \\ &\leq \liminf_{m \to \infty} f(u_n) \\ &= \liminf_{m \to \infty} p(u_n, Tu_n) \\ &\leq \lim_{n \to \infty} p(u_n, u_{n+1}) = 0, \end{aligned} \tag{2.13}$$

which implies $p(v, Tv) = 0$. By Lemma 1.4, we get $v \in \mathcal{F}(T)$.

Suppose that (H3) holds. Since $\{x_n\}$ is convergent in X, $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$. Since

$$\begin{aligned} d(v, Tv) &= g(v) \\ &\leq \liminf_{m \to \infty} d(u_n, Tu_n) \\ &\leq \lim_{n \to \infty} d(u_n, u_{n+1}) = 0, \end{aligned} \tag{2.14}$$

we have $d(v, Tv) = 0$ and hence $v \in \mathcal{F}(T)$.

If (H4) holds, by (2.11), there exists $\{a_n\} \subset \{u_n\}$ with $\lim_{n\to\infty} \sup\{p(a_n, a_m) : m > n\} = 0$ and $\{b_n\} \subset Tv$ such that $\lim_{n\to\infty} p(a_n, b_n) = 0$. By ($\tau$3), $\lim_{n\to\infty} d(a_n, b_n) = 0$. Since $d(b_n, v) \leq d(b_n, a_n) + d(a_n, v)$, it follows that $b_n \to v$ as $n \to \infty$. By the closedness of Tv, we get $v \in Tv$ or $v \in \mathcal{F}(T)$.

Finally, assume that (H5) holds. On the contrary, suppose that $v \notin Tv$. Then, by (2.10) and (2.12), we obtain

$$\begin{aligned} 0 &< \inf_{x \in X} \{p(x, v) + p(x, Tx)\} \\ &\leq \inf_{n \in \mathbb{N}} \{p(u_n, v) + p(u_n, Tu_n)\} \\ &\leq \inf_{n \in \mathbb{N}} \{p(u_n, v) + p(u_n, u_{n+1})\} \\ &\leq \lim_{n \to \infty} 2\alpha_n \\ &= 0, \end{aligned} \tag{2.15}$$

a contradiction. Therefore $v \in \mathcal{F}(T)$. The proof is completed. □

If we put $p \equiv d$ in Theorem 2.1, then we have the following result.

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that*

($\mathcal{B}1$) *there exist two functions $\varphi, \tau : [0,\infty) \to [0,1)$ such that τ is an $\mathcal{MT}$-function and for each $x \in X$, it holds*

$$3d(y, Ty) \leq \varphi(d(x,y))d(x,Tx) + \tau(d(x,y))d(x,Ty) \quad \forall y \in Tx. \tag{2.16}$$

($\mathcal{B}2$) *T further satisfies one of the following conditions:*

(h1) *T is closed;*

(h2) *the map $g : X \to [0,\infty)$ defined by $g(x) = d(x,Tx)$ is l.s.c.;*

(h3) *for any sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = v$, one has $\lim_{n\to\infty} d(x_n, Tv) = 0$;*

(h4) $\inf\{d(x,z) + d(x,Tx) : x \in X\} > 0$ *for every $z \notin \mathcal{F}(T)$.*

Then $\mathcal{F}(T) \neq \emptyset$.

The following result is immediate from Theorem 2.1.

Theorem 2.3. *Let (X, d) be a complete metric space, p be a τ^0-function, and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that the condition ($\mathcal{A}2$) holds and further assume that*

($\mathcal{A}3$) *there exists an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ such that for each $x \in X$,*

$$3p(y, Ty) \leq \alpha(p(x,y))(p(x,Tx) + p(x,Ty)) \quad \forall y \in Tx, \tag{2.17}$$

then $\mathcal{F}(T) \neq \emptyset$.

Corollary 2.4. *Let (X, d) be a complete metric space and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that the condition ($\mathcal{B}2$) holds and further assume that*

($\mathcal{B}3$) *there exists an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ such that for each $x \in X$,*

$$3d(y, Ty) \leq \alpha(d(x,y))(d(x,Tx) + d(x,Ty)) \quad \forall y \in Tx, \tag{2.18}$$

then $\mathcal{F}(T) \neq \emptyset$.

Theorem 2.5. *Let (X, d) be a complete metric space, p be a τ^0-function, and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that the condition ($\mathcal{A}2$) holds and further assume that*

($\mathcal{A}4$) *there exist $\alpha, \beta \in [0,1)$ such that for each $x \in X$, $3p(y,Ty) \leq \alpha p(x,Tx) + \beta p(x,Ty)$ for all $y \in Tx$,*

then $\mathcal{F}(T) \neq \emptyset$.

