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Research Article

Blow-Up Time for Nonlinear Heat Equations with Transcendental Nonlinearity

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A blow-up time for nonlinear heat equations with transcendental nonlinearity is investigated. An upper bound of the first blow-up time is presented. It is pointed out that the upper bound of the first blow-up time depends on the support of the initial datum.

1. Introduction

We are concerned with the initial value problem of nonstationary nonlinear heat equations:

$$\begin{gathered} \frac{\partial}{\partial t} u(x,t) - \Delta u(x,t) = F(u(x,t)), \\ u(x,0) = u_0(x), \end{gathered} \tag{1.1}$$

where $x \in \mathbb{R}^n$, F is a given nonlinear function and u is unknown. Due to the mathematical and physical importance, existence and uniqueness theories of solutions of nonlinear heat equations have been extensively studied by many mathematicians and physicists, for example, [1–10] and references therein. Unlike other studies, we focus on the nonlinear heat equations with *transcendental* nonlinearities such as

$$\frac{\partial}{\partial t} u - \Delta u = |u|^p e^{|u|^q}, \tag{1.2}$$

for some positive real numbers p, q. The nonlinearity in the above problem grows so fast that the solutions may blow up very fast. We are interested in *how fast*! Even though we present only one problem with the specific nonlinear function $F(u) \equiv |u|^p e^{|u|^q}$, this nonlinearity exemplifies (analytic) nonlinearities with rapid growth.

The study of the blow-up problem has attracted a considerable attention in recent years. The latest developments for the case of power type nonlinear terms $F(u) \equiv |u|^{p-1}u$ are mainly devoted to the subjects of blow-up rate, set, profiles, and the possible continuation after blow-up. The continuity with respect to the initial data also has been studied.

The studies on finite time blow-up rates were conducted in [11–21]. For example, it has been proved that for $1 < p < (n+2)/(n-2)$, there exists a uniform constant C such that

$$\|u(t)\|_{L^\infty} \leq Ct^{-1/(p-1)} \tag{1.3}$$

under certain constraints before the blow-up, see [19, 22]. It also has been noticed after the blow-up that for such *subcritical* cases $1 < p < (n+2)/(n-2)$ the blow-up is *complete*, that is to say, a proper continuation of the solution beyond the blow-up point identically equals $+\infty$ in the whole space $\mathbb{R}^n$. The first main contribution in this direction seems to be the work of Baras and Cohen [23] who looked into the complete blow-up of semi-linear heat equations with subcritical power type nonlinear terms, and thus established the validity of a conjecture of H. Brezis (page 143 in [23]). Further results were obtained in [18, 24, 25]; see also the references therein.

It seems to be very natural and important to find the explicit blow-up time in study of the blow-up problem. To the author's knowledge, explicit blow-up time has not been uncovered yet—even for the case of power type nonlinearity. One only began to understand that the blow-up time is continuous with respect to the initial data u_0 (for a certain topological sense) for details, see [8, 23, 24, 26–28].

This paper is mainly concerned with the blow-up time. For the power type nonlinearity, when the blow-up phenomena are established, a partial representation for an upper bound of the (first) blow-up time can be found in Section 9 in [29] and also in [30]. One preliminary observation of this research is that an upper bound of the blow-up time for the case of the power type nonlinear term is related with the explicit solution of the classical Bernoulli's equations (see (3.5) below). For the case of transcendental nonlinearities, we prove a series of ordinary differential inequalities and equations to disclose an effective upper bound of the blow-up time for positive solutions with a large initial datum. We have found that the blow-up time (of the positive solutions) *may* depend not only on the norm of given initial datum but also on the area of the *support of the initial datum*.

The upper bound of the blow-up time we present here is universal in the sense that it is an upper bound for many popular function spaces as explained at Remark 2.3. A better upper bound and a lower bound in a special space, for example the Lebesgue space L^∞, are of obvious interest.