Proof. Let $\varphi, \tau : [0,\infty) \to [0,1)$ be defined by $\varphi(t) = \alpha$ and $\tau(t) = \beta$ for all $t \in [0,\infty)$. Then ($\mathcal{A}4$) implies ($\mathcal{A}1$) and the conclusion follows from Theorem 2.1. □

Corollary 2.6. *Let (X,d) be a complete metric space and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that the condition ($\mathcal{B}2$) holds and further assume that*

($\mathcal{B}4$) *there exist $\alpha,\beta \in [0,1)$ such that for each $x \in X$, $3d(y,Ty) \leq \alpha d(x,Tx) + \beta d(x,Ty)$ for all $y \in Tx$,*

then $\mathcal{F}(T) \neq \emptyset$.

Theorem 2.7. *Let (X,d) be a complete metric space, p be a τ^0-function, and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that the condition ($\mathcal{A}2$) holds and further assume that*

($\mathcal{A}5$) *there exists $\gamma \in [0,1)$ such that for each $x \in X$, $3p(y,Ty) \leq \gamma(p(x,Tx) + p(x,Ty))$ for all $y \in Tx$,*

then $\mathcal{F}(T) \neq \emptyset$.

Proof. Let $\alpha = \beta = \gamma$. Then ($\mathcal{A}5$) implies ($\mathcal{A}4$) and the conclusion follows from Theorem 2.5. □

Remark 2.8. ($\mathcal{A}4$) and ($\mathcal{A}5$) are equivalent. Indeed, in the proof of Theorem 2.7, we have shown that ($\mathcal{A}5$) implies ($\mathcal{A}4$). If ($\mathcal{A}4$) holds, then put $\gamma = \max\{\alpha,\beta\}$. So $\gamma \in [0,1)$ and ($\mathcal{A}5$) holds. Hence ($\mathcal{A}4$) and ($\mathcal{A}5$) are equivalent. Therefore Theorem 2.5 can also be proved by using Theorem 2.7 and we know that Theorems 2.5 and 2.7 are indeed equivalent.

Corollary 2.9. *Let (X,d) be a complete metric space and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that the condition ($\mathcal{B}2$) holds and further assume that*

($\mathcal{B}5$) *there exists $\gamma \in [0,1)$ such that for each $x \in X$, $3d(y,Ty) \leq \gamma(d(x,Tx) + d(x,Ty))$ for all $y \in Tx$,*

then $\mathcal{F}(T) \neq \emptyset$.

Remark 2.10. Corollaries 2.6 and 2.9 are equivalent.

3. Applications of Theorem 2.1 to the Existence of Coincidence Points

By applying Theorem 2.1, we can prove easily the following new coincidence point theorem.

Theorem 3.1. *Let (X,d) be a complete metric space, p be a τ^0-function, $g : X \to X$ be a self-map, $T : X \to \mathcal{C}(X)$ be a multivalued map, and $L \geq 0$. Suppose that the condition ($\mathcal{A}2$) holds and further assume that*

($\mathcal{A}6$) *Tx is g-invariant (i.e., $g(Tx) \subseteq Tx$) for each $x \in X$;*

($\mathcal{A}7$) *there exist two functions $\varphi,\tau : [0,\infty) \to [0,1)$ such that τ is an $\mathcal{MT}$-function and it holds*

$$3p(y,Ty) \leq \varphi(p(x,y))p(x,Tx) + \tau(p(x,y))p(x,Ty) + Lp(gy,Tx) \quad \forall x,y \in X, \tag{3.1}$$

then $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Proof. For each $x \in X$, if $y \in Tx$, from ($\mathcal{A}6$), we have $gy \in Tx$. So $p(gy,Tx) = 0$. Hence ($\mathcal{A}7$) implies ($\mathcal{A}1$). Applying Theorem 2.1, $\mathcal{F}(T) \neq \emptyset$. So there exists $v \in X$ such that $v \in Tv$. By ($\mathcal{A}6$), $gv \in Tv$. Therefore, $v \in \mathcal{COP}(g,T) \cap \mathcal{F}(T)$ and the proof is complete. □

The following result is immediate from Theorem 3.1.