2. The Main Theorem

Let u_0 be a function with compact support in $\mathbb{R}^n$ and let u be a (smooth) solution of (1.2) inside of $\operatorname{supp} u_0$ with a homogeneous Dirichlet's boundary condition and the initial condition $u(x,0) = u_0(x)$. It is clear that $\operatorname{supp} u(t) \subset \operatorname{supp} u_0$ for all $t \geq 0$ if we employ the trivial extension of u to the whole space $\mathbb{R}^n$. By virtue of maximum principle, if the initial source u_0 is nonnegative, so is u. It is also well known that a positive solution u of (1.2) with sufficiently large initial datum blows up within a finite time; that is, there exists a positive constant T^* (the maximal existence time) so that $\lim_{t\uparrow T^*}\|u(t)\|_X = \infty$ in an appropriate function space X. We choose an open ball B_δ of radius δ that contains the support of u_0. We proceed by choosing an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ for $L^2(B_\delta)$, where $w_j \in H_0^1(B_\delta)$ is an eigenfunction corresponding to each eigenvalue λ_j of $-\Delta$:

$$\begin{aligned} -\Delta w_j = \lambda_j w_j \quad &\text{in } B_\delta, \\ w_j = 0 \quad &\text{on } \partial B_\delta, \end{aligned} \tag{2.1}$$

for $j = 1, 2, \ldots$. In particular, we are interested in the eigenfunctions corresponding to the principal eigenvalue $\lambda_1 > 0$.

We recall a relationship between the volume of the domain and the principal eigenvalue of the Laplacian, which says that

$$\lambda_1 = \frac{r_0^2}{\delta^2}, \tag{2.2}$$

where $r_0 > 0$ is the first positive zero of the Bessel function $J_{n/2-1}$ of order $(n/2) - 1$ which can be expressed by elementary functions (for $n \geq 2$, see page 45 in [31]). Also, we may choose an eigenfunction w_1 satisfying

$$w_1 > 0 \quad \text{in } B_\delta, \quad \int_{B_\delta} w_1(x)\, dx = 1. \tag{2.3}$$

A smooth solution u in $H_0^1(B_\delta)$ can be expressed by a linear combination of $\{w_j\}_{j=1}^{\infty}$: $u(x,t) = \sum_{j=1}^{\infty} a_j(t) w_j(x)$ $(0 \leq t < T^*,\ x \in B_\delta)$, where $a_j(t) = \int_{B_\delta} u(x,t) w_j(x) dx$. In particular, we denote the *eigen-coefficient* of u with respect to the eigenfunction w_1 by $\eta(t) \equiv a_1(t)$.

We introduce two specific real numbers m_1 and c_0 as follows: m_1 is the smallest positive integer among m satisfying $qm + p > 1$, and c_0 is the smallest nonnegative number such that $t^p e^{t^q} > \lambda_1 t$ holds for all $t > c_0$.

Theorem 2.1. *Let the spatial dimension n be greater than 1. With the notations above, assume that the given initial source u_0 is large enough that the initial eigen-coefficient $\eta_0 \equiv \int_{B_\delta} u_0(x) w_1(x) dx$ is greater than both $(m_1!\lambda_1)^{1/(qm_1+p-1)}$ and c_0. Then the (first) blow-up time T_η^* of the first eigen-coefficient $\eta(t)$ is less than or equal to the positive number*

$$\frac{\delta^2}{(qm_1 + p - 1) r_0^2} \ln\left(\frac{\delta^2 \eta_0^{qm_1+p-1}}{\delta^2 \eta_0^{qm_1+p-1} - m_1! r_0^2} \right), \tag{2.4}$$

where $\delta = (1/2)\max\{|x - y| : x, y \in \operatorname{supp} u_0\}$.

Remark 2.2. We notice that as the diameter δ of the support of u_0 gets bigger, (2.4) converges to

$$\frac{m_1!}{(qm_1 + p - 1)\eta_0^{qm_1+p-1}}. \tag{2.5}$$

Remark 2.3. By virtue of Hölder's inequality on $\eta(t) = \int_{B_\delta} u(x,t) w_1(x) dx$, it is noted that the blow-up time T^* of $\|u\|_X$ cannot exceed the (first) blow-up time T_η^* of $\eta(t)$. Here the space X can be one of any function spaces that obey Hölder's inequality together with the dual space X'. Classical Lebesgue spaces, BMO, Besov spaces, Triebel-Lizorkin spaces, and Orlicz spaces are some of the examples.

3. The Arguments

The monotone convergence theorem implies that

$$\begin{aligned} \frac{d}{dt}\eta(t) &= \int_{B_\delta} u_t w_1 dx = \int_{B_\delta} \left(\Delta u + |u|^p e^{|u|^q}\right) w_1 dx \\ &= -\lambda_1 \eta(t) + \sum_{k=0}^{\infty} \frac{1}{k!} \int_{B_\delta} |u|^{qk+p} w_1 dx. \end{aligned} \tag{3.1}$$

Hölder's inequality and (2.3), on the other hand, yield that for each k

$$\begin{aligned} |\eta(t)| &\le \int_{B_\delta} |u| w_1 dx \le \left(\int_{B_\delta} |u|^{qk+p} w_1 dx\right)^{1/(qk+p)} \left(\int_{B_\delta} w_1 dx\right)^{(qk+p-1)/(qk+p)} \\ &= \left(\int_{B_\delta} |u|^{qk+p} w_1 dx\right)^{1/(qk+p)}. \end{aligned} \tag{3.2}$$

Therefore we have $|\ \eta(t)|^{qk+p} \le \int_{B_\delta} |u|^{qk+p} w_1 dx$. Apply this inequality on (3.1) to find that for $0 \le t < T^*$,

$$\frac{d}{dt}\eta(t) \ge -\lambda_1 \eta(t) + \sum_{k=0}^{\infty} \frac{1}{k!} |\eta(t)|^{qk+p} = -\lambda_1 \eta(t) + |\eta(t)|^p e^{|\eta(t)|^q}. \tag{3.3}$$

We are now going to find a lower bound function for $\eta(t)$. To do it, take ϕ to be a solution of the ordinary differential equation:

$$\frac{d}{dt}\phi(t) = -\lambda_1 \phi(t) + |\phi(t)|^p e^{|\phi(t)|^q} \tag{3.4}$$

with $\eta(0) = \phi(0)$. We also define a real-valued function f by $f(t) \equiv -\lambda_1 t + |t|^p e^{|t|^q}$. A closer look at (3.3) and a chain of considerations on the choice of c_0 deliver that $\eta(t) \ge \eta(0) = \eta_0 > c_0$, which in turn implies that $(d/dt)\eta(t)/f(\eta(t)) \geq 1$. Integrate both sides with respect to t, and we have $\int_0^t ((d/ds)\eta(s)/f(\eta(s)))ds \geq t$. Consider an indefinite integral F of $1/f(x)$ to get $F(\eta(t)) - F(\eta(0)) \geq t$. Similarly, we can obtain $F(\phi(t)) - F(\phi(0)) = t$. Hence these facts together with $\eta(0) = \phi(0)$ yield that $F(\eta(t)) \geq F(\eta(0)) + t = F(\phi(0)) + t = F(\phi(t))$. Note that F is monotone increasing on (c_0, ∞), and so we can deduce that $\eta(t) \geq \phi(t)$ for $0 \leq t < T^*$.

We will find the first blow-up time for $\phi(t)$. First, for a *fixed* $k \in \mathbb{N}$, we consider two real-valued functions $g, h : [0, \infty) \to \mathbb{R}$ defined by $g(x) \equiv -\lambda_1 x + |x|^p e^{|x|^q}$ and $h(x) \equiv -\lambda_1 x + \sum_{m=0}^{k} (1/m!)\, |x|^{qm+p}$. Then it is clear that $g(x) \geq h(x)$ for all $0 \leq x < \infty$. Let ρ_k be a solution for a Bernoulli-type equation:

$$\frac{d}{dt}\rho_k(t) = h(\rho_k(t)) \tag{3.5}$$

with the initial condition:

$$\rho_k(0) = \left(1 - \frac{1}{k+2}\right)\phi(0). \tag{3.6}$$

Lemma 3.1. *For each k, $\phi(t) \geq \rho_k(t)$ for all $t \in [0, T^*)$.*

Proof. We choose indefinite integrals G and H of $1/g$ and $1/h$, respectively, with the conditions that $G(0) = 0$ and $H(\rho_k(0)) = G(\phi(0))$. We have $G(x) \leq H(x)$ for all x, which follows from the facts that $g(x) \geq h(x)$ for all x and $\rho_k(0) < \phi(0)$. On the other hand, the argument used above leads to get $G(\phi(t)) - G(\phi(0)) = t$, and similarly $t = H(\rho_k(t)) - H(\rho_k(0))$. Hence we arrive at $G(\phi(t)) = H(\rho_k(t))$. From this together with the fact that the function G is dominated by H, we can realize that ρ_k should be dominated by ϕ, that is, $\phi(t) \geq \rho_k(t)$ for all $0 \leq t < T^*$. □