Corollary 3.2. *Let (X, d) be a complete metric space, $g : X \to X$ be a self-map, $T : X \to \mathcal{C}(X)$ be a multivalued map, and $L \geq 0$. Suppose that the condition $(\mathcal{B}2)$ holds and further assume*

$(\mathcal{B}6)$ *Tx is g-invariant (i.e., $g(Tx) \subseteq Tx$) for each $x \in X$;*

$(\mathcal{B}7)$ *there exist two functions $\varphi, \tau : [0, \infty) \to [0, 1)$ such that τ is an $\mathcal{MT}$-function and it holds*

$$3d(y, Ty) \leq \varphi(d(x, y))d(x, Tx) + \tau(d(x, y))d(x, Ty) + Ld(gy, Tx) \quad \forall x, y \in X, \tag{3.2}$$

then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.

As an application of Theorem 3.1, one has the following fixed point theorem.

Theorem 3.3. *Let (X, d) be a complete metric space, p be a τ^0-function, $T : X \to \mathcal{C}(X)$ be a multivalued map, and $L \geq 0$. Suppose that the condition $(\mathcal{A}2)$ holds and further assume that*

$(\mathcal{A}8)$ *there exist two functions $\varphi, \tau : [0, \infty) \to [0, 1)$ such that τ is an $\mathcal{MT}$-function and it holds*

$$3p(y, Ty) \leq \varphi(p(x, y))p(x, Tx) + \tau(p(x, y))p(x, Ty) + Lp(y, Tx) \quad \forall x, y \in X, \tag{3.3}$$

then $\mathcal{F}(T) \neq \emptyset$.

Corollary 3.4. *Let (X, d) be a complete metric space, $T : X \to \mathcal{C}(X)$ be a multivalued map and $L \geq 0$. Suppose that the condition $(\mathcal{B}2)$ holds and further assume that*

$(\mathcal{A}8)$ *there exist two functions $\varphi, \tau : [0, \infty) \to [0, 1)$ such that τ is an $\mathcal{MT}$-function and it holds*

$$3d(y, Ty) \leq \varphi(d(x, y))d(x, Tx) + \tau(d(x, y))d(x, Ty) + Ld(y, Tx) \quad \forall x, y \in X, \tag{3.4}$$

then $\mathcal{F}(T) \neq \emptyset$.

Theorem 3.5. *Let (X, d) be a complete metric space, p be a τ^0-function, $\mathfrak{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a self-map, and $L \geq 0$. Suppose that the conditions $(\mathcal{A}2)$ and $(\mathcal{A}6)$ hold and further assume that*

$(\mathcal{A}9)$ *there exist two functions $\varphi, \tau : [0, \infty) \to [0, 1)$ such that τ is an $\mathcal{MT}$-function and it holds*

$$\mathfrak{D}_p(Tx, Ty) \leq \varphi(p(x, y))p(x, Tx) + \tau(p(x, y))p(x, Ty) + Lp(gy, Tx) \quad \forall x, y \in X, \tag{3.5}$$

then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.

Corollary 3.6. *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $g : X \to X$ be a self-map, and $L \geq 0$. Suppose that the conditions $(\mathcal{B}2)$ and $(\mathcal{B}6)$ hold and further assume that*

($\mathcal{B}$9) *there exist two functions* $\varphi, \tau : [0,\infty) \to [0,1)$ *such that* τ *is an* $\mathcal{MT}$*-function and it holds*

$$\mathcal{H}(Tx,Ty) \le \varphi(d(x,y))d(x,Tx) + \tau(d(x,y))d(x,Ty) + Ld(gy,Tx) \quad \forall x,y \in X, \qquad (3.6)$$

then $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Theorem 3.7. *Let* (X,d) *be a complete metric space,* p *be a* τ^0*-function,* $\mathfrak{D}_p$ *be a* τ^0*-metric on* $\mathcal{CB}(X)$*,* $T : X \to \mathcal{CB}(X)$ *be a multivalued map, and* $L \ge 0$*. Suppose that the condition* ($\mathcal{A}$2) *holds and further assume that*

($\mathcal{A}$10) *there exist two functions* $\varphi, \tau : [0,\infty) \to [0,1)$ *such that* τ *is an* $\mathcal{MT}$*-function and it holds*

$$\mathfrak{D}_p(Tx,Ty) \le \varphi(p(x,y))p(x,Tx) + \tau(p(x,y))p(x,Ty) + Lp(y,Tx) \quad \forall x,y \in X, \qquad (3.7)$$

then $\mathcal{F}(T) \neq \emptyset$.