We assert that the sequence $\{\rho_k(t)\}_{k=1}^{\infty}$ is monotone increasing and converges to $\phi(t)$ for $t \in [0, T^*)$. In fact, by the same argument used in Lemma 3.1, it can be noticed that $\{\rho_k(t)\}_{k=1}^{\infty}$ is monotone increasing and bounded above by $\phi(t)$, and so it converges to some $\xi(t)$. The integral representation of (3.5) can be written as

$$\rho_k(t) = \rho_k(0) - \lambda_1 \int_0^t \rho_k(\tau)d\tau + \int_0^t \sum_{m=0}^{k} \frac{1}{m!} \left|\rho_k(\tau)\right|^{qm+p} d\tau. \tag{3.7}$$

Lebesgue dominated convergence theorem together with Lemma 3.1 leads to the (pointwise) limit of (3.7): $\xi(t) = \phi(0) - \lambda_1 \int_0^t \xi(\tau)d\tau + \int_0^t |\xi(t)|^p e^{|\xi(t)|^q} d\tau$, which implies that ξ is the solution of (3.4). The uniqueness of the solution for (3.4) yields that $\xi = \phi$.

We can explicitly compute the solutions ρ_k by observing that $\rho_k = \sum_{m=0}^{k} \varrho_m$, where ϱ_m are solutions for *classical Bernoulli's equations*: $(d/dt)\varrho_m = -\lambda_1 \varrho_m + (1/m!)\varrho_m^{qm+p}$ with initial values:

$$\varrho_m(0) = \left(\frac{1}{m+1} - \frac{1}{m+2}\right)\phi(0). \tag{3.8}$$

By solving each Bernoulli's equation and summing up the solutions, we obtain

$$\rho_k(t) = \sum_{m=0}^{k} \left(\frac{\lambda_1 m!}{\lambda_1 m! - \varrho_m(0)^{qm+p-1}\left(1 - e^{-(qm+p-1)\lambda_1 t}\right)} \right)^{1/(qm+p-1)} e^{-\lambda_1 t} \varrho_m(0), \tag{3.9}$$

provided that the denominator is not zero. In case $(1-p)/q$ is a positive integer, to say m_0, then the m_0-th term in the summation above should be replaced by $\varrho_{m_0}(0)e^{((1/m_0!)-\lambda_1)t}$. Therefore we obtain

$$\phi(t) = \sum_{m=0}^{\infty} \left(\frac{\lambda_1 m!}{\lambda_1 m! - \varrho_m(0)^{qm+p-1}\left(1 - e^{-(qm+p-1)\lambda_1 t}\right)} \right)^{1/(qm+p-1)} e^{-\lambda_1 t} \varrho_m(0). \tag{3.10}$$

The first blow-up time at the right hand side of (3.10) is

$$T_1 \equiv -\frac{1}{(qm_1 + p - 1)\lambda_1} \ln\left(1 - \frac{\{(m_1+1)(m_1+2)\}^{qm_1+p-1} m_1! \lambda_1}{\eta(0)^{qm_1+p-1}}\right), \tag{3.11}$$

(m_1 is defined at page 3) which implies that $T^* \le T_1$, and so the solution blows up before the finite time T_1.

We now present a better upper bound than T_1 of the blow-up time T^*. In fact, the number "$\{(m_1+1)(m_1+2)\}^{qm_1+p-1}$" in (3.11) can be improved by taking another initial data in (3.6) and (3.8). We choose a strictly increasing sequence of real numbers $\{a_k\}_{k=1}^{\infty}$ satisfying $0 = a_1 < a_2 < \cdots < \lim_{k\to\infty} a_k = 1$. Then by replacing the initial conditions in (3.6) and (3.8) with $\rho_k(0) = a_{k+2}\ \phi(0)$ and $\varrho_m(0) = (a_{m+2} - a_{m+1})\phi(0)$, respectively, we have

$$T_\eta^* \le -\frac{1}{(qm_1 + p - 1)\lambda_1} \ln\left(1 - \frac{m_1! \lambda_1}{\{a_{m_1+2} - a_{m_1+1}\}^{qm_1+p-1} \eta(0)^{qm_1+p-1}}\right) \tag{3.12}$$

instead of (3.11). The estimate (3.12) holds for any sequence $\{a_m\}_{m=1}^{\infty}$ with $0 < a_{m_1+1} < a_{m_1+2} < 1$. Therefore letting the number $a_{m_1+2} - a_{m_1+1}$ go to 1, we finally get a better upper bound

$$\frac{1}{(qm_1 + p - 1)\lambda_1} \ln\left(\frac{\eta(0)^{qm_1+p-1}}{\eta(0)^{qm_1+p-1} - m_1! \lambda_1}\right) \tag{3.13}$$

of T^*. This completes the proof.

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