Corollary 3.8. *Let* (X,d) *be a complete metric space,* $T : X \to \mathcal{CB}(X)$ *be a multivalued map and* $L \ge 0$*. Suppose that the condition* ($\mathcal{B}$2) *holds and further assume that*

($\mathcal{B}$10) *there exist two functions* $\varphi, \tau : [0,\infty) \to [0,1)$ *such that* τ *is an* $\mathcal{MT}$*-function and it holds*

$$\mathcal{H}(Tx,Ty) \le \varphi(d(x,y))d(x,Tx) + \tau(d(x,y))d(x,Ty) + Ld(y,Tx) \quad \forall x,y \in X, \qquad (3.8)$$

then $\mathcal{F}(T) \neq \emptyset$.

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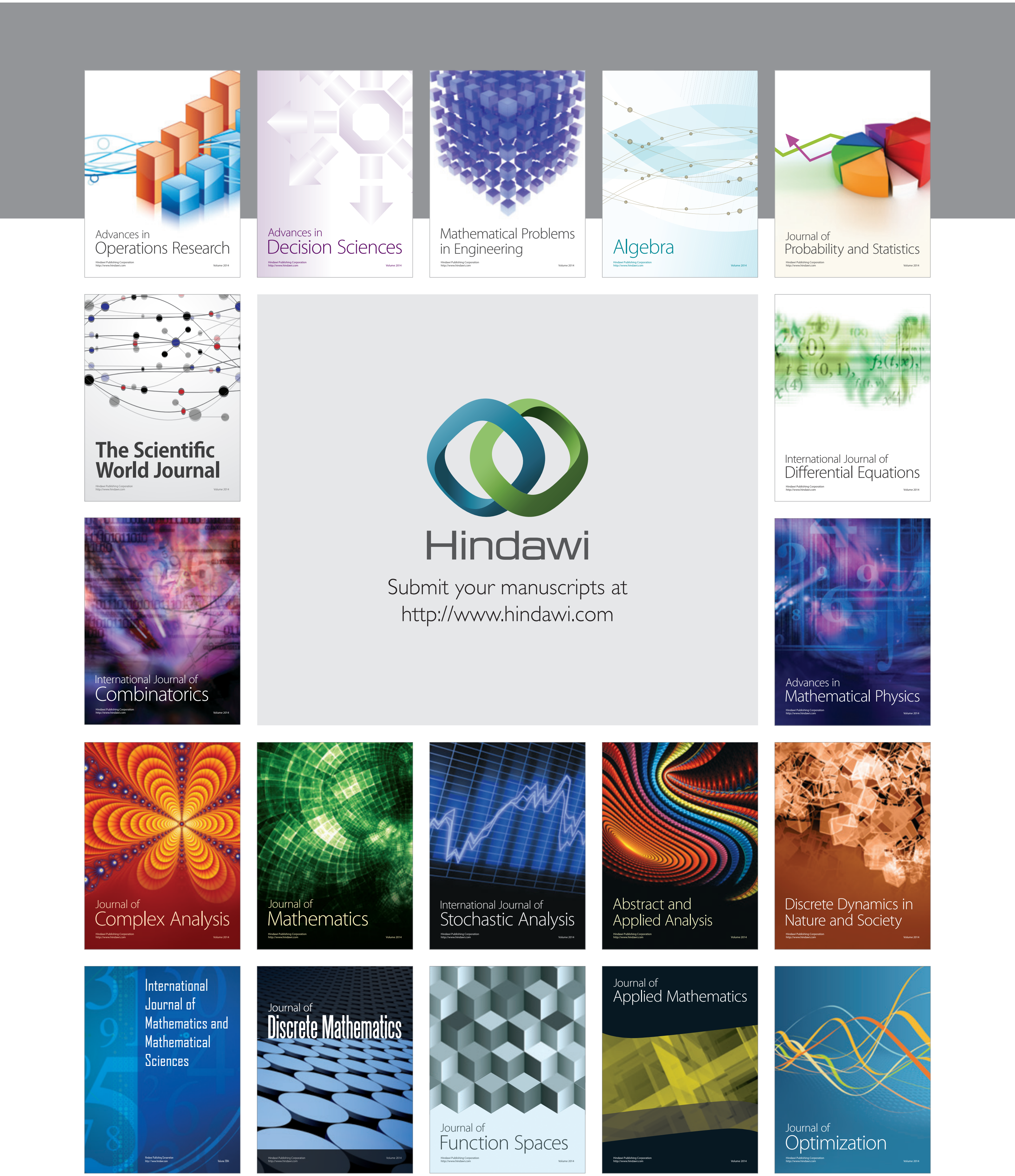
